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LOCAL TO GLOBAL PRINCIPLE FOR HIGHER MOMENTS OVER FUNCTION FIELDS

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Abstract

We establish a local to global principle for higher moments over holomorphy rings of global function fields and use it to compute the higher moments of rectangular unimodular matrices and Eisenstein polynomials with coefficients in such rings.

1. Introduction

A classical problem in number theory is to compute the natural density of subsets of the integers. The natural density of a subset $A \subseteq \mathbb{Z}^d$ is given by considering the number of points in $A \cap [-H; H)^d$, normalizing it by $(2H)^d$ - the number of points in the whole "box" $[-H; H)^d$ - and then taking $H \to \infty$. A convenient tool to compute such densities in special situations was developed in [16, Lemma 20]. If the set A can be characterized locally in the sense that $A = \bigcap_p (\mathbb{Z}^d \cap U_p)$ for some defining sets $U_p \subseteq Z_p^d$ (which is the case if A is defined by equations modulo prime powers), then under certain conditions the natural density of A can be expressed in terms of the Haar measures of the U_p. One would naturally expect that such a local to global principle for natural density should hold for any global field. Indeed, a similar result was established for number fields in [1, Proposition 3.2]. Finally, the case of global function fields was covered in [7, Theorem 2.1].

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One should think of the natural density as a substitute for a finite Haar measure on the integers. It then becomes natural to ask whether one can make sense of the notion of expected value (or any higher moment). If $U_p \subseteq \mathbb{Z}_p^d$ are again the defining sets of our set in $\mathbb{Z}^d$, then the expected value is defined as

$$\lim_{H\to\infty} \sum_{a\in[-H;H)^d} \frac{|\{p \ : \ p \text{ prime}, \ a \in U_p\}|}{(2H)^d}.$$

This means, for the elements in the box $[-H;H)^d$ we count the number of defining sets in which it is contained, average over the total number of points in the box, and let the side length of the box go to infinity. This notion was considered in [5] for Eisenstein polynomials.

One can make a similar definition of expected value over the ring of algebraic integers of number fields or for higher moments. In [11, 12], a local to global principle for higher moments over number fields was established based on a local to global principle for natural densities [1, 16]. In [8] the notion of natural density for holomorphy rings of global function fields was introduced (slightly different from the one in [14]), where the boxes are replaced by Riemann-Roch spaces (see Section 2 for precise definitions). For local to global principles for densities over global Dedekind domains, we direct the interested reader to [3]. In this paper we will prove a local to global principle for higher moments over global function fields and show how to use this tool for some interesting examples.

This paper is organized as follows. In Section 2 we recall the local to global principle for the natural density in global function fields as introduced in [7]. In Section 3 we will prove our main theorem, the local to global principle for higher moments over function fields, and in Section 4 we will apply it to some examples (coprime pairs, affine Eisenstein polynomials, and rectangular unimodular matrices, all with coefficients in the holomorphy ring of some global function field).

2. Preliminaries

In this section we recall the basic definitions and results from [7]. We follow the terminology of [17] for function fields and related concepts.

Let F be a global function field, that is, a finite extension $F/\mathbb{F}_q(X)$, where $\mathbb{F}_q$ denotes a finite field with q elements. We denote by $\mathcal{O}_P$ a valuation ring of F, having maximal ideal P. Such an ideal is called a *place* of the corresponding function field. The set of all places of F will be denoted by $\mathbb{P}_F$. If $\emptyset \neq S \subsetneq \mathbb{P}_F$ and $t \in \mathbb{N}$, we define

$$S_t = \{P \in S \ : \ \deg(P) \geq t\}. \tag{1}$$

Moreover, we write $\mathcal{O}_S$ to denote the holomorphy ring of S,

$$\mathcal{O}_S = \bigcap_{P\in S} \mathcal{O}_P.$$

The easiest example of a holomorphy ring is $\mathbb{F}_q[x]$, as this consists of the intersection of all the valuation rings of $\mathbb{F}_q(x)$ different from the infinite place (see [17, Section 1.2]). We denote by $\mathrm{Div}(F)$ the set of divisors of F, i.e. the free abelian group on the set $\mathbb{P}_F$. Furthermore, for $D = \sum_{P\in\mathbb{P}_F} n_P P \in \mathrm{Div}(F)$ we define $v_P(D) = n_P$. The *support* of D, $\mathrm{supp}(D)$, is defined as the finite subset of $\mathbb{P}_F$ for which $v_P(D)$ is non-zero. For $D, \widetilde{D} \in \mathrm{Div}(F)$ we write $D \leq \widetilde{D}$ if and only if $v_P(D) \leq v_P(\widetilde{D})$ for all $P \in \mathbb{P}_F$. Note that this defines a partial order on $\mathrm{Div}(F)$. Moreover, we will write $D \geq 0$ whenever $v_P(D) \geq 0$ for all P in $\mathbb{P}_F$. Let

$$\mathrm{Div}^+(F) = \{D \in \mathrm{Div}(F) \mid D \geq 0\}.$$

For $S \subseteq \mathbb{P}_F$, let $\mathcal{D}_S$ be the subset of divisors of $\mathrm{Div}^+(F)$ having support contained in the complement of S.

Let $(a_D)_{D\in\mathcal{D}_S} \subseteq \mathbb{R}$ then, for $a \in \mathbb{R}$, we write

$$\lim_{D\in\mathcal{D}_S} a_D = a$$

if for every $\varepsilon > 0$ there exists $D_\varepsilon \in \mathcal{D}_S$ such that, for all $D \in \mathcal{D}_S$ with $D \geq D_\varepsilon$, one has $|a - a_D| < \varepsilon$. Similarly one defines $\limsup_{D\in\mathcal{D}_S} a_D$ and $\liminf_{D\in\mathcal{D}_S} a_D$. For further information on Moore–Smith convergence, see [4, Chapter 2].

We define the *upper density* for $A \subseteq \mathcal{O}_S^d$ as

$$\overline{\rho}_S(A) := \limsup_{D\in\mathcal{D}_S} \frac{|A \cap \mathcal{L}(D)^d|}{q^{\ell(D)d}},$$

where $\mathcal{L}(D)$ is the Riemann-Roch space attached to the divisor D and $\ell(D) = \dim_{\mathbb{F}_q}(\mathcal{L}(D))$. Analogously, one can give a notion of *lower density* $\underline{\rho}_S$ by replacing the limit superior by the limit inferior. Whenever these two quantities coincide, we define the density of A as $\overline{\rho}_S(A) = \underline{\rho}_S(A) = \rho_S(A)$.

This definition of density coincides with the classical definition,

$$\lim_{d\to\infty} \frac{|A \cap \{f \in \mathbb{F}_q[x] \ : \ \deg(f) \leq d\}|}{|\{f \in \mathbb{F}_q[x] \ : \ \deg(f) \leq d\}|},$$

when $\mathcal{O}_S = \mathbb{F}_q[x]$ (that is, $F = \mathbb{F}_q(x)$ and S is all the places except the infinite place). This was used in [14] to compute the density of square-free, multivariate polynomials with coefficients in $\mathbb{F}_q[x]$. A similar notion of density was used in [15] for homogeneous polynomials.

For a valuation ring $\mathcal{O}_P$, let us denote by $\widehat{\mathcal{O}}_P$ its completion. As F is a global function field, $\widehat{\mathcal{O}}_P$ admits a normalized Haar measure, which we denote by μ_P. By abuse of notation, we will denote the product measure on $\widehat{\mathcal{O}}_P$ also by μ_P. For $U \subseteq \widehat{\mathcal{O}}_P^n$ we denote by ∂U the boundary of U with respect to the P-adic metric.

In [7], extending [14], the following local to global principle for densities over global function fields was established.

Theorem 1 ([7, Theorem 2.1])**.** *Let d be a positive integer, S be a proper, nonempty subset of $\mathbb{P}_F$, S_t as defined in Equation* (1) *and $\mathcal{O}_S$ the holomorphy ring of S. For any $P \in S$, let $U_P \subseteq \widehat{\mathcal{O}}_P^d$ be a Borel-measurable set such that $\mu_P(\partial U_P) = 0$. Suppose that*

$$\lim_{t\to\infty} \overline{\rho}_S(\{a \in \mathcal{O}_S^d \mid a \in U_P \text{ for some } P \in S_t\}) = 0. \tag{2}$$

Let $\pi : \mathcal{O}_S^d \longrightarrow 2^S$ be defined by $\pi(a) = \{P \in S : a \in U_P\} \in 2^S$. Then

(i) $\sum_{P\in S} \mu_P(U_P)$ *is convergent.*

(ii) Let $\Gamma \subseteq 2^S$. Then $\nu(\Gamma) := \rho_S(\pi^{-1}(\Gamma))$ exists and ν defines a measure on 2^S.

