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RAMSEY THEORY ON THE INTEGER GRID: THE "L" PROBLEM

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Abstract

In an $[n] \times [n]$ integer grid, a monochromatic L is any set of points $\{(i,j),(i,j+t),(i+t,j+t)\}$ for some positive integer t, where $1 \leq i,j,i+t,j+t \leq n$. In this paper, we investigate the upper bound for the smallest integer n such that a 3-colored $n \times n$ grid is guaranteed to contain a monochromatic L. We use various methods, such as counting intervals on the main diagonal and using Golomb rulers, to improve the upper bound. This bound originally sat at 2593, and we improve it first to 1803, then to 1573, then to 772, and finally to 493.

1. Introduction to the Problem

The problem we deal with in this paper is a Ramsey-type problem on the integer grid. Namely, we deal with the following corollary of the Gallai-Witt theorem.

Theorem 1 ([11]). *For all positive integers k, there exists a positive integer n such that, for all k-colorings of $[n] \times [n]$, there is a monochromatic L. That is, there exist positive integers x, y, and d such that:*

1. *$(x,y),(x+d,y)$, and $(x+d,y+d)$ are all in $[n] \times [n]$, and*
2. *$(x,y),(x+d,y)$, and $(x+d,y+d)$ are all the same color.*

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To elaborate, an $[n] \times [n]$ *integer grid* is an integer lattice with n rows, n columns, and a set of points such that, for each pair of integers i, j with $1 \leq i, j \leq n$, there is exactly one point p lying in row i and column j. In such a grid, the rows are counted going downward and the columns are counted left-to-right. Such a point p is said to be *located at* (i, j).

An interesting problem is to find $n = R_k(L)$, the least positive integer such that the $n \times n$ integer grid with k colors must contain a monochromatic L. From our definition, it is obvious that $R_1(L) = 2$, since a monochromatic L can only appear in a grid of at least length 2 and is guaranteed to appear in a monochromatic 2×2 grid at the points $(1, 1), (2, 1)$, and $(2, 2)$. It is also known that $R_2(L) = 5$ [7]. As a result, we focus mostly on $R_3(L)$, of which much less is known.

In 2015, Canacki et al. [2] found that $21 \leq R_3(L) \leq 2593$. This paper improves substantially on the upper bound, the methods of which are detailed in the following sections. In Section 2.1, we prove $R_3(L) \leq 1804$, in Section 2.2, we lower this upper bound to 1573, and in Section 2.3, we lower this bound to 772 and finally 493. As we do not improve upon the lower bound, discussion of it is omitted from this paper. However, information on the lower bound as well as speculation on how to improve it can be found in the arxiv version of this paper [8].

2. The Upper Bound

To aid in our arguments, we will open with the proof in Canacki et al. [2] that $R_3(L) \leq 2593$. However, before continuing, we will introduce diagonals and subdiagonals, as these are crucial to all our proofs. The *main diagonal* on an $n \times n$ grid is the series of points $(1, 1), (2, 2), \ldots, (n, n)$. A *subdiagonal* is a series of points in the $n \times n$ integer grid that follow the form of either $(1, k), (2, k + 1), \ldots, (1 + n - k, n)$ or $(k, 1), (k + 1, 2), \ldots, (n, 1 + n - k)$ for some integer k where $1 \leq k \leq n$. We label S_k as the subdiagonal containing the point $(1, k + 1)$ (that is, the subdiagonal k points below the main diagonal). The subdiagonals above the main diagonal are rarely discussed.

In the proofs that follow, we assume our 3 colors in a 3-colored grid to be red, green, and blue.

Theorem 2 ([2]). *The value of $R_3(L)$ is at most 2593.*

Proof. Assume we have a 3-coloring of the $n \times n$ grid. Consequentially, there are n points on the main diagonal, and since the diagonal is 3-colored, there exists $\frac{n}{3}$ points of a single color on the main diagonal. Without loss of generality, assume this color is red. For each pair of red points on the main diagonal, there is a unique point in the grid below this main diagonal such that if colored red, this point and the two selected red points form a monochromatic L. Thus, there are $\binom{n/3}{2}$ points in the grid that must be colored either blue or green.

