A NOTE ON CONGRUENCES FOR GENERALIZED CUBIC PARTITIONS MODULO PRIMES

Russelle Guadalupe
Institute of Mathematics, University of the Philippines, Diliman, Quezon City, Philippines
rguadalupe@math.upd.edu.ph

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Abstract

Recently, Amdeberhan, Sellers, and Singh introduced the notion of a generalized cubic partition function $a_c(n)$ and proved two isolated congruences via modular forms, namely, $a_3(7n+4) \equiv 0 \pmod 7$ and $a_5(11n+10) \equiv 0 \pmod{11}$. In this paper, we provide another proof of these congruences by using classical q-series manipulations. We also give infinite families of congruences for $a_c(n)$ for primes $p \not\equiv 1 \pmod 8$.

1. Introduction

Throughout this paper, let $f_m := \prod_{n\geq 1}(1-q^{mn})$ for a positive integer m and a complex number q with $|q| < 1$. Recall that a *partition* of a positive integer n is a nonincreasing sequence of positive integers, known as its parts, whose sum is n. A *cubic partition* of n is a partition of n whose even parts may appear in two colors. Let $a(n)$ be the number of cubic partitions of n and set $a(0) := 1$. Then the generating function of $a(n)$ is

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{f_1 f_2}.$$

Chan [3] showed that $a(3n+2) \equiv 0 \pmod 3$ by establishing the remarkable identity

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{f_3^3 f_6^3}{f_1^4 f_2^4},$$

which is an analogue of the identity of Ramanujan [4, pp. 210–213] given by

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6},$$

where $p(n)$ is the number of partitions of n.

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Recently, Amdeberhan, Sellers, and Singh [2] introduced the notion of a *generalized cubic partition* of n, which is a partition of n whose even parts may appear in $c \geq 1$ different colors. Let $a_c(n)$ be the number of such generalized cubic partitions of n and set $a_c(0) := 1$. Then the generating function of $a_c(n)$ is

$$\sum_{n=0}^{\infty} a_c(n) q^n = \frac{1}{f_1 f_2^{c-1}}.$$

Using the theory of modular forms, they proved the following congruences for $a_c(n)$ modulo 7 and 11.

Theorem 1 ([2]). *For all $n \geq 0$,*

$$a_3(7n+4) \equiv 0 \pmod{7}, \tag{1}$$

$$a_5(11n+10) \equiv 0 \pmod{11}. \tag{2}$$

In this paper, we offer another proof of Theorem 1 using classical q-series manipulations. We also provide infinite families of congruences for $a_c(n)$ modulo primes $p \not\equiv 1 \pmod{8}$.

The rest of the paper is organized as follows. In Section 2, we present another proof of Theorem 1 using the identities of Euler and Ramanujan. In Section 3, we prove two infinite families of congruences for $a_c(n)$ modulo primes $p \not\equiv 1 \pmod{8}$ using another identity of Ramanujan and the result of Ahlgren [1].

2. Another Proof of Theorem 1

Proof. To prove Theorem 1, we recall the identity of Euler [5, (1.7.1)]

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \tag{3}$$

and the identity of Ramanujan [5, (10.7.3)]

$$\frac{f_1^5}{f_2^2} = \sum_{n=-\infty}^{\infty} (-1)^n (6n+1) q^{n(3n+1)/2}. \tag{4}$$

By the binomial theorem, we have $f_7 \equiv f_1^7 \pmod{7}$, so from Equations (3) and (4),

$$\begin{aligned}\sum_{n=0}^{\infty} a_3(n) q^n &= \frac{1}{f_1 f_2^2} \equiv \frac{1}{f_7} \cdot \frac{f_1^5}{f_2^2} \cdot f_1 \\ &\equiv \frac{1}{f_7} \sum_{m,\, n=-\infty}^{\infty} (-1)^{m+n} (6m+1) q^{(3m^2+m)/2+(3n^2+n)/2} \pmod{7}. \end{aligned} \tag{5}$$

We consider the equation

$$\frac{3m^2+m}{2}+\frac{3n^2+n}{2}\equiv 4 \pmod 7,$$

which is equivalent to

$$(6m+1)^2+(6n+1)^2\equiv 0 \pmod 7. \tag{6}$$

Since $7\equiv 3 \pmod 4$, it follows that -1 is a quadratic nonresidue modulo 7. Thus, the solution of Equation (6) is $6m+1\equiv 6n+1\equiv 0 \pmod 7$. Extracting the terms containing q^{7n+4} on both sides of Equation (5), dividing by q^4, and then replacing q^7 with q, we get Equation (1).

On the other hand, we have $f_{11}\equiv f_1^{11} \pmod{11}$, so Equation (4) implies that

$$\begin{aligned}\sum_{n=0}^{\infty} a_5(n)q^n &= \frac{1}{f_1f_2^4}\equiv \frac{1}{f_{11}}\cdot\left(\frac{f_1^5}{f_2^2}\right)^2\\ &\equiv \frac{1}{f_{11}}\sum_{m,\,n=-\infty}^{\infty}(-1)^{m+n}(6m+1)(6n+1)q^{(3m^2+m)/2+(3n^2+n)/2} \pmod{11}.\end{aligned} \tag{7}$$

We consider the equation

$$\frac{3m^2+m}{2}+\frac{3n^2+n}{2}\equiv 10 \pmod{11},$$

which is equivalent to

$$(6m+1)^2+(6n+1)^2\equiv 0 \pmod{11}. \tag{8}$$

Since $11\equiv 3 \pmod 4$, we have that -1 is a quadratic nonresidue modulo 11. Thus, the solution of Equation (8) is $6m+1\equiv 6n+1\equiv 0 \pmod{11}$. Extracting the terms containing q^{11n+10} on both sides of Equation (7), dividing by q^{10}, and then replacing q^{11} with q, we arrive at Equation (2). □

3. Congruences for $a_c(n)$ Modulo Primes $p\not\equiv 1 \pmod 8$

We now prove two infinite families of congruences for $a_c(n)$ modulo primes $p\not\equiv 1 \pmod 8$. We first give the following result for primes $p\equiv 5,7 \pmod 8$, which generalizes Equation (1) in Theorem 1.