(iii) The measure ν is concentrated on finite subsets of S. In addition, if $T \subseteq S$ is finite we have

$$\nu(\{T\}) = \left(\prod_{P\in T} \mu_P(U_P)\right) \prod_{P\in S\setminus T} (1 - \mu_P(U_P)). \tag{3}$$

In the same paper, the following variant of Ekedahl's sieve was proved as a tool to verify assumption (2).

Theorem 2 ([7, Theorem 2.2])**.** *Let F be a global function field and S be a proper, nonempty subset of $\mathbb{P}_F$. Let $\mathcal{O}_S$ be the holomorphy ring of S. Furthermore, let $f, g \in \mathcal{O}_S[x_1, \ldots, x_d]$ be coprime polynomials. Then*

$$\lim_{t\to\infty} \overline{\rho}_S\left(\{y \in \mathcal{O}_S^d : \quad f(y) \equiv g(y) \equiv 0 \mod P \quad \text{for some } P \in S_t\}\right) = 0. \tag{4}$$

Remark 1. Note that in [7] the theorem is only stated for sets S having finite complement; however, the assumption is not needed in the proof. An alternative proof was sketched in [14, Theorem 8.1].

The next corollary follows from Theorem 1. It relates the density of the defining sets to their Haar measures and will play a crucial role in the proof of our main theorem.

Corollary 1. *Let F be a global function field with full field of constants equal to $\mathbb{F}_q$. Let d be a positive integer, S be a proper, nonempty subset of $\mathbb{P}_F$, and $\mathcal{O}_S$*

the holomorphy ring of S. Let $P_1, ..., P_n \in S$ be distinct and for $j = 1, ..., n$, let $U_{P_j} \subseteq \widehat{\mathcal{O}}_{P_j}^d$ be a Borel-measurable set with $\mu_{P_j}(\partial U_{P_j}) = 0$. Then

$$\rho_S\left(\bigcap_{j=1}^{n}\left(U_{P_j} \cap \mathcal{O}_S^d\right)\right) = \prod_{j=1}^{n} \mu_{P_j}(U_{P_j}). \tag{5}$$

Proof. Define $V_P = U_P$ for $P \in \{P_1, \ldots, P_n\}$ and $V_P = \emptyset$ otherwise. Then $(V_P)_{P\in S}$ satisfies the assumption of Theorem 1 and Equation (5) corresponds to (3) with $T = \{P_1, \ldots, P_n\}$. □

3. Higher Moments

If $\mathcal{O}_S^d$ carried a Haar measure, we could consider a random element $a \in \mathcal{O}_S$ and define the expected number of places $P \in S$ such that $a \in U_P$. The next definition is the analogue notion of expected value, respectively higher moments, in the case where we only have the natural density.

Definition 1. Let F be a global function field with full field of constants equal to $\mathbb{F}_q$. Let n and d be positive integers, $\emptyset \neq S \subsetneq \mathbb{P}_F$. Suppose $U_P \subseteq \widehat{\mathcal{O}}_P$. Then we define *the n-th moment of the system $(U_P)_{P\in S}$* to be

$$\mu_n = \lim_{D\in\mathcal{D}_S} \frac{\sum\limits_{a\in\mathcal{L}(D)^d} |\{P \in S \mid a \in U_P\}|^n}{q^{\ell(D)d}} \tag{6}$$

if it exists. We call μ_1 the *expected value* of the system $(U_P)_{P\in S}$.

Our main theorem gives an easy way to compute higher moments for a large class of systems.

Theorem 3. *Let d and n be positive integers. Let F be a global function field with full field of constants equal to $\mathbb{F}_q$, S be a proper, nonempty subset of $\mathbb{P}_F$, and $\mathcal{O}_S$ be the holomorphy ring of S. For each $P \in S$, let $U_P \subseteq \widehat{\mathcal{O}}_P^d$ be a measurable set such that $\mu_P(\partial(U_P)) = 0$. Let $S_t \coloneqq \{P \in S \mid \deg(P) \geq t\}$. If*

$$\lim_{t\to\infty} \overline{\rho}_S\left(\{a \in \mathcal{O}_S^d \mid a \in U_P \text{ for some } P \in S_t\}\right) = 0 \tag{7}$$

is satisfied, and for some $\alpha \in [0, \infty)$ there exist absolute constants $c', c, \tilde{c} \in \mathbb{Z}$, such that for all $D \in \mathcal{D}_S$ with $\deg(D) \geq \tilde{c}$ and for all $a \in \mathcal{L}(D)^d$ one has that

$$\left|\left\{P \in S \mid \deg(P) > c' \deg(D)^\alpha, a \in U_P \cap \mathcal{L}(D)^d\right\}\right| < c \tag{8}$$

and that there exists a sequence $(v_P)_{P\in S} \subseteq \mathbb{R}_{>0}$, such that for all $m \in \{1, \ldots, n\}$ and all $\deg(P_1), \ldots, \deg(P_m) \leq c' \deg(D)^\alpha$ with $P_j \in S$ pairwise distinct one has *that*

$$\left| \bigcap_{j=1}^{m} U_{P_j} \cap \mathcal{L}(D)^d \right| \leq q^{\ell(D)d} \prod_{j=1}^{m} v_{P_j}, \tag{9}$$

$$\sum_{P \in S} v_P \qquad \textit{converges}, \tag{10}$$

then it follows that

$$\mu_n = \lim_{D \in \mathcal{D}_S} \frac{\sum\limits_{a \in \mathcal{L}(D)^d} |\{P \in S \mid a \in U_P\}|^n}{q^{\ell(D)d}}$$

exists and $\mu_n < \infty$.

For $l \in \mathbb{N}_{\geq 1}$ *we denote by* $\left\{ {n \atop l} \right\}$ *the number of partitions of the set* $\{1, \ldots, n\}$ *with exactly* l *subsets. Then we have the formula*

$$\mu_n = \sum_{l=1}^{n} \left\{ {n \atop l} \right\} \sum_{\substack{P_1, \ldots, P_l \in S \\ \forall i < j \in \{1, \ldots, l\},\ P_i \neq P_j}} \prod_{m=1}^{l} \mu_{P_m}(U_{P_m}). \tag{11}$$

Remark 2.

1. Let $(U_P)_{P \in S}, (\widetilde{U}_P)_{P \in S}$ be systems satisfying the assumptions of Theorem 3 for a moment r. If $\mu_P(U_P) = \mu_P(\widetilde{U}_P)$ and we have another system $(V_P)_{P \in S}$ such that $U_P \subseteq V_P \subseteq \widetilde{U}_P$, then the r-th moment of $(V_P)_{P \in S}$ exists too and is given by Equation (11).

2. The coefficient $\left\{ {n \atop l} \right\}$ in Equation (11) is the Stirling number of the second kind and can be computed as follows:

$$\left\{ {n \atop l} \right\} = \frac{1}{l!} \sum_{k=0}^{l} (-1)^k \binom{l}{k} (l-k)^n.$$

3. One could weaken the assumptions (8), (9), and (10). Namely, it would be enough to assume that for every $D \in \mathcal{D}_S$, there exists $\widetilde{D} \in \mathcal{D}_S$ with $\widetilde{D} \geq D$ such that for all $D' \in \mathcal{D}_S$ with $D' \geq \widetilde{D}$, the Conditions (8), (9), and (10) hold true. The statement could be proved using the same ideas as in the proof below.

4. We could also include the case $\alpha = \infty$, meaning that we could drop assumption (8) and require instead that (9) and (10) hold for all places in S.