As there are $\binom{n/3}{2}$ blue or green points across $n-1$ subdiagonals, there lie $\frac{\binom{n/3}{2}}{n-1}$ blue or green points on some subdiagonal $S_{\mathcal{B}}$. Since these points are 2-colored, there are either $\frac{\binom{n/3}{2}}{2(n-1)}$ blue or $\frac{\binom{n/3}{2}}{2(n-1)}$ green points on this diagonal. Without loss of generality, assume the majority are blue, and let us define $b = \frac{\binom{n/3}{2}}{2(n-1)}$. As with the red points, each pair of these blue points corresponds to a unique point in the subdiagonals below such that if colored blue, this point and the pair of blue points form a monochromatic L. Moreover, this point cannot be colored red, since it will form a monochromatic L with the red points that force this pair of blue points to be blue. So there are $\binom{b}{2}$ points that must be colored green.

Since there are at most $n-2$ subdiagonals below $S_{\mathcal{B}}$, one of these subdiagonals $S_{\mathcal{G}}$ must contain $\frac{\binom{b}{2}}{n-2}$ green points. For each of these green points on $S_{\mathcal{G}}$, there is a corresponding point that cannot be colored green, or else it will form a monochromatic L with its pair on $S_{\mathcal{G}}$. Moreover, it cannot be colored red or blue, or it will form a monochromatic L with the red and/or blue points that forced the pair on $S_{\mathcal{G}}$ to be green. So if there is more than one green point on $S_{\mathcal{G}}$ (that is, if $\frac{\binom{b}{2}}{n-2} > 1$), we reach a contradiction. The smallest n such that $\frac{\binom{b}{2}}{n-2} > 1$ is 2593 [2]. $\square$

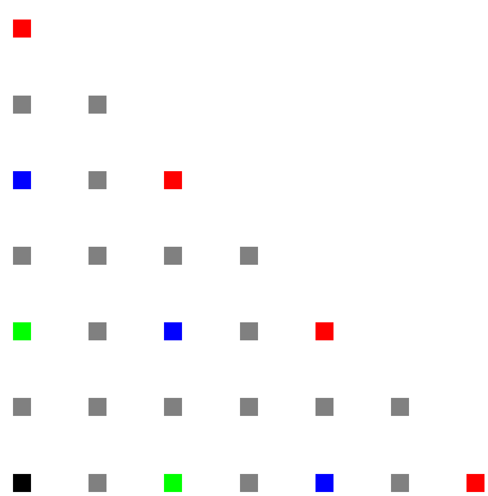

Figure 1: An illustration of the contradiction we reach in Theorem 1. Points with unspecified color are grey, and the point that cannot be any color without forming a monochromatic L is black.

2.1. Intervals

We begin this section with a definition and a lemma concerning the minimum number of blue points on a given diagonal needed to force a contradiction.

Definition 1. Let $n \geq 3$. Assume there is a 3-coloring of the $n \times n$ grid. Let p be a point below the main diagonal. If (1) p is colored blue, and (2) there are two red points on the main diagonal such that those points and p form an L, then p is *forced by red to be blue.*

Lemma 1. *Let $n \geq 3$. Let*

$$b = \left\lceil \sqrt{2n - \frac{15}{4}} + \frac{1}{2} \right\rceil .$$

If there are at least b blue points on a diagonal which are forced by red to be blue, then there exists a monochromatic L in the $n \times n$ grid.

Proof. Assume there is a 3-coloring of the $n \times n$ grid. We show that b blue points of the $n \times n$ grid fulfill these constraints. Assume there are b blue points p such that p together with 2 red points on the diagonal form an L. Take any pair $\{p_1, p_2\}$ of these blue points. Let q be the point such that $\{p_1, p_2, q\}$ form an L with q at the corner. Then note (a) if q is blue there is a blue L, and (b) if q is red there is a red L. Hence q must be green. We see there are $\binom{b}{2}$ green points forced by p and $n-2$ subdiagonals on which these green points can occur, and so if $\frac{\binom{b}{2}}{n-2} > 1$, a monochromatic L is forced. If b is the least integer such that this inequality holds, we see the following:

$$\binom{b}{2} = \frac{b(b-1)}{2} > n-2 \Longrightarrow b^2 - b > 2(n-2) \Longrightarrow b^2 - b + \frac{1}{4} > 2n - \frac{15}{4}$$
$$\Longrightarrow b - \frac{1}{2} > \sqrt{2n - \frac{15}{4}} \Longrightarrow b = \left\lceil \sqrt{2n - \frac{15}{4}} + \frac{1}{2} \right\rceil .$$

