SERIES ASSOCIATED WITH A FORGOTTEN IDENTITY OF NÖRLUND

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Abstract

We apply a seemingly forgotten series expression of Nörlund for the psi function to express infinite series involving inverse factorials in closed form. Many of the series contain products of Catalan numbers and (odd) harmonic numbers. We also prove some new series for π.

1. Motivation and Preliminaries

The Gamma function, $\Gamma(z)$, is defined for $\Re(z) > 0$ by the integral [11]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

The function $\Gamma(z)$ has the property that it extends the classical factorial function to the complex plane by $(z-1)! = \Gamma(z)$. The Gamma function can be extended to the whole complex plane by analytic continuation. Closely related to the Gamma function is the psi (or digamma) function defined by $\psi(z) = \Gamma'(z)/\Gamma(z)$. It possesses a range of integral representations and infinite series expressions. Two such series expressions are [11, p. 14]

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+z} \right), \tag{1}$$

and

$$\psi(z+n) = \psi(z) + \sum_{k=1}^{n} \frac{1}{z+k-1},$$

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where in the series (1) γ denotes the Euler-Mascheroni constant. Coffey [4] has obtained a variety of other series and integral representations of $\psi(z)$. Summations over digamma and polygamma functions were studied, among others, by Alzer et al. [1] and Coffey [5].

A series expression that seems to be not well known is the expression

$$\psi(z+h)-\psi(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\,h\,(h-1)\cdots(h-n)}{(n+1)\,z\,(z+1)\cdots(z+n)}, \tag{2}$$

which holds for all $z,h\in\mathbb{C}$ with $\Re(z)>0$ and $\Re(z+h)>0$. Identity (2) is stated by Hille in his paper [6] from 1927 and is attributed to Nörlund.

The seemingly forgotten Identity (2) will play the key role in this paper. The common feature of all series studied here is the appearance of inverse factorials. Similar series were studied by Sofo [8, 9] and more recently by Karp and Prilepkina [7], and also by Boyadzhiev in his papers from 2020 and 2023 [2, 3]. A finite variant was the subject of another paper by Sofo [10].

Among other things, we will establish connections with harmonic numbers H_α and odd harmonic numbers O_α defined for $0\neq\alpha\in\mathbb{C}\setminus\mathbb{Z}^-$ by the recurrence relations

$$H_\alpha=H_{\alpha-1}+\frac{1}{\alpha}\qquad\text{and}\qquad O_\alpha=O_{\alpha-1}+\frac{1}{2\alpha-1},$$

with $H_0=0$ and $O_0=0$. Harmonic numbers are connected to the digamma function through the fundamental relation

$$H_\alpha=\psi(\alpha+1)+\gamma.$$

Generalized harmonic numbers $H_\alpha^{(m)}$ and generalized odd harmonic numbers $O_\alpha^{(m)}$ of order $m\in\mathbb{C}$ are defined by

$$H_\alpha^{(m)}=H_{\alpha-1}^{(m)}+\frac{1}{\alpha^m}\qquad\text{and}\qquad O_\alpha^{(m)}=O_{\alpha-1}^{(m)}+\frac{1}{(2\alpha-1)^m},$$

with $H_0^{(m)}=0$ and $O_0^{(m)}=0$ so that $H_\alpha=H_\alpha^{(1)}$ and $O_\alpha=O_\alpha^{(1)}$. The recurrence relations imply that if $\alpha=n$ is a non-negative integer, then

$$H_n^{(m)}=\sum_{j=1}^{n}\frac{1}{j^m}\qquad\text{and}\qquad O_n^{(m)}=\sum_{j=1}^{n}\frac{1}{(2j-1)^m}.$$

Generalized harmonic numbers are linked to the polygamma functions $\psi^{(r)}(z)$ of order r defined by

$$\psi^{(r)}(z)=\frac{d^r}{dz^r}\psi(z)=(-1)^{r+1}r!\sum_{j=0}^{\infty}\frac{1}{(j+z)^{r+1}},$$

through

$$H_z^{(r)} = \zeta(r) + \frac{(-1)^{r-1}}{(r-1)!}\psi^{(r-1)}(z+1),$$

where $\zeta(y)$ is the Riemann zeta function.