Proof of Theorem 3. We fix S throughout the proof, and write $\mathcal{O}$, ρ, and $\mathcal{D}$ in place of $\mathcal{O}_S$, ρ_S, and $\mathcal{D}_S$, respectively. For $a \in \mathcal{O}^d$ and $P \in \mathbb{P}_F$, we define

$$\tau(a,P) = \begin{cases} 1, & a \in U_P, \\ 0, & a \notin U_P. \end{cases}$$

For $M \in \mathbb{N}$, we have

$$\sum_{a \in \mathcal{L}(D)^d} \frac{\left(\sum_{P \in S} \tau(a,P)\right)^n}{q^{\ell(D)d}} = \sum_{j=0}^{n} \binom{n}{j} R_j(M,D),$$

where for all $j \in \{0, \dots, n\}$, we define

$$R_j(M,D) \coloneqq \sum_{a \in \mathcal{L}(D)^d} \frac{\left(\sum_{P \in S_M} \tau(a,P)\right)^{n-j} \left(\sum_{P \in S,\ \deg(P)<M} \tau(a,P)\right)^j}{q^{\ell(D)d}}.$$

First we show that, for all $j \in \{0, \dots, n-1\}$, the term $R_j(M,D)$ is negligible for M going to infinity. We define

$$l_{a,D} \coloneqq \left|\left\{P \in S \mid \deg(P) > c' \deg(D)^\alpha, a \in U_P \cap \mathcal{L}(D)^d\right\}\right|.$$

Then by Assumption (8) there exist $c, \tilde{c} > 0$ such that for all $a \in \mathcal{O}^d$ and all $D \in \mathcal{D}$ with $\deg(D) \geq \tilde{c}$ we have $l_{a,D} \leq c$. We define

$$\Theta_n(M,D) \coloneqq q^{-\ell(D)d} \sum_{a \in \mathcal{L}(D)^d} \left(\sum_{P \in S_M} \tau(a,P)\right)^n.$$

Thus, for $M \geq \tilde{c}$, we can express $q^{\ell(D)d}\Theta_n(M,D)$ as

$$\sum_{i=0}^{n} \binom{n}{i} \sum_{\substack{a \in \mathcal{L}(D)^d: \\ a \in \bigcup_{P_1,\dots,P_n \in S_M} \bigcap_{j=1}^n U_{P_j}}} \left|\left\{(P_j)_{j=1}^n \in S^n : \begin{matrix} M < \deg(P_k) \leq c' \deg(D)^\alpha & \text{if } 1 \leq k \leq i \\ c' \deg(D)^\alpha < \deg(P_k) & \text{if } i < k \leq n \end{matrix}\right\}\right|$$

$$\leq \sum_{i=0}^{n} \binom{n}{i} \sum_{\substack{a \in \mathcal{L}(D)^d: \\ a \in \bigcup_{P_1,\dots,P_n \in S_M} \bigcap_{j=1}^n U_{P_j}}} l_{a,D}^{n-i} \left|\{(P_j)_{j=1}^i \in S^i : M < \deg(P_j) < c' \deg(D)^\alpha\}\right|$$

$$\leq \left|\mathcal{L}(D)^d \cap \bigcup_{P \in S_M} U_P\right| + \sum_{i=1}^{n} \binom{n}{i} c^{n-i} \sum_{\substack{(P_1,\dots,P_i) \in S^i \\ M < \deg(P_1),\dots,\deg(P_i) \leq c' \deg(D)^\alpha}} \left|\mathcal{L}(D)^d \cap \bigcap_{j=1}^{i} U_{P_j}\right|.$$

Using Assumption (9), we further have that

$$\Theta_n(M,D) \le c^n \frac{|\mathcal{L}(D)^d \cap \bigcup_{P\in S_M} U_P|}{q^{\ell(D)d}} + \sum_{i=1}^{n}\sum_{j=1}^{i} 2^i \binom{n}{i} c^{n-i} \sum_{\substack{(P_1,\dots,P_j)\in S^j \\ M<\deg(P_1),\dots,\deg(P_j)<c'\deg(D)^\alpha}} \left(\prod_{k=1}^{j} v_{P_k}\right)$$
$$\le c^n \frac{|\mathcal{L}(D)^d \cap \bigcup_{P\in S_M} U_P|}{q^{\ell(D)d}} + \sum_{i=1}^{n}\sum_{j=1}^{i} 2^i \binom{n}{i} c^{n-i} \left(\sum_{P\in S_M} v_P\right)^j.$$

This implies that

$$\limsup_{D\in\mathcal{D}} \Theta_n(M,D) \le c^n \overline{\rho}\left(\mathcal{O}^d \cap \bigcup_{P\in S_M} U_P\right) + \sum_{i=1}^{n}\sum_{j=1}^{i} 2^i \binom{n}{i} c^{n-i} \left(\sum_{P\in S_M} v_P\right)^j.$$

Thus, we get from Equations (7) and (10) that

$$\lim_{M\to\infty} \limsup_{D\in\mathcal{D}} \Theta_n(M,D) = 0. \tag{12}$$

Using Hölder's inequality, we obtain, for $j \in \{1,\dots,n-1\}$,

$$R_j(M,D) \le \Theta_n(M,D)^{(n-j)/n} R_n(M,D)^{j/n}.$$

Hence, if we can show that $\lim_{M\to\infty} \lim_{D\in\mathcal{D}} R_n(M,D)$ exists, then we get, for all $j \in \{0,\dots,n-1\}$,

$$\lim_{M\to\infty} \limsup_{D\in\mathcal{D}} R_j(M,D) = 0,$$

and thus, μ_n exists as well and we have

$$\mu_n = \lim_{M\to\infty} \lim_{D\in\mathcal{D}} R_n(M,D).$$

In order to show that $\lim_{M\to\infty} \lim_{D\in\mathcal{D}} R_n(M,D)$ exists, we need to evaluate expressions of the form

$$\lim_{D\in\mathcal{D}} \frac{|\mathcal{L}(D)^d \cap \bigcap_{j=1}^{n} U_{P_j}|}{q^{\ell(D)d}}.$$

We would like to use Corollary 1, however, this only applies if the places are pairwise distinct. For $l \in \mathbb{N}_{\geq 1}$, we denote by $\left\{ n \atop l \right\}$ the number of partitions of $\{1, \ldots, n\}$ which contain exactly l subsets. Note that $\{P_1, \ldots, P_n\} = \{Q_1, \ldots, Q_l\}$ with $Q_1, \ldots, Q_k$ pairwise distinct if and only if

$$\{1, \ldots, n\} = \bigsqcup_{k=1}^{l} \{j \in \{1, \ldots, n\} \ : \ P_j = Q_k\}.$$

Thus, we get

$$\begin{aligned} R_n(M, D) &= \sum_{\substack{P_1, \ldots, P_{\ell(\tau)} \in S \\ \deg(P_1), \ldots, \deg(P_{l(\tau)}) < M}} \frac{|\mathcal{L}(D)^d \cap \bigcap_{j=1}^{n} U_{P_j}|}{q^{\ell(D)d}} \\ &= \sum_{l=1}^{n} \left\{ n \atop l \right\} \sum_{\substack{P_1, \ldots, P_l \in S \\ \deg(P_1), \ldots, \deg(P_l) < M \\ \forall i < j \in \{1, \ldots, l(\tau)\}, \ P_i \neq P_j}} \frac{|\mathcal{L}(D)^d \cap \bigcap_{j=1}^{l} U_{P_j}|}{q^{\ell(D)d}}. \end{aligned}$$

As all the sums are finite, we can pull the limit over the divisors inside of the sums and get, with Corollary 1,

$$\begin{aligned} \lim_{D \in \mathcal{D}} R_n(M, D) &= \sum_{l=1}^{n} \left\{ n \atop l \right\} \sum_{\substack{P_1, \ldots, P_{\ell(\tau)} \in S \\ \deg(P_1), \ldots, \deg(P_{l(\tau)}) < M \\ \forall i < j \in \{1, \ldots, l(\tau)\}, \ P_i \neq P_j}} \rho \left(\bigcap_{j=1}^{l(\tau)} U_{P_j} \cap \mathcal{O}^d \right) \\ &= \sum_{l=1}^{n} \left\{ n \atop l \right\} \sum_{\substack{P_1, \ldots, P_l \in S \\ \deg(P_1), \ldots, \deg(P_l) < M \\ \forall i < j \in \{1, \ldots, l(\tau)\}, \ P_i \neq P_j}} \prod_{j=1}^{l} \mu_{P_j}(U_{P_j}). \end{aligned}$$

Taking $M \to \infty$ yields Equation (11). Using Condition (10) and the crude estimate $\left\{ n \atop l \right\} \leq n^n$, one gets

$$\mu_n \leq n^{n+1} \left(1 + \sum_{P \in S} \mu_P(U_P) \right)^n < \infty. \qquad \square$$

Remark 3. We briefly compare this with the results in [11, 12]. There, an alternative definition of expected value (respectively, higher moment) was used. Namely, under the same assumptions for $(U_P)_{P \in S}$ as in Definition 1, they define

$$I = \{a \in \mathcal{O}_S^d \mid a \in U_P \text{ for infinitely many } P \in S\} \tag{13}$$

and $\mathcal{L}(D)^d_I \coloneqq \mathcal{L}(D)^d \setminus I$. Then they define *the n-th moment of the system $(U_P)_{P\in S}$* to be

$$\lim_{D\in\mathcal{D}_S} \frac{\sum_{a\in\mathcal{L}(D)^d_I} |\{P \in S \mid a \in U_P\}|^n}{q^{\ell(D)d}}, \tag{14}$$

if it exists. Let us call this the *renormalized* n-th moment.