Theorem 2. *Let $p\equiv 5,7 \pmod 8$ be a prime and $0\le l\le p-1$ be a nonnegative integer with $p\mid 8l+3$. Then for all $n\ge 0$,*

$$a_{p-4}(pn+l)\equiv 0 \pmod p. \tag{9}$$

Proof. We start with the following identity of Ramanujan [5, (10.7.7)]

$$\frac{f_2^5}{f_1^2} = \sum_{n=-\infty}^{\infty} (-1)^n (3n+1) q^{3n^2+2n}. \tag{10}$$

With $f_{2p} \equiv f_2^p \pmod{p}$, we see from Equations (3) and (10) that

$$\begin{aligned}\sum_{n=0}^{\infty} a_{p-4}(n) q^n = \frac{1}{f_1 f_2^{p-5}} &\equiv \frac{1}{f_{2p}} \cdot \frac{f_2^5}{f_1^2} \cdot f_1 \\ &\equiv \frac{1}{f_{2p}} \sum_{m,\, n=-\infty}^{\infty} (-1)^{m+n} (3m+1) q^{3m^2+2m+n(3n+1)/2} \pmod{p}.\end{aligned} \tag{11}$$

We now consider the equation

$$3m^2 + 2m + \frac{n(3n+1)}{2} \equiv l \pmod{p},$$

which can be written as

$$2(6m+2)^2 + (6n+1)^2 \equiv 3(8l+3) \equiv 0 \pmod{p}. \tag{12}$$

Since $p \equiv 5, 7 \pmod{8}$, we see that -2 is a quadratic nonresidue modulo p. Thus, the solution of Equation (12) is $6m+2 \equiv 6n+1 \equiv 0 \pmod{p}$. We get $3m+1 \equiv 0 \pmod{p}$, and extracting the terms containing q^{pn+l} on both sides of Equation (11), dividing by q^l, and then replacing q^p with q yield Equation (9). □

We next prove the analogous result for primes $p \equiv 3, 7 \pmod{8}$, which may be seen as a generalization of Equation (2) in Theorem 1.

Theorem 3. *Let $p \geq 7$ be a prime with $p \equiv 3, 7 \pmod{8}$. Then for all $n \geq 0$,*

$$a_{p-6}\left(pn + \frac{13(p^2-1)}{24}\right) \equiv 0 \pmod{p}. \tag{13}$$

Proof. Since $f_{2p} \equiv f_2^p \pmod{p}$,

$$\sum_{n=0}^{\infty} a_{p-6}(n) q^n = \frac{1}{f_1 f_2^{p-7}} \equiv \frac{1}{f_{2p}} \cdot \frac{f_2^7}{f_1} \pmod{p}. \tag{14}$$

Let

$$\sum_{n=0}^{\infty} A(n) q^n := \frac{f_2^7}{f_1}.$$

Then we have the following identity [1, p. 223]

$$A\left(pn+\frac{13(p^2-1)}{24}\right)=\epsilon p^2 A\left(\frac{n}{p}\right), \tag{15}$$

where $p \equiv 7, 11 \pmod{12}$ is a prime and

$$\epsilon = \begin{cases} 1 & \text{if } p \equiv 7 \pmod{8}, \\ -1 & \text{if } p \equiv 3 \pmod{8}. \end{cases}$$

As $p \geq 7$ and $p \equiv 3, 7 \pmod{8}$, we know that $p \equiv 7, 11, 19, 23 \pmod{24}$, so $p \equiv 7, 11 \pmod{12}$. Thus, applying Equation (15) to Equation (14) yields

$$a_{p-6}\left(pn+\frac{13(p^2-1)}{24}\right) \equiv A\left(pn+\frac{13(p^2-1)}{24}\right) \equiv 0 \pmod{p}$$

for any $n \geq 0$, completing the proof of Equation (13). □

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References

[1] S. Ahlgren, Multiplicative relations in powers of Euler's product, *J. Number Theory* **89** (2) (2001), 222-233.

[2] T. Amdeberhan, J. A. Sellers, and A. Singh, Arithmetic properties for generalized cubic partitions and overpartitions modulo a prime, *Aequationes Math.* (2024), `doi.org/10.1007/s00010-024-01116-7`.

[3] H.-C. Chan, Ramanujan's cubic continued fraction and an analog of his 'most beautiful identity', *Int. J. Number Theory* **6** (3) (2010), 673-680.

[4] G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson, *Collected Papers of Srinivasa Ramanujan*, Cambridge University Press, 1927.

[5] M. D. Hirschhorn, *The Power of q: A Personal Journey*, Developments in Mathematics vol. 49, Springer, Cham, 2017.