To whet the reader's appetite for reading on, we present the following samples from our results:

$$1 + \sum_{n=1}^{\infty} \frac{1}{n+1} \prod_{j=1}^{n} \frac{3j-1}{3j+1} = \frac{\pi}{\sqrt{3}},$$

$$1 + \sum_{n=1}^{\infty} \frac{2^n}{n+1} \prod_{j=1}^{n} \frac{5j-2}{10j+3} = \frac{3\pi}{4}\sqrt{\frac{\sqrt{5}}{\alpha^3}}, \qquad (\alpha = (1+\sqrt{5})/2)$$

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{\binom{n}{h}\binom{2(n+1)}{(n+1)}(n+1)^2} = \frac{\pi}{4}\,\frac{H_{h-1/2} - H_{-1/2}}{\sin(\pi h)}, \quad h \in \mathbb{C}\setminus\mathbb{Z},\ 2h \notin \mathbb{Z}^-,$$

and for non-negative integer h,

$$\sum_{n=h}^{\infty} \frac{\binom{2(n-h)}{n-h}}{\binom{n}{h}\binom{2(n+1)}{n+1}} \frac{O_{n-h}}{(n+1)^2} = -(-1)^h \frac{\zeta(2) - H_h^{(2)}}{2(h+1)\binom{2(h+1)}{h+1}} + (-1)^h \frac{H_h + 2\ln 2}{(h+1)\binom{2(h+1)}{h+1}} O_{h+1} - \sum_{n=0}^{h-1} \frac{(-1)^{n-h}\binom{h}{n}}{\binom{2(h-n)}{h-n}\binom{2(n+1)}{n+1}} \frac{O_{n-h}}{(n+1)^2}.$$

2. First New Series Associated with Identity (2)

We first prove two results that are immediate consequences of Identity (2) and which yield some possibly new series for π. Two such series provide unexpected expressions involving the golden section $\alpha = (1+\sqrt{5})/2$. We then evaluate several series of certain ratios involving Catalan numbers and binomial coefficients in closed form.

Theorem 1. *For each integer $k \geq 2$ we have the expression*

$$(k-2)\left(1 + \sum_{n=1}^{\infty} \frac{1}{n+1} \prod_{j=1}^{n} \frac{kj-(k-2)}{kj+1}\right) = \pi \cot\left(\frac{\pi}{k}\right).$$

In particular, we have the curious expressions for π in the form

$$1+\sum_{n=1}^{\infty}\frac{1}{n+1}\prod_{j=1}^{n}\frac{3j-1}{3j+1}=\frac{\pi}{\sqrt{3}},$$
$$1+\sum_{n=1}^{\infty}\frac{2^n}{n+1}\prod_{j=1}^{n}\frac{2j-1}{4j+1}=\frac{\pi}{2}, \tag{3}$$
$$1+\sum_{n=1}^{\infty}\frac{1}{n+1}\prod_{j=1}^{n}\frac{5j-3}{5j+1}=\frac{\pi}{3}\sqrt{\frac{\alpha^3}{\sqrt{5}}}$$

and

$$1+\sum_{n=1}^{\infty}\frac{2^n}{n+1}\prod_{j=1}^{n}\frac{3j-2}{6j+1}=\frac{\sqrt{3}\pi}{4}. \tag{4}$$

Proof. Set $h=1-2z$ in Identity (2). Then, for all $z\in\mathbb{C}$ with $0<\Re(z)<1$, we have

$$\pi\cot(\pi z)=\psi(1-z)-\psi(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\prod_{j=0}^{n}\left(\frac{1-z}{j+z}-1\right)$$

and with $z=1/k$, $k\geq 2$, the claim follows. □

Theorem 2. *For all $z\in\mathbb{C}$ with $-1/2<\Re(z)<1/2$, we have*

$$\sum_{n=0}^{\infty}\frac{2^{n+1}}{n+1}\prod_{j=0}^{n}\frac{2z-j}{2z-(2j+1)}=-\pi\tan(\pi z). \tag{5}$$

In particular, when $z=1/3$ then Identity (5) *gives Identity* (4)*, and when $z=1/4$ then we get Identity* (3)*. When $z=1/5$, then Identity* (5) *yields*

$$1+\sum_{n=1}^{\infty}\frac{2^n}{n+1}\prod_{j=1}^{n}\frac{5j-2}{10j+3}=\frac{3\pi}{4}\sqrt{\frac{\sqrt{5}}{\alpha^3}}.$$