□

Having defined b, we now define intervals and discuss how we will use them to improve our bounds. An *interval* a_k on the main diagonal, where $k \in \mathbb{Z}$, $1 \leq k \leq \lceil \frac{n}{3} \rceil$, is the number of points between the kth and $(k+1)$th red points on the main diagonal. In general, if the kth and $(k+1)$th red points are separated by c points, then $a_k = c$, and their corresponding point that cannot be colored red without forcing a monochromatic L lies on S_{c+1}. For example, red points right next to each other have an interval of length 0, red points separated by one non-red point have an interval of length 1, and so on.

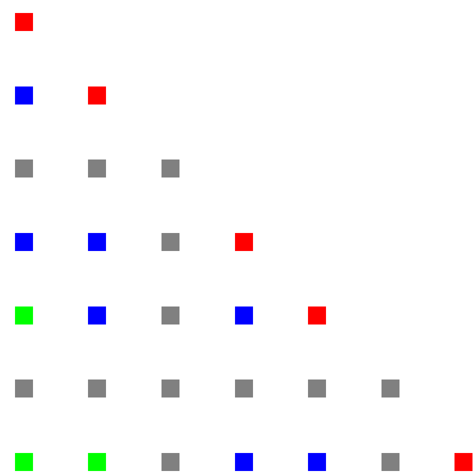

Figure 2: An illustration of how certain intervals force points on corresponding diagonals to be either blue or green. For example, two adjacent red points force a blue/green point on the subdiagonal right below.

Now, consider our $n \times n$ integer grid. We cannot have more than $2(b-1)$ forced blue or green points in one diagonal, or else we must have either b blue or b green points, which forces a monochromatic L. Let k_i be the number of pairs of consecutive red points on the main diagonal separated by i points that are not red. Observe that $k_i \leq 2(b-1)$ for all possible k_i, or else b blue points (we assume majority blue) will lie on S_{i+1}. The space s taken up on the main diagonal by the red points and the intervals between consecutive red points is the following:

$$s = \left\lceil \frac{n}{3} \right\rceil + 0k_0 + 1k_1 + \cdots + (n-2)k_{n-2}.$$

In the above formula, $\lceil \frac{n}{3} \rceil$ represents the space taken up by our red points (we have $\lceil \frac{n}{3} \rceil$ of them and will prove this later), $0k_0$ represents the amount of space taken up by the intervals of length 0 (as each of these intervals takes up 0 space), $1k_1$ represents the amount of space taken up by the intervals of length 1, and so on.

The way to get a lower bound on s is to make these intervals as small as possible while still maintaining that there are not b blue or b green points on any subdiagonal created by these intervals. This means that $2(b-1)$ of these intervals will be length 0, $2(b-1)$ will be length 1, and so on until we run out of intervals. In other words, we take $k_0, k_1, \ldots, k_{q-1} = 2(b-1)$, and $k_q = r$ where

$$q = \left\lfloor \frac{\lceil \frac{n}{3} \rceil - 1}{2(b-1)} \right\rfloor, r = \left(\left\lceil \frac{n}{3} \right\rceil - 1 \right) - q \cdot 2(b-1) = \left\lceil \frac{n}{3} \right\rceil - 1 \pmod{2(b-1)}.$$

As such, we get the following lower bound for s (which we call $s_{\min}$):

$$\begin{aligned} s_{\min} &= \left\lceil \frac{n}{3} \right\rceil + (0k_0 + 1k_1 + \cdots + (q-1)k_{q-1}) + qk_q \\ &= \left\lceil \frac{n}{3} \right\rceil + 2(b-1)(0 + 1 + \cdots + (q-1)) + qr \\ &= \left\lceil \frac{n}{3} \right\rceil + 2(b-1)\binom{q}{2} + qr. \end{aligned}$$