This means that they consider, in our language, the moments of the sets $(U_P \setminus I)_{P\in S}$. The restriction to the complement of I prevents the moment from being infinity in a trivial fashion. The rationale in [11, 12] is that the moment of a random variable with respect to a probability measure does not change when altered on a null set. Furthermore, for system $(U_P)_{P\in S}$ satisfying Condition (2), one can show that the set I of elements which lie in infinitely many sets U_P has density zero (see the lemma below). Hence, the renormalized moment should be seen as a renormalized version of the more natural Definition 1. If $(U_P)_{P\in S}$ satisfies Condition (7), $(U_P \setminus I)_{P\in S}$ satisfies all the conditions of Theorem 1 and I is closed in all $\widehat{\mathcal{O}}_P$ for $P \in S$, then the n-th moment of $(U_P \setminus I)_{P\in S}$ coincides with the renormalized n-th moment. This happens for all the examples we have worked out in Section 4.

Lemma 1. *Let F a global function field with full field of constants equal to $\mathbb{F}_q$, d a positive integer, S be a proper, nonempty subset of $\mathbb{P}_F$, S_t as defined in Equation* (1)*, and $\mathcal{O}_S$ the holomorphy ring of S. For any $P \in S$, let $U_P \subseteq \widehat{\mathcal{O}}_P^d$ be Borel-measurable and I defined as in Equation* (13)*.*

1. *For all $P \in S$, we have*

$$\mu_P(U_P \setminus I) = \mu_P(U_P).$$

2. *If $(U_P)_{P\in S}$ satisfies Condition* (7)*, then*

$$\rho_S(I) = \rho_S(\{a \in \mathcal{O}_S^d \mid a \in U_P \textit{ for infinitely many } P \in S\}) = 0.$$

Proof. For the first part we note that F is a finite extension of $\mathbb{F}_q(x)$ and therefore countable. Thus, $I \subseteq \mathcal{O}_S \subseteq F$ is countable too. Recall that μ_P is a Haar measure and hence, $\mu_P(I) = 0$.

If $a \in I$, then for any integer t, $a \in U_P$ for some $P \in S_t$, that is

$$I \subseteq \{a \in \mathcal{O}_S^d \mid a \in U_P \text{ for some } P \in S_t\}$$

for all positive integer t. So we have

$$\overline{\rho}_S(I) \le \lim_{t\to\infty} \overline{\rho}_S\{a \in \mathcal{O}_S^d \mid a \in U_P \text{ for some } P \in S_t\} = 0,$$

where the last equality follows from Equation (7). Thus, we have $\rho_S(I) = 0$ as desired. $\square$

4. Applications

In this section, we will verify the assumptions of Theorem 3, i.e., compute all higher moments, for various examples that were considered in the existing literature.

4.1. Coprime n-Tuples

In this subsection we will compute all higher moments of coprime n-tuples. Here a *coprime n-tuple* denotes an n-tuple such that all entries are coprime over a specified ring. The computation of the natural density of coprime pairs over the integers is classical and goes back to Mertens [6] and Césaro [2] in the 1870's.

The densities of coprime n-tuples over holomorphy rings have been calculated in [8]. Previously this has also been considered over $\mathbb{F}_q[x]$ in [18]. We now compute all corresponding higher moments over holomorphy rings.

Theorem 4. *Let F be a global function field with full field of constants equal to $\mathbb{F}_q$. Let $n \geq 2$ be a positive integer. Let $\emptyset \neq S \subsetneq \mathbb{P}_F$ and let $\mathcal{O}_S$ be the holomorphy ring of S. Define the system $U_P = (P\widehat{\mathcal{O}}_P)^n \setminus \{0\}$ for each $P \in S$. Then all moments exist and are given by Equation* (11)*, where*

$$\mu_P(U_P) = q^{-n \deg(P)}. \tag{15}$$

Proof. We show that the system satisfies the assumptions of Theorem 3. We first check that Condition (7) is satisfied using Theorem 2. Consider the polynomials $f(x_1, x_2, \ldots, x_n) = x_1$ and $g(x_1, x_2, \ldots, x_n) = x_2$. Then for positive integers t, define

$$\begin{aligned} S_t(f, g) &= \{a \in \mathcal{O}_S^n \mid f(a) \in P \text{ and } g(a) \in P \text{ for some } P \in S_t\} \\ &= \{(a_1, a_2, \ldots, a_n) \in \mathcal{O}_S^n \mid a_1 \in P \text{ and } a_2 \in P \text{ for some } P \in S_t\}. \end{aligned}$$

Note that $A_t = \{a \in \mathcal{O}_S^d \mid a \in U_P \text{ for some } P \in S_t\}$ is a subset $A_t \subset S_t(f, g)$. Thus, by Theorem 2, we have

$$\lim_{t \to \infty} \overline{\rho}_S(A_t) \leq \lim_{t \to \infty} \overline{\rho}_S(S_t(f, g)) = 0.$$

So $\lim_{t \to \infty} \overline{\rho}_S(A_t) = 0$.

Next we check that Condition (8) is satisfied. Let $\alpha = 1$. Fix $a = (a_1, a_2, \ldots, a_n) \in \mathcal{L}(D)^n \setminus \{(0, \ldots, 0)\}$. As $a \neq (0, \ldots, 0)$, we can without loss of generality assume that $a_1 \neq 0$. Now by [17, Theorem 1.4.11], we have

$$\sum_{P \in S} \deg(P) v_P(a_1) = - \sum_{P \in \mathbb{P}_F \setminus S} \deg(P) v_P(a_1).$$

Recall that $x \in \mathcal{L}(D)$ implies that $v_P(x) \geq -v_P(D)$. So we have for $D \in \mathcal{D}_S$

$$\begin{aligned}\sum_{P \in S} \deg(P) v_P(a_1) &= -\sum_{P \in \mathbb{P}_F \setminus S} \deg(P) v_P(a_1) \\ &\leq \sum_{P \in \mathbb{P}_F \setminus S} \deg(P) v_P(D) \\ &\leq \deg(D).\end{aligned}$$

For any constant $c' > 0$, we obtain

$$\sum_{\substack{P \in S \\ \deg(P) > c' \deg(D)}} \deg(P) v_P(a_1) \leq \sum_{P \in S} \deg(P) v_P(a_1) \leq \deg(D).$$

Since $P \setminus \{0\} = \{x \in F \mid v_P(x) \geq 1\}$ and $a_1 \neq 0$, we have

$$c' \deg(D) \cdot |\{P \in S \ : \ \deg(P) > c' \deg(D), a \in U_P \cap \mathcal{L}(D)^n\}| \leq \deg(D).$$

Together with the observation that $(0, \dots, 0)$ is not contained in any U_p, we obtain, for all $a \in \mathcal{L}(D)^n$,

$$|\{P \in S \ : \ \deg(P) > c' \deg(D), a \in U_P \cap \mathcal{L}(D)^n\}| \leq \frac{1}{c'}.$$

Hence, Condition (8) is satisfied for any choice of $c' > 0$.

Now we verify Conditions (9) and (10). Let $P_1, \dots, P_r \in S$ be pairwise distinct places. Then we get for $D \in \mathcal{D}_S$

$$\mathcal{L}(D) \cap \bigcap_{j=1}^{r} (P_j \mathcal{O}_{P_j}) = \mathcal{L}(D_{P_1, \dots, P_r}),$$

where $D_{P_1, \dots, P_r}$ is defined by

$$v_{\widetilde{P}}(D_{P_1, \dots, P_r}) = \begin{cases} -1 & \text{if } \widetilde{P} \in \{P_1, \dots, P_r\}, \\ v_{\widetilde{P}}(D) & \text{otherwise.} \end{cases}$$

If $\deg(P_1), \dots, \deg(P_r) \leq c' \deg(D)$, then we obtain

$$\deg(D_{P_1, \dots, P_r}) = \deg(D) - \sum_{j=1}^{r} \deg(P_j) \geq \deg(D) - rc' \deg(D) = (1 - rc') \deg(D).$$

Hence, if we pick $0 < c' < \frac{1}{r}$, then we can use the Riemann-Roch theorem [17, Theorem 1.4.17 (b.)] and obtain that there exists a constant $C > 0$ depending only on F and c' such that for all $D \in \mathcal{D}_S$ with $\deg(D) \geq C$, we have

$$\ell(D_{P_1,\dots,P_r}) = \deg(D_{P_1,\dots,P_r}) + 1 - g = \deg(D) + 1 - g - \sum_{j=1}^{r} \deg(P_j)$$
$$= \ell(D) - \sum_{j=1}^{r} \deg(P_j),$$

where g is the genus of F. Thus, we obtain for $\deg(D) \geq C$

$$|\mathcal{L}(D) \cap \bigcap_{j=1}^{r} P_j| = |\mathcal{L}(D_{P_1,\dots,P_r})| = q^{\ell(D)} \prod_{j=1}^{r} q^{-\deg(P_j)}.$$