Proof. In Identity (2) make the replacements $z\mapsto 1/2-z$ and $h\mapsto 2z$, while keeping in mind that

$$\psi\left(\frac{1}{2}+z\right)-\psi\left(\frac{1}{2}-z\right)=\pi\tan(\pi z).$$

□

Since

$$h(h-1)\cdots(h-n)=\prod_{j=0}^{n}(h-j)=(-1)^{n+1}\frac{\Gamma(n+1-h)}{\Gamma(-h)}$$

and

$$z(z+1)\cdots(z+n)=\prod_{j=0}^{n}(z+j)=\frac{\Gamma(n+1+z)}{\Gamma(z)},$$

Identity (2) can be written in terms of harmonic numbers via Identity (1) as

$$H_{z+h-1}-H_{z-1}=-\frac{\Gamma(z)}{\Gamma(-h)}\sum_{n=0}^{\infty}\frac{\Gamma(n+1-h)}{(n+1)\Gamma(n+1+z)}. \tag{6}$$

Lemma 1. *We have*

$$\begin{aligned} H_{k-1/2}&=2O_k-2\ln 2,\\ H^{(2)}_{k-1/2}&=-2\zeta\left(2\right)+4O^{(2)}_k,\\ H^{(3)}_{-1/2}&=-6\zeta\left(3\right),\\ H^{(m+1)}_{k-1/2}-H^{(m+1)}_{-1/2}&=2^{m+1}O^{(m+1)}_k. \end{aligned}$$

Lemma 2. *We have*

$$\begin{aligned} (r-1/2)!&=\frac{\sqrt{\pi}}{2^{2r}}\binom{2r}{r}r!,\quad r\in\mathbb{C}\setminus\mathbb{Z}^-,\\ (-r-1/2)!&=\sqrt{\pi}\frac{(-1)^r}{r!}\frac{2^{2r}}{\binom{2r}{r}},\quad r\in\mathbb{N}_0. \end{aligned}$$

Theorem 3. *We have*

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{C_n}{2^{2n}\left(n+1\right)}&=4\left(1-\ln 2\right),\\ \sum_{n=0}^{\infty}\frac{C_n}{2^{2n}\left(n+1\right)\left(n+2\right)}&=\frac{10}{3}-4\ln 2, \end{aligned}$$

and more generally, if $0\neq z\in\mathbb{C}\setminus\mathbb{Z}^-$, $2z\notin\mathbb{Z}^-$, *then*

$$\sum_{n=0}^{\infty}\frac{C_n}{2^{2n}\binom{n+z}{n}}=-2z\left(H_{z-1}-H_{z-1/2}\right), \tag{7}$$

where here and throughout this paper C_j *are Catalan numbers, defined for* $j\in\mathbb{N}_0$ *by*

$$C_j=\frac{\binom{2j}{j}}{j+1}.$$

Proof. Set $h=1/2$ in Identity (6), express the Gamma functions as factorials and then arrange. The result is

$$-\sum_{n=0}^{\infty}\frac{(n-1/2)!}{n!\,(n+1)}\frac{1}{z}\frac{z!n!}{(z+n)!}=2\sqrt{\pi}(H_{z-1}-H_{z-1/2}),$$

that is,

$$-\sum_{n=0}^{\infty}\frac{\sqrt{\pi}}{2^{2n}}\frac{\binom{2n}{n}}{n+1}\frac{1}{z}\frac{1}{\binom{z+n}{n}}=2\sqrt{\pi}(H_{z-1}-H_{z-1/2}),$$

and hence Identity (7) follows. ☐

Theorem 4. *We have*

$$\sum_{n=0}^{\infty}\frac{O_{n+1}}{(n+1)(2n+1)}=\frac{\pi^2}{6}, \tag{8}$$

$$\sum_{n=0}^{\infty}\frac{C_n}{2^{2n}}\frac{H_{n+1}}{n+1}=8-\frac{2\pi^2}{3}, \tag{9}$$

and more generally, if $0\neq z\in\mathbb{C}\setminus\mathbb{Z}^-$, $2z\notin\mathbb{Z}^-$, *then*

$$\sum_{n=0}^{\infty}\frac{C_n}{2^{2n}}\frac{H_{n+z}}{\binom{n+z}{n}}=2(H_{z-1}-H_{z-1/2})(1-zH_z)+2z\left(-H_{z-1}^{(2)}+H_{z-1/2}^{(2)}\right). \tag{10}$$