Hence, there exists a constant $C > 0$ such that for all $D \in \mathcal{D}_S$ with $\deg(D) \geq C$ and all pairwise distinct places $P_1, \dots, P_r \in S$ with $\deg(P_1), \dots, \deg(P_r) \leq \frac{1}{2r}\deg(D)$, we have

$$|\mathcal{L}(D) \cap \bigcap_{j=1}^{r} P_j| = q^{\ell(D)} \prod_{j=1}^{r} q^{-\deg(P_j)}. \tag{16}$$

Thus, we get

$$|\mathcal{L}(D)^n \cap \bigcap_{j=1}^{r} U_{P_j}| = |\mathcal{L}(D) \cap \bigcap_{j=1}^{r} P_j|^n = q^{n\ell(D)} \prod_{j=1}^{r} q^{-n\deg(P_j)}.$$

So for each $P \in S$, we choose $v_P = q^{-n\deg(P)}$, which satisfies Condition (9). The series $\sum_{P\in S} q^{-n\deg(P)}$ is dominated by the Zeta function $Z(q^{-n}) = \sum_{P\in\mathbb{P}_F} q^{-n\deg(P)}$. For $n \geq 2$, the Zeta function converges by [17, Proposition 5.1.6], hence Condition (10) is satisfied. Thus, we can invoke Theorem 3. Note that $(P\widehat{\mathcal{O}}_P)^n$ is a subgroup of $\widehat{\mathcal{O}}_P^n$ and singletons are null sets, so (15) holds true as

$$\mu_P((P\widehat{\mathcal{O}}_P)^n \setminus \{0\}) = \mu_P((P\widehat{\mathcal{O}}_P)^n) = |\widehat{\mathcal{O}}_P^n/(P\widehat{\mathcal{O}}_P)^n|^{-1} = q^{-n\deg(P)}. \qquad \square$$

4.2. Affine Eisenstein Polynomials

In this subsection, we will compute all higher moments of affine Eisenstein polynomials. The affine Eisenstein polynomials over holomorphy rings can be used to study totally ramified extensions, see [7] and the references therein for more details. In said paper, the density of affine Eisenstein polynomials is also computed (see [7, Theorem 3.6]). The density of the shifted/affine Eisenstein polynomials over number fields have been computed in [9] and the higher moments over number fields have been considered in [11]. For this section we will by abuse of notation identify polynomials of degree d with the corresponding $(d+1)$-tuple of coefficients.

Let $\emptyset \neq S \subsetneq \mathbb{P}_F$ and $P \in S$. A polynomial $f(x) \in \mathcal{O}_S[x]$ of degree d, say $f(x) = \sum_{j=0}^{d} a_j x^j$, is said to be *P-Eisenstein* if

$$a_d \notin P, a_0 \notin P^2 \text{ and } a_i \in P \quad \text{for all } i \in \{0, \ldots, d-1\}.$$

In addition, $f(x)$ is said to be *Eisenstein* if there exists $P \in S$ such that $f(x)$ is P-Eisenstein. We define for $P \in S$

$$U_P = \left(P\widehat{\mathcal{O}}_P \setminus P^2\widehat{\mathcal{O}}_P\right) \times \left(P\widehat{\mathcal{O}}_P\right)^{d-1} \times \left(\widehat{\mathcal{O}}_P \setminus P\widehat{\mathcal{O}}_P\right). \tag{17}$$

This will be the system for Eisenstein polynomials in $\mathcal{O}_S[x]$ as $U_P \cap \mathcal{O}_S^{d+1}$ represents exactly the P-Eisenstein polynomials.

Next we introduce shifted Eisenstein polynomials. For $P \in S$ we say $f(x) \in \mathcal{O}_S[x]$ is a *shifted P-Eisenstein polynomial* if there exists $t \in \mathcal{O}_S$ such that $f(x+t)$ is a P-Eisenstein polynomial.

For $t \in \widehat{\mathcal{O}}_P$ we denote by σ_t the map

$$\sigma_t : \widehat{\mathcal{O}}_P^{d+1} \to \widehat{\mathcal{O}}_P^{d+1}, f(x) \mapsto f(x+t). \tag{18}$$

We define for $P \in S$

$$V_P = \bigcup_{t \in \mathcal{O}_S} \sigma_t(U_P). \tag{19}$$

This yields a system for shifted Eisenstein polynomials as $V_P \cap \mathcal{O}_S^{d+1}$ represents the shifted P-Eisenstein polynomials.

For a commutative ring R, $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_{2\times 2}(R)$ and $f(x) \in R[x]$ of degree d we define

$$(f * A)(x) = (\gamma x + \delta)^d f\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \in R[x].$$

Let P be a prime ideal in R. We call $f(x) \in R[x]$ *affine P-Eisenstein* if there exists $A \in \mathrm{GL}_{2\times 2}(R)$ such that $(f * A)(x) \in R[x]$ is P-Eisenstein. Furthermore, a polynomial $f(x) \in \mathcal{O}_S[x]$ is called *affine Eisenstein* if there exists $P \in S$ such that $(f * A)(x)$ is affine P-Eisenstein.

It turns out that only particular affine transformations are needed to realize all affine Eisenstein polynomials. The following lemma is a consequence of [7, Corollary 3.4].

Lemma 2. *Let F be a global function field, $\emptyset \neq S \subsetneq \mathbb{P}_F$ and $P \in S$.*

(a) Let σ_t denote the shift introduced in (18) *and U_P as in* (17)*. Let $s, t \in \mathcal{O}_P$. Then the following are equivalent:*

(i) $\sigma_s(U_P) \cap \sigma_t(U_P) \neq \emptyset$

(ii) $\sigma_t(U_P) = \sigma_s(U_P)$

(iii) $s - t \in P$.

(b) *Let $f \in \mathcal{O}_S[x]$ be a polynomial of degree $d \geq 2$. Then the following are equivalent:*

(i) *$f(x)$ is affine P-Eisenstein*

(ii) *There exists $t \in \mathcal{O}_S$ such that $f(x+t)$ is P-Eisenstein or $x^d f(1/x)$ is P-Eisenstein.*

(c) *Let $f \in \mathcal{O}_S[x]$ be a polynomial of degree $d \geq 2$, such that $x^d f(1/x)$ is P-Eisenstein. Then $f(x)$ is not a shifted P-Eisenstein polynomial.*

Proof. We start with the proof of part (a). As $\sigma_t^{-1}\sigma_s = \sigma_{s-t}$, we can without loss of generality assume that $s = 0$. Clearly we have $\sigma_t(U_P) = U_P$ for $t \in P$. Hence, $(c) \Rightarrow (b) \Rightarrow (a)$ holds true.

Let us now assume that $f(x) \in \sigma_t(U_P) \cap U_P$. Write $f(x) = \sum_{j=0}^d a_j x^j$. As $f(x) \in U_P$, we get $a_{d-1} \equiv 0 \pmod{P\widehat{\mathcal{O}}_P}$ and a_d is invertible $\pmod{P\widehat{\mathcal{O}}_P}$. However, $f(x) \in \sigma_t(U_P)$ and thus, looking at the constant coefficient of $f(x+t)$, we get that

$$a_d t^d \equiv \sum_{j=0}^{d} a_j t^j \equiv 0 \pmod{P\widehat{\mathcal{O}}_P}$$

as $a_0, \ldots, a_{d-1} \in P$. Hence, we get $t \in (P\widehat{\mathcal{O}}_P) \cap \mathcal{O}_P = P\mathcal{O}_P$.

Moving onto the proof of part (b), clearly we have $(b) \Rightarrow (a)$. For the other direction we recall that, by [7, Corollary 3.4], for every affine P-Eisenstein polynomial $f(x)$, either $x^d f(1/x)$ is P-Eisenstein, or there exists $t \in \mathcal{O}_P$ such that $f(x+t)$ is P-Eisenstein. We are left to prove that we can choose $t \in \mathcal{O}_S$. Let G_P be a set of representatives in $\mathcal{O}_P$ of $\mathcal{O}_P/P$ and let H_P be a set of representatives in $\mathcal{O}_S$ of $\mathcal{O}_S/(P \cap \mathcal{O}_S)$. By 1.$(c)$ we can write

$$\bigsqcup_{t \in H_P} \sigma_t(U_P) = \bigcup_{t \in \mathcal{O}_S} \sigma_t(U_P) \subseteq \bigcup_{\widetilde{t} \in \mathcal{O}_P} \sigma_{\widetilde{t}}(U_P) = \bigsqcup_{\widetilde{t} \in G_P} \sigma_{\widetilde{t}}(U_P).$$

Now using 1. and the fact that $|H_P| = |\mathcal{O}_S/(P \cap \mathcal{O}_S)| = |\mathcal{O}_P/P| = |G_P|$ we get

$$\bigsqcup_{t \in H_P} \sigma_t(U_P) = \bigcup_{t \in \mathcal{O}_S} \sigma_t(U_P) = \bigcup_{\widetilde{t} \in \mathcal{O}_P} \sigma_{\widetilde{t}}(U_P) = \bigsqcup_{\widetilde{t} \in G_P} \sigma_{\widetilde{t}}(U_P), \tag{20}$$

which yields the claim.