Proof. Differentiate Identity (7) with respect to z to obtain

$$\sum_{n=0}^{\infty}\frac{C_n}{2^{2n}}\frac{(H_{n+z}-H_z)}{\binom{n+z}{n}}=2H_{z-1}-2H_{z-1/2}+2z\left(-H_{z-1}^{(2)}+H_{z-1/2}^{(2)}\right);$$

from which Identity (10) follows upon a second use of (7). Evaluation of (10) at $z=1/2$ gives (8) while evaluation at $z=1$ gives (9). ☐

Theorem 5. *We have*

$$\sum_{n=0}^{\infty}\frac{2n+1}{n+1}\frac{C_n}{2^{2n}}=4\ln 2,$$

$$\sum_{n=0}^{\infty}\frac{2n+1}{(n+1)(n+2)}\frac{C_n}{2^{2n}}=4\ln 2-2,$$

and more generally, if $z\in\mathbb{C}$ *with* $z-3/2\notin\mathbb{Z}^-$, *then*

$$\sum_{n=0}^{\infty}\frac{2n+1}{2^{2n}}\frac{C_n}{\binom{n+z}{n}}=2z\left(H_{z-1}-H_{z-3/2}\right). \tag{11}$$

Proof. The proof is very similar to the proof of Theorem 3. Set $h=-1/2$ in Identity (2) and use $\Gamma(1/2)=\sqrt{\pi}$. ☐

Corollary 1. *If* $z\in\mathbb{C}$ *with* $z-3/2\notin\mathbb{Z}^-$, *then*

$$\sum_{n=0}^{\infty}\frac{n}{2^{2n}}\frac{C_n}{\binom{n+z}{n}}=2zH_{z-1}-z\left(H_{z-1/2}+H_{z-3/2}\right). \tag{12}$$

Proof. Combine Theorems 3 and 5. □

Remark 1. We have

$$\sum_{n=0}^{\infty} \frac{n}{(n+1)} \frac{C_n}{2^{2n}} = \sum_{n=0}^{\infty} \frac{2n+1}{(n+1)(n+2)} \frac{C_n}{2^{2n}} = 4\ln 2 - 2.$$

Theorem 6. *If $z \in \mathbb{C}$ with $z - 3/2 \notin \mathbb{Z}^-$, then*

$$\sum_{n=0}^{\infty} \frac{2n+1}{2^{2n}} \frac{C_n H_{n+z}}{\binom{n+z}{n}} = 2(zH_z - 1)\left(H_{z-1} - H_{z-3/2}\right) + 2z\left(H_{z-1}^{(2)} - H_{z-3/2}^{(2)}\right). \quad (13)$$

In particular,

$$\sum_{n=0}^{\infty} \frac{2n+1}{n+1} \frac{C_n H_{n+1}}{2^{2n}} = \frac{2\pi^2}{3},$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{(n+2)(n+1)} \frac{C_n H_{n+2}}{2^{2n}} = \frac{2\pi^2}{3} + 4\ln 2 - 8,$$

and

$$\sum_{n=0}^{\infty} \frac{O_{n+2}}{(n+1)(2n+3)} = 4 - 2\ln 2 - \frac{\pi^2}{6}. \quad (14)$$

Proof. Differentiate Identity (11) with respect to z and simplify. Identity (14) is obtained by setting $z = 3/2$ in Identity (13) and using Lemma 1. □

Theorem 7. *If $z \in \mathbb{C}$ with $z - 3/2 \notin \mathbb{Z}^-$, then*

$$\sum_{n=0}^{\infty} \frac{n}{2^{2n}} \frac{C_n H_{n+z}}{\binom{n+z}{n}} = (zH_z - 1)\left(2H_{z-1} - H_{z-1/2} - H_{z-3/2}\right) + z\left(2H_{z-1}^{(2)} - H_{z-1/2}^{(2)} - H_{z-3/2}^{(2)}\right). \quad (15)$$

In particular, at $z = 1$ and at $z = 3/2$ we obtain

$$\sum_{n=1}^{\infty} \frac{nC_n}{2^{2n}} \frac{H_{n+1}}{n+1} = \frac{2\pi^2}{3} - 4$$

and

$$\sum_{n=1}^{\infty} \frac{nO_{n+2}}{(n+1)(2n+1)(2n+3)} = \frac{13}{4} - 2\ln 2 - \frac{\pi^2}{6}, \quad (16)$$

respectively.

Proof. Differentiate Identity (12) with respect to z and simplify. Note that we used

$$\binom{n+3/2}{n} = \frac{(2n+1)(2n+3)}{2^{2n}\, 3} \binom{2n}{n}$$

in Identity (15) to derive the special case in (16). □

3. More New Series Associated with Identity (2)

Theorem 8. *We have*

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{\binom{2(n+1)}{n+1}(n+1)^2} = \frac{\pi^2}{8}, \tag{17}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(n+1)} = 2\ln 2, \tag{18}$$

and, more generally, for $h \notin \mathbb{Z}$, $2h \notin \mathbb{Z}^-$,

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{\binom{n}{h}\binom{2(n+1)}{(n+1)}(n+1)^2} = \frac{\pi}{4}\,\frac{H_{h-1/2} - H_{-1/2}}{\sin(\pi h)}. \tag{19}$$

Proof. Set $z = 1/2$ in Identity (6) and express as factorials to obtain

$$\sum_{n=0}^{\infty} \frac{(n-h)!h!}{n!}\,\frac{n!\sqrt{\pi}}{(n+1/2)!}\,\frac{1}{n+1} = h!\,(-1-h)!\left(H_{-1/2} - H_{h-1/2}\right),$$

from which Identity (19) follows upon using Lemma 2 and the Euler reflection formula:

$$(-r)!\,(r-1)! = \frac{\pi}{\sin(\pi r)}, \quad r \notin \mathbb{Z}.$$

Identities (17) and (18) are special cases of (19) at $h = 0$ and $h = 1/2$. Note that in deriving Identity (17), we used L'Hospital's rule:

$$\begin{aligned}\lim_{h\to 0} \frac{H_{h-1/2} - H_{-1/2}}{\sin(\pi h)} &= \lim_{h\to 0} \frac{\psi\,(h+1/2) - \psi\,(1/2)}{\sin(\pi h)} \\ &= \lim_{h\to 0} \frac{\psi^{(1)}\,(h+1/2)}{\pi\cos(\pi h)} = \frac{\pi}{2}.\end{aligned}$$

□

Corollary 2. *If h is a non-negative integer, then*

$$\sum_{n=h}^{\infty} \frac{\binom{2(n-h)}{n-h}}{\binom{n}{h}\binom{2(n+1)}{(n+1)}(n+1)^2} = (-1)^h \frac{H_h + 2\ln 2}{(h+1)\binom{2(h+1)}{h+1}} - \sum_{n=0}^{h-1} \frac{(-1)^{n-h}\binom{h}{n}}{\binom{2(h-n)}{(h-n)}\binom{2(n+1)}{n+1}(n+1)^2}. \tag{20}$$

Proof. Write $h + 1/2$ for h in Identity (19) and use the fact that if n and h are non-negative integers, then

$$\binom{n}{h+1/2} = \frac{2^{2n+2}}{\pi\,(h+1)} \begin{cases} \binom{n}{h}\binom{2(n-h)}{n-h}^{-1}\binom{2(h+1)}{h+1}^{-1} & \text{if } n \geq h, \\ (-1)^{h-n}\binom{2(h-n)}{h-n}\binom{h}{n}^{-1}\binom{2(h+1)}{h+1}^{-1} & \text{if } n \leq h, \end{cases} \tag{21}$$

since

$$\binom{n}{h+1/2} = \begin{cases} \frac{n!}{(n+1/2)!(n-h-1/2)!} & \text{if } n \geq h, \\ \frac{n!}{(n+1/2)!(-(h-n)-1/2)!} & \text{if } n \leq h. \end{cases}$$

□

Theorem 9. *If h is a non-negative integer, then*

$$\sum_{n=h}^{\infty} \frac{\binom{2(n-h)}{n-h}}{\binom{n}{h}\binom{2(n+1)}{n+1}} \frac{O_{n-h}}{(n+1)^2} = -(-1)^h \frac{\zeta(2) - H_h^{(2)}}{2(h+1)\binom{2(h+1)}{h+1}} + (-1)^h \frac{H_h + 2\ln 2}{(h+1)\binom{2(h+1)}{h+1}} O_{h+1} - \sum_{n=0}^{h-1} \frac{(-1)^{n-h}\binom{h}{n}}{\binom{2(h-n)}{h-n}\binom{2(n+1)}{n+1}} \frac{O_{h-n}}{(n+1)^2}. \tag{22}$$