For part (c), notice that If $f(x) = \sum_{j=0}^d a_j x^j$ is a shifted P-Eisenstein polynomial, then $a_d \notin P$. However, if $x^d f(1/x) = \sum_{j=0}^d a_{d-j} x^j$ is P-Eisenstein, then $a_d \in P$. □

For every $P \in S$, we define

$$\operatorname{inv} : \widehat{\mathcal{O}}_P^{d+1} \to \widehat{\mathcal{O}}_P^{d+1}, f(x) \mapsto x^d f(1/x).$$

For $P \in S$, let V_P be as in (19); then we define

$$W_P = V_P \cup \operatorname{inv}(U_P). \tag{21}$$

This yields a system for affine Eisenstein polynomials as $W_P \cap \mathcal{O}_S^{d+1}$ represents affine P-Eisenstein polynomials by Lemma 2.

Now we are ready to compute the higher moments of the Eisenstein polynomials, the shifted Eisenstein polynomials and the affine Eisenstein polynomials.

Theorem 5. *Let F be a global function field with full field of constants given by $\mathbb{F}_q$. Let $d \geq 3$ be a positive integer. Let $\emptyset \neq S \subsetneq \mathbb{P}_F$ and let $\mathcal{O}_S$ be the holomorphy ring of S. Define the systems $(U_P)_{P\in S}, (V_P)_{P\in S}$, and $(W_P)_{P\in S}$ as in Equations* (17), (19) *and* (21)*, respectively. Then all moments exist for all three of these systems and are given by Equation* (11)*, where*

$$\mu_P(U_P) = \frac{\left(1 - q^{-\deg(P)}\right)^2}{q^{d\deg(P)}} \tag{22}$$

for the system $(U_P)_{P\in S}$,

$$\mu_P(V_P) = \frac{\left(1 - q^{-\deg(P)}\right)^2}{q^{(d-1)\deg(P)}} \tag{23}$$

for the system $(V_P)_{P\in S}$ and

$$\mu_P(W_P) = \frac{(1 - q^{-\deg(P)})^2 \left(1 + q^{\deg(P)}\right)}{q^{d\deg(P)}} \tag{24}$$

for the system $(W_P)_{P\in S}$.

Proof. First we note that U_P is clopen and that inv and σ_t are homeomorphisms, and thus V_P and W_P are clopen too. Hence, the boundary of each of those sets is empty. Next, we compute $\mu_P(U_P), \mu_P(V_P)$ and $\mu_P(W_P)$. Clearly we have

$$\begin{aligned}
\mu_P(U_P) &= \left(1 - \mu_P(P\widehat{\mathcal{O}}_P)\right) \mu_P(P\widehat{\mathcal{O}}_P)^{d-1} \left(\mu_P(P\widehat{\mathcal{O}}_P) - \mu_P(P\widehat{\mathcal{O}}_P)^2\right) \\
&= \left(1 - q^{-\deg(P)}\right) q^{-(d-1)\deg(P)} \left(q^{-\deg(P)} - q^{-2\deg(P)}\right) \\
&= \frac{(1 - q^{-\deg(P)})^2}{q^{d\deg(P)}}.
\end{aligned}$$

Using Equation (20) and the fact that shifts preserve the measure, we obtain

$$\mu_P(V_P) = |\mathcal{O}_S/(P \cap \mathcal{O}_S)|\mu_P(U_P) = \frac{\left(1 - q^{-\deg(P)}\right)^2}{q^{(d-1)\deg(P)}}.$$

Again using Equation (20) and Lemma 2 (c) we get

$$W_P = \operatorname{inv}(U_P) \sqcup \bigsqcup_{t \in H_P} \sigma_t(U_P).$$

As the shifts and inv preserve the measure, we get

$$\mu_P(W_P) = (1 + |G_P|)\mu_P(U_P) = (1 + q^{\deg(P)})\frac{(1 - q^{-\deg(P)})^2}{q^{d\deg(P)}}.$$

We are only checking the Conditions (7), (8), (9), (10) for the system $(W_P)_{P \in S}$ corresponding to affine Eisenstein polynomials. The estimates for $(U_P)_{P \in S}$ and $(V_P)_{P \in S}$ follow similarly due to the inclusions $U_P \subseteq V_P \subseteq W_P$.

Let $P \in S$, $D \in \mathcal{D}_S$, and suppose $f \in W_P \cap \mathcal{L}(D)^{d+1}$. Either $x^d f(1/x) \in U_P \cap \mathcal{L}(D)^{d+1}$ or $f(x) \in V_P \cap \mathcal{L}(D)^{d+1}$. For any $c' > 0$ we have

$$\begin{aligned}
&|\{P \in S \mid \deg(P) > c' \deg(D), f \in W_P \cap \mathcal{L}(D)^{d+1}\}| \\
&\quad \leq \left|\left\{P \in S \mid \deg(P) > c' \deg(D), f \in V_P \cap \mathcal{L}(D)^{d+1}\right\}\right| \\
&\quad\quad + |\{P \in S \mid \deg(P) > c' \deg(D), x^d f(1/x) \in U_P \cap \mathcal{L}(D)^{d+1}\}| \\
&\quad =: I + II.
\end{aligned}$$

Note that the coefficients of $x^d f(1/x)$ are just a permutation of the coefficients of $f(x)$ and hence, $II \leq I$. Thus, it is enough to estimate I.

Let $f \in V_P \cap \mathcal{L}(D)^{d+1}$ and denote by $\operatorname{Disc}(f(x))$ the discriminant of $f(x)$. We first consider the case $\operatorname{Disc}(f(x)) \neq 0$. Let $b \in \mathcal{O}_S$ be such that $f(x+b) \in U_P$. Since the discriminant is invariant under a shift, $\operatorname{Disc}(f(x)) = \operatorname{Disc}(f(x+b))$. As the discriminant is the Sylvester matrix of the resultant of $f(x+b)$ and $(f(x+b))'$, we get $\operatorname{Disc}(f(x)) \in P$ as all but the leading coefficients of $f(x+b)$ and $(f(x+b))'$ are elements of P. Furthermore, as $f \in \mathcal{L}(D)$ and the discriminant is a homogeneous polynomial of degree $d(d-1)$ in the coefficients of f, we have $\operatorname{Disc}(f(x)) \in \mathcal{L}(d(d-1)D)$. Hence, we obtain by the same reasoning as for the coprime n-tuples in the proof of Theorem 4,

$$\begin{aligned}
&|\{P \in S \mid \deg(P) > c' \deg(D), f \in V_P \cap \mathcal{L}(D)^{d+1}\}| \\
&\quad \leq 2|\{P \in S \mid \deg(P) > c' \deg(D), \operatorname{Disc}(f(x)) \in P \cap \mathcal{L}(d(d-1)D)\}| \\
&\quad \leq \frac{2d(d-1)}{c'}.
\end{aligned}$$

Now suppose $\operatorname{Disc}(f(x)) = 0$. Then $f(x)$ is inseparable. Therefore, we can write $f(x) = g(x^{p^k})$ for some $k \in \mathbb{N}$, where $g(x)$ is separable. Hence $\operatorname{Disc}(g(x)) \neq 0$. By assumption there exists some b such that $f(x+b)$ is P-Eisenstein and thus, we get that

$$f(x+b) = g((x+b)^{p^k}) = g(x^{p^k} + b^{p^k})$$

is P-Eisenstein. Hence, $g(x+b^{p^k})$ is P-Eisenstein. Thus, we can apply the previous argument for g and obtain

$$\begin{aligned}
&|\{P \in S \mid \deg(P) > c' \deg(D), f \in V_P \cap \mathcal{L}(D)^d\}| \\
&\quad \le |\{P \in S \mid \deg(P) > c' \deg(D), g \in V_P \cap \mathcal{L}(D)^{d/p^k}\}| \\
&\quad \le \frac{2d(d-1)}{c'}.
\end{aligned}$$

Next we are going to verify Conditions (9), (10). For this we fix some moment $0 \neq n \in \mathbb{N}$.

As for coprime pairs we choose $\alpha = 1$ and $c' = \frac{1}{2n}$. We estimate the size of intersections of $W_{P_1}, W_{P_2}, \ldots, W_{P_n}$ for distinct $P_1, \ldots, P_n$ with $\deg(P_j) \le \frac{1}{2n}\deg(D)$ for $j \in \{1, 2, \ldots, n\}$. As before $D \in \mathcal{D}_S$ with $\deg(D) \ge C$ for some constant C depending only on n, d. Let $f \in \bigcap_{i=1}^n W_{P_i}$ for each P_i, then either $f(x+t_i)$ is P_i-Eisenstein for some $t_i \in \mathcal{O}_S$, or $x^d f(1/x)$ is P_i-Eisenstein. If $f(x+t_i)$ is P_i-Eisenstein then we have

$$a_d(x+t_i)^d + a_{d-1}(x+t_i)^{d-1} + \cdots + a_0 = a'_d x^d + a'_{d-1}x^{d-1} + \cdots + a'_0$$

for some $a'_d, a'_{d-1}, \ldots, a'_0 \in \mathcal{O}_S$. Thus, we can express $a'_{d-1}, a'_{d-2}, a'_{d-3}$ as functions of $a_{d-3}, \ldots, a_d$ and t_i:

$$\begin{aligned}
a'_{d-1} &= a_{d-1} + \binom{d}{1} t_i a_d \\
a'_{d-2} &= a_{d-2} + \binom{d-1}{1} t_i a_{d-1} + \binom{d}{2} t_i^2 a_d \\
a'_{d-3} &= a_{d-3} + \binom{d-2}{1} t_i a_{d-2} + \binom{d-1}{2} t_i^2 a_{d-1} + \binom{d}{3} t_i^3 a_d.
\end{aligned}$$

As $f(x+t_i)$ is P_i-Eisenstein, we get

$$\begin{aligned}
a_{d-1} + \binom{d}{1} t_i a_d &\equiv 0 \pmod{P_i} \\
a_{d-2} + \binom{d-1}{1} t_i a_{d-1} + \binom{d}{2} t_i^2 a_d &\equiv 0 \pmod{P_i} \\
a_{d-3} + \binom{d-2}{1} t_i a_{d-2} + \binom{d-1}{2} t_i^2 a_{d-1} + \binom{d}{3} t_i^3 a_d &\equiv 0 \pmod{P_i}.
\end{aligned}$$

Therefore, given a_d and t_i, from the first equation, $a_{d-1} \pmod{P_i}$ is fixed and we denote it by $\overline{a}_{d-1}(a_d, t_i)$. Using the other equations, we also get that a_{d-2} and a_{d-3} are uniquely determined modulo P_i by a_d and t_i and we denote the corresponding elements modulo P_i by $\overline{a}_{d-2}(a_d, t_i)$ and $\overline{a}_{d-3}(a_d, t_i)$ respectively. If $x^d f(1/x)$ is P_i-Eisenstein then we must have $a_{d-1}, a_{d-1}, a_{d-3} \in P_i$ as $d \geq 3$.

We partition $P_1, P_2, \ldots, P_n$ into $Q_1, Q_2, \ldots, Q_{n-k}$ and $S_1, S_2, \ldots, S_k$, where $k \in \{0, n\}$ are valid choices. Suppose $x^d f(1/x)$ is Eisenstein with respect to Q_1, Q_2, ..., Q_{n-k} and $f(x)$ is shifted Eisenstein with respect to S_1, $S_2, \ldots,$ S_k. We count the number of such f for a given partition. The restriction on $x^d f(1/x)$ implies that a_{d-1} and a_{d-2} both are in $Q_1, Q_2, \ldots, Q_{n-k}$. Then, since f is shifted Eisenstein with respect to each S_i, for each S_i there exists some t_i such that $f(x + t_i)$ is S_i-Eisenstein. This implies that the coefficients of f satisfy the following system of equations:

$$\begin{aligned}
a_{d-1} &= \overline{a}_{d-1}(a_d, t_i) && \pmod{S_i} \\
a_{d-2} &= \overline{a}_{d-2}(a_d, t_i) && \pmod{S_i} \\
a_{d-3} &= \overline{a}_{d-3}(a_d, t_i) && \pmod{S_i} \\
a_{d-1} &= 0 && \pmod{Q_m} \\
a_{d-2} &= 0 && \pmod{Q_m}
\end{aligned} \tag{25}$$

for each S_i for all $i \in \{1, \ldots, k\}$ and each Q_m with $m \in \{1, \ldots, n-k\}$. Using the Chinese Remainder Theorem and the fact that $P_i \cap \mathcal{O}_S$ and $P_j \cap \mathcal{O}_S$ are coprime for $P_i \neq P_j$ (by [17, Proposition 3.2.9.]), we see that the coefficients a_{d-1}, a_{d-2} are uniquely determined in $\mathcal{O}_S/(\mathcal{O}_S \cap \bigcap_{i=1}^k S_i \cap \bigcap_{m=1}^{n-k} Q_m)$ and a_{d-3} is uniquely determined in $\mathcal{O}_S/(\mathcal{O}_S \cap \bigcap_{i=1}^k S_i)$, once we have fixed $a_d \in \mathcal{L}(D)$ and $t_i \in \mathcal{O}_S/P_i$. This readily implies that

$$\begin{aligned}
&|\{a \in \mathcal{L}(D)^{d+1} \ : \ f(x) \in \bigcap_{i=1}^{k} V_{P_i}, x^d f(1/x) \in \bigcap_{m=1}^{n-k} U_{Q_m}\}| \\
&\leq |\mathcal{L}(D)| \cdot q^{\sum_{i=1}^k \deg(S_i)} \cdot |\mathcal{L}(D) \cap \bigcap_{i=1}^{k} S_i \cap \bigcap_{m=1}^{n-k} Q_m|^2 \cdot |\mathcal{L}(D) \cap \bigcap_{i=1}^{k} S_i| \cdot |\mathcal{L}(D)|^{d-3} \\
&= |\mathcal{L}(D)|^{d+1} \prod_{j=1}^{n} q^{-2\deg(P_j)},
\end{aligned} \tag{26}$$

where we have used for the first inequality that $a_d \in \mathcal{L}(D), a_0, \ldots, a_{d-4} \in \mathcal{L}(D)$ and that we can restrict ourselves to $|\mathcal{O}_{S_i}/S_i| = q^{\deg(S_i)}$ choices for t_i by Lemma 2. Equation (16) allows us to pass to the third line of Equation (26). Summing over all possible partitions, we get

$$|\mathcal{L}(D)^{d+1} \cap \bigcap_{j=1}^{n} W_{P_j}| \leq 2^n q^{\ell(D)(d+1)} \prod_{j=1}^{n} q^{-2\deg(P_j)}.$$

Since $\sum_{P \in \mathbb{P}_F} q^{-2\deg(P)}$ is dominated by the Zeta function $Z(q^{-2})$, and since $Z(q^{-2})$ converges (see [17, Proposition 5.1.6.]), $\sum_{P \in \mathbb{P}_F} q^{-2\deg(P)}$ converges too. □

4.3. Rectangular Unimodular Matrices

In this subsection we compute all higher moments of rectangular unimodular matrices over function fields. Rectangular unimodular matrices have already been considered in the literature in similar situations. Namely, their density over number fields was calculated in [13]; their density over function fields was done in [7, 10], and their expected value over rationals was established in [12].

Let us recall the definition of rectangular unimodular matrices over a Dedekind domain. Let $\mathcal{D}$ be a Dedekind domain and $n, m \in \mathbb{N}$ with $n < m$. A matrix $M \in \mathrm{Mat}_{n\times m}(\mathcal{D})$ is called *rectangular unimodular*, if and only if $M \mod P$ has full rank for all non-zero prime ideals P of $\mathcal{D}$ [13, Proposition 3]. This is equivalent to saying that the matrix has a basic minor which is not contained in $P\widehat{\mathcal{O}}_P$, where a basic minor of a matrix is the determinant of a square submatrix of maximal size. Note that rectangular unimodular matrices in the case $n = 1$ correspond to coprime pairs. We will use Theorem 3 and ideas of [12] to compute all higher moments of rectangular unimodular matrices.

Theorem 6. *Let n, m be positive integers such that $n < m$ and F be a global function field with full field of constants equal to $\mathbb{F}_q$ and $\emptyset \neq S \subsetneq \mathbb{P}_F$. For any $P \in S$, let V_P be the set of matrices in $Mat_{n\times m}(\widehat{\mathcal{O}}_P)$ for which the ideal generated by its basic minors is contained in $P\hat{\mathcal{O}}_P$, and let I be the set of matrices contained in infinitely many V_P. We define $U_P = V_P \setminus I$.*

Then the system $(U_P)_{P\in S}$ satisfies the conditions of Theorem 3, and the higher moments are given by Equation (11)*, where*

$$\mu_P(U_P) = 1 - \prod_{i=0}^{n-1} \left(1 - q^{-deg(P)(m-i)}\right). \tag{27}$$

Proof. Equation (27) holds true due to the computation in the proof of [7, Theorem 4.4] and the fact that $\mu_P(V_P \cap I) = 0$ (by Lemma 1). We are left to check that the assumptions of Theorem 3 are satisfied.