Proof. Differentiate Identity (19) with respect to h to obtain

$$\sum_{n=0}^{\infty} \frac{2^{2n}(H_{n-h} - H_h)}{\binom{n}{h}\binom{2(n+1)}{n+1}(n+1)^2} = \frac{\pi^2 \cos(\pi h)}{4\sin^2(\pi h)} \left(H_{h-1/2} - H_{-1/2}\right) - \frac{\pi}{4\sin(\pi h)} \left(\zeta(2) - H_{h-1/2}^{(2)}\right).$$

Now write $h + 1/2$ for h and use Lemma 1 and Identities (20) and (21). □

Note that when $h = 0$ we get

$$\sum_{n=0}^{\infty} \frac{O_n}{(n+1)(2n+1)} = -\frac{\pi^2}{12} + 2\ln 2,$$

which in view of Identity (8) also gives

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)^2} = \frac{\pi^2}{4} - 2\ln 2.$$

When $h = 1$ we see that

$$\sum_{n=1}^{\infty} \frac{O_{n-1}}{(n+1)(2n+1)(2n-1)} = \frac{\pi^2}{36} - \frac{8}{9}\ln 2 + \frac{7}{18}.$$

Theorem 10. *If m is a positive integer, then*

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{(n+1)\binom{n+m}{m}\binom{2(n+m)}{n+m}} = \frac{m}{4} \frac{3\zeta(2) - 4\,O_{m-1}^{(2)}}{\binom{2(m-1)}{m-1}}. \tag{23}$$

In particular, we have

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{(n+1)^2\binom{2(n+1)}{n+1}} = \frac{\pi^2}{8}.$$

Proof. Arrange Identity (6) as

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1-h)}{(n+1)\,\Gamma(n+1+z)} = \Gamma(-h)\left(H_{z-1} - H_{z+h-1}\right)\frac{1}{\Gamma(z)}$$

and take the limit as h approaches zero, using

$$\begin{aligned}\lim_{h\to 0} \Gamma(-h)\left(H_{z-1} - H_{z+h-1}\right) &= \lim_{h\to 0} \Gamma(-h)\left(\psi(z) - \psi(z+h)\right)\\ &= \psi^{(1)}(z) = \zeta(2) - H_{z-1}^{(2)},\end{aligned}$$

to obtain

$$\sum_{n=0}^{\infty} \frac{n!}{(n+1)\,\Gamma(n+1+z)} = \frac{1}{\Gamma(z)}\left(\zeta(2) - H_{z-1}^{(2)}\right).$$

Now replace z with $z - 1/2$ and use

$$\Gamma(r+1/2) = \binom{r-1/2}{r} r!\sqrt{\pi}, \tag{24}$$

to obtain

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)\binom{n+z}{z}\binom{n+z-1/2}{n+z}} = \frac{z}{\binom{z-3/2}{z-1}}\left(\zeta(2) - H_{z-3/2}^{(2)}\right), \tag{25}$$

which is valid for $z \in \mathbb{C} \setminus \mathbb{Z}^-$ such that $2z \notin \mathbb{Z}^-$, $z \neq 0$ and $z \neq 1/2$. Finally set $z = m$, a positive integer, in Identity (25) and use

$$\binom{p-1/2}{p} = \frac{\binom{2p}{p}}{2^{2p}}; \tag{26}$$

which gives Identity (23) on account of Lemma 1. □

Theorem 11. *If m is a positive integer, then*

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{2^{2n} O_{n+m}}{(n+1)\binom{n+m}{m}\binom{2(n+m)}{n+m}}\\ &\quad = \frac{m}{4}\binom{2(m-1)}{m-1}^{-1}\left(7\,\zeta(3) - 8\,O_{m-1}^{(3)} + O_{m-1}(3\,\zeta(2) - 4\,O_{m-1}^{(2)})\right).\end{aligned} \tag{27}$$

In particular, we have

$$\sum_{n=0}^{\infty} \frac{2^{2n} O_{n+1}}{\binom{2(n+1)}{n+1}(n+1)^2} = \frac{7}{4}\,\zeta(3).$$