We start by noting that V_P is clopen and I is closed, and hence $\mu_P(\partial U_P) \leq \mu_P(I) = 0$ by Lemma 1. Condition (7) is satisfied as shown in [7, Theorem 13]. We want to show Condition (8) is satisfied for $\alpha = 1$. When $M \in U_P$, by definition, there exist some $n \times n$ submatrix A such that $\det(A) \neq 0$. As $M \in U_P \cap \mathcal{L}(D)^{n\times m}$, we have $A \in \mathcal{L}(D)^{n\times n}$. Thus,

$$\begin{aligned} v_P(\det(A)) &= v_P\left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}\right) \\ &\geq \min_{\substack{\sigma\in S_n \\ \prod_{i=1}^{n} A_{i,\sigma(i)}\neq 0}} v_P\left(\prod_{i=1}^{n} A_{i,\sigma(i)}\right) \\ &\geq -n v_P(D). \end{aligned}$$

This implies, as for coprime n-tuples, that for $D \in \mathcal{D}_S$ we have

$$\begin{aligned} \sum_{P\in S} \deg(P) v_P(\det(A)) &= -\sum_{P\in \mathbb{P}_F\setminus S} \deg(P) v_P(\det(A)) \\ &\leq \sum_{P\in \mathbb{P}_F\setminus S} \deg(P) n v_P(D) \\ &\leq n \deg(D). \end{aligned}$$

Hence, we get for for any constant $c' > 0$,

$$\sum_{\substack{P\in S \\ \deg(P)\geq c'\deg(D)}} \deg(P) v_P(\det(A)) \leq \sum_{P\in S} \deg(P) v_P(\det(A)) \leq n \deg(D).$$

Since $0 \neq \det(A) \in P\widehat{\mathcal{O}}_P$, and $P = \{x \in F \mid v_P(x) \geq 1\}$, we have

$$c' \deg(D) |\{P \in S : \deg(P) > c' \deg(D), M \in U_P \cap \mathcal{L}(D)_I^{n\times m}\}| \leq n \deg(D).$$

Hence Condition (8) is satisfied for any $c' > 0$.

Now we check Conditions (9) and (10). For each $A \in \mathcal{L}(D)^{n\times n}$ such that $0 \neq \det(A) \in \bigcap_{i=1}^{r} P_i$, by definition, there exists $M \in \bigcap_{i=1}^{r} U_{P_i} \cap {\mathcal{L}(D)_I}^{n\times m}$ containing A as a submatrix. There are at most $\binom{m}{n} |\mathcal{L}(D)|^{nm-n^2}$ such choices per matrix M. So, we have

$$|\bigcap_{i=1}^{r} U_{P_i} \cap {\mathcal{L}(D)_I}^{n\times m}| \leq \binom{m}{n} |\mathcal{L}(D)|^{nm-n^2} |\{A \in \mathcal{L}(D)^{n\times n} \mid 0 \neq \det(A) \in \bigcap_{i=1}^{r} P_i\}|.$$

Fix an arbitrary $D \in \mathcal{D}_S$. Define ϕ_1 to be the inclusion map $\mathrm{Mat}_{n\times n}(\mathcal{L}(D)) \to \mathrm{Mat}_{n\times n}(\mathcal{O}_S)$, and denote the quotient map $\mathrm{Mat}_{n\times n}(\mathcal{O}_S) \to \mathrm{Mat}_{n\times n}(\mathcal{O}_S/\bigcap_{i=1}^{r} P_i)$ by ϕ_2. Let $\phi = \phi_2 \circ \phi_1$. As $\{A \in \mathcal{L}(D)^{n\times n} \mid 0 \neq \det(A) \in \bigcap_{i=1}^{r} P_i\}$ is a subset of

$\mathrm{Mat}_{n\times n}(\mathcal{L}(D))$, we get

$$\begin{aligned}
|\{A \in & \mathcal{L}(D)^{n\times n} \mid 0 \neq \det(A) \in \bigcap_{i=1}^{r} P_i\}| \\
&\leq |\{B \in \mathrm{Mat}_{n\times n}(\mathcal{O}_S/\textstyle\bigcap_{i=1}^{r} P_i) \mid \det(B) = 0\}| \cdot |\ker(\phi)| \\
&= |\{B \in \mathrm{Mat}_{n\times n}(\mathcal{O}_S/\textstyle\bigcap_{i=1}^{r} P_i) \mid \det(B) = 0\}| \cdot |\mathcal{L}(D) \cap \displaystyle\bigcap_{i=1}^{r} P_i|^{n^2}.
\end{aligned}$$

Recall that $P_i \cap \mathcal{O}_S$ and $P_j \cap \mathcal{O}_S$ are distinct maximal ideals in $\mathcal{O}_S$ for $P_i \neq P_j$ (see [17, Proposition 3.2.9.]) and therefore, by the Chinese Remainder Theorem, the following is an isomorphism of $\mathcal{O}_S$-modules:

$$\begin{aligned}
&\pi : \mathrm{Mat}_{n\times n}(\mathcal{O}_S/\textstyle\bigcap_{i=1}^{r} P_i) \to \displaystyle\prod_{i=1}^{r} \mathrm{Mat}_{n\times n}(\mathcal{O}_S/P_i), \\
&\left(a_{jk} + \bigcap_{i=1}^{r} P_i\right)_{1\leq j,k\leq n} \mapsto \left((a_{jk}+P_1)_{1\leq j,k\leq n}, \ldots, (a_{jk}+P_r)_{1\leq j,k\leq n}\right).
\end{aligned}$$

Clearly we have that $\det(a_{jk} + \bigcap_{i=1}^{r} P_i) \equiv 0 \pmod{\bigcap_{i=1}^{r} P_i}$ if and only if $\det(a_{jk} + P_i) \equiv 0 \pmod{P_i}$ for all $i \in \{1, \ldots, r\}$. Therefore,

$$|\{B \in \mathrm{Mat}_{n\times n}(\mathcal{O}_S/\textstyle\bigcap_{i=1}^{r} P_i) \mid \det(B) = 0\}| = \displaystyle\prod_{i=1}^{r} |\{B_i \in \mathrm{Mat}_{n\times n}(\mathcal{O}_S/P_i) \mid \det(B_i) = 0\}|.$$

We know that the quotient ring $\mathcal{O}_S/P_i$ is isomorphic to $\mathbb{F}_{q^{\deg(P_i)}}$ (see [17, Proposition 3.2.9.]). Hence, we have

$$\begin{aligned}
|\{B_i \in \mathrm{Mat}_{n\times n}(\mathcal{O}_S/P_i) \mid \det(B_i) = 0\}| &= |\mathrm{Mat}_{n\times n}(\mathbb{F}_{q^{\deg(P_i)}}) \setminus \mathrm{GL}_{n\times n}(\mathbb{F}_{q^{\deg(P_i)}})| \\
&= q^{deg(P_i)n^2} - \prod_{k=0}^{n-1} (q^{\deg(P_i)n} - q^{\deg(P_i)k}) \\
&\leq 2^n q^{\deg(P_i)n(n-1)}.
\end{aligned}$$

By Equation (16), there exist a constant $C > 0$ such that for all $D \in \mathcal{D}_S$ with $\deg(D) \geq C$, we have

$$|\mathcal{L}(D) \cap \bigcap_{i=1}^{r} P_i| = q^{\ell(D)} \prod_{j=1}^{r} q^{-\deg(P_j)}.$$

Now, combining everything, we have

$$\begin{aligned}
&|\bigcap_{i=1}^{r} U_{P_i} \cap \mathcal{L}(D)^{n\times m}| \\
&\qquad \leq \binom{m}{n} q^{(nm-n^2)\deg(D)} q^{\ell(D)n^2} \prod_{j=1}^{r} q^{-n^2\deg(P_j)} \prod_{i=1}^{r} \left(2^n q^{\deg(P_i)n(n-1)}\right) \\
&\qquad = C' q^{nm\ell(D)} \prod_{j=1}^{r} q^{-n\deg(P)}
\end{aligned}$$

for a constant $C' > 0$ depending only on n, m. Observe that $\sum_{P\in\mathbb{P}_F} q^{-n\deg(P)}$ is the Zeta function $Z(q^{-n})$. By [17, Proposition 5.1.6.], it converges when $n > 1$. The case $n = 1$ corresponds to coprime m-tuples and is covered by Theorem 4. □

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