Proof. Differentiate Identity (25) with respect to z to obtain

$$\sum_{n=0}^{\infty}\frac{H_{n+z-1/2}}{(n+1)\binom{n+z-1/2}{n+z}\binom{n+z}{z}}=\frac{z}{\binom{z-3/2}{z-1}}\left(\zeta(2)-H_{z-3/2}^{(2)}\right)\left(H_{z-3/2}-H_{z-1}\right)+\frac{2z}{\binom{z-3/2}{z-1}}\left(\zeta(3)-H_{z-3/2}^{(3)}\right)+\frac{zH_{z-1}}{\binom{z-3/2}{z-1}}\left(\zeta(2)-H_{z-3/2}^{(2)}\right)$$

and, hence, Identity (27) holds by using Lemma 1 and Identity (26). □

Theorem 12. *If m is a positive integer, then*

$$\sum_{n=0}^{\infty}\frac{C_n}{\binom{n+m}{n}\binom{2(n+m)}{n+m}}=\frac{m}{2}\,\frac{(H_{m-1}-2\,O_{m-1}+2\,\ln 2)}{\binom{2(m-1)}{m-1}}. \tag{28}$$

Proof. Set $h=1/2$, write $z-1/2$ for z in Identity (6) and use Identity (24) to obtain

$$\sum_{n=0}^{\infty}\frac{\binom{n-1/2}{n}}{(n+1)\binom{n+z-1/2}{n+z}\binom{n+z}{z}}=\frac{2z}{\binom{z-3/2}{z-1}}\left(H_{z-1}-H_{z-3/2}\right), \tag{29}$$

from which Identity (28) follows after setting $z=m$, a positive integer, and using Identity (26) and Lemma 1. □

Theorem 13. *If m is a positive integer, then*

$$\sum_{n=0}^{\infty}\frac{C_nO_{n+m}}{\binom{n+m}{m}\binom{2(n+m)}{n+m}}=\frac{m}{4}\binom{2(m-1)}{m-1}^{-1}\Big(H_{m-1}^{(2)}+2\,\zeta(2)-4\,O_{m-1}^{(2)}+2\,O_{m-1}\left(H_{m-1}-2\,O_{m-1}+2\ln 2\right)\Big). \tag{30}$$

Proof. Differentiate Identity (29) with respect to z, obtaining

$$\sum_{n=0}^{\infty}\frac{\binom{n-1/2}{n}H_{n+z-1/2}}{(n+1)\binom{n+z-1/2}{n+z}\binom{n+z}{z}}=\frac{2z}{\binom{z-3/2}{z-1}}\left(H_{z-1}^{(2)}-H_{z-3/2}^{(2)}-\left(H_{z-1}-H_{z-3/2}\right)^2\right)+\frac{2zH_{z-1}}{\binom{z-3/2}{z-1}}\left(H_{z-1}-H_{z-3/2}\right),$$

from which Identity (30) follows. □

Remark 2. When $m=1$ in Theorem 13 we recover Identity (8).

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References

[1] H. Alzer, D. Karayannakis, and H. M. Srivastava, Series representations for some mathematical constants, *J. Math. Anal. Appl.* **320** (2006), 145-162.

[2] K. N. Boyadzhiev, New series identities with Cauchy, Stirling, and harmonic numbers, and Laguerre polynomials, *J. Integer Seq.* **23** (2020), Article 20.11.7.

[3] K. N. Boyadzhiev, Stirling numbers and inverse factorial series, *Contrib. Math.* **7** (2023), 24-33.

[4] M. W. Coffey, Integral and series representations of the digamma and polygamma functions, *Analysis* **32** (2012), 317-337.

[5] M. W. Coffey, On one-dimensional digamma and polygamma series related to the evaluation of Feynman diagrams, *J. Comput. Appl. Math.* **183** (2005), 84-100.

[6] E. Hille, On the logarithmic derivatives of the Gamma function, *Det Kgl. Danske Videnskabernes Selskab*, Mathematisk-fysiske Meddelelser. **VIII** (1) (1927), 17 pp.

[7] D. B. Karp and E. G. Prilepkina, An inverse factorial series for a general gamma ratio and related properties of the Nörlund-Bernoulli polynomials, *J. Math. Sci.* **234** (2018), 680-696.

[8] A. Sofo, General properties involving reciprocals of binomial coefficients, *J. Integer Seq.* **9** (2006), Article 06.4.5.

[9] A. Sofo, Integrals and polygamma representations for binomial sums, *J. Integer Seq.* **13** (2010), Article 10.2.8.

[10] A. Sofo, Reciprocal power sums, *Integers* **12** (2012), #A39, 9 pp.

[11] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, 2001.