To verify that $s_{\min}$ is minimized when we have as few red points on the main diagonal as possible, suppose there are $m > \lceil \frac{n}{3} \rceil$ points on the main diagonal. Then we have

$$q_m = \left\lfloor \frac{m-1}{2(b-1)} \right\rfloor, r_m = m - 1 \pmod{2(b-1)},$$

and so the minimum amount of space they take up (using the above argument) is

$$\begin{aligned} &m + (0k_0 + 1k_1 + \cdots + (q-1)k_{q-1} + qk_q + \cdots + qk_{q_m}) \\ &> \left\lceil \frac{n}{3} \right\rceil + (0k_0 + 1k_1 + \cdots + (q-1)k_{q-1}) + qk_q = s_{\min}. \end{aligned}$$

Thus, $s_{\min}$ occurs when we have as few red points on the main diagonal as possible. If $s_{\min} > n$, then the lower bound for the necessary space on the main diagonal is larger than the diagonal itself, meaning the red points cannot "fit" on this diagonal without forcing a monochromatic L. The following theorem tells us which number this is.

Theorem 3. *The value of $R_3(L)$ is at most* 1804.

Proof. Assume there is a 3-coloring of the $n \times n$ grid. We wish to find the least $n \in \mathbb{N}$ such that $s_{\min} > n$. The above problem is an optimization problem, and so it is possible to solve it with a Python script. The program works in the following way: we start with a value of n for which we can be sure this contradiction does not arise (say, $n = 100$). We then set $s_{\min}$ and run a while loop while $s_{\min} \leq n$. For each iteration, we increase n by 1 and set all variables outlined in the proof (b, $s_{\min}$, q, and r) for the given n. This algorithm increases n by 1 until we have an n such that $s_{\min} > n$. This n is 1804, and so the smallest grid that must have a monochromatic L in any of its 3-colorings is at most 1804×1804. □

Algorithm 1 on the page below contains the pseudocode for Theorem 3.

2.2. Unaccounted Intervals

One may notice that in the above proof, we only accounted for intervals between consecutive points. This is to make the proof simpler, but we can even improve on this by taking into account other intervals.

Algorithm 1 Proving $R_3(L) \leq 1804$

```
n = 100
while s_min ≤ n do
    n = n + 1
    b = ⌈ sqrt(2n − 15/4) + 1/2 ⌉
    q = ⌊ (⌈n/3⌉ − 1) / (2(b − 1)) ⌋
    r = ⌈n/3⌉ − 1 (mod 2(b − 1))
    s_min = ⌈n/3⌉ + 2(b − 1) binom(q,2) + qr
end while
```

$n = 100$
while $s_{\min} \leq n$ **do**
$n = n + 1$
$b = \left\lceil \sqrt{2n - \frac{15}{4}} + \frac{1}{2} \right\rceil$
$q = \left\lfloor \frac{\lceil \frac{n}{3} \rceil - 1}{2(b-1)} \right\rfloor$
$r = \lceil \frac{n}{3} \rceil - 1 \pmod{2(b-1)}$
$s_{\min} = \lceil \frac{n}{3} \rceil + 2(b-1)\binom{q}{2} + qr$
end while

To give an idea of how this might be done, let us define the *interval* $a_{j,k}$ as the number of points between the jth and kth red points as. From how we defined a_j, we note that $a_{j,k} = a_j$. Now, consider the space between the first and third red points. Between these, there exists the second red point and $a_1 + a_2$ points that are not red. Thus, the interval between these points $a_{1,3}$ is of size $a_1 + a_2 + 1$. To generalize, between points j and k, there exist $k - j - 1$ red points as well as intervals $a_j, a_{j+1}, \ldots, a_{k-1}$. Thus,

$$a_{j,k} = \sum_{i=j}^{k-1} a_i + k - j - 1.$$

Let us extrapolate our property concerning intervals in Theorem 3 to these intervals $a_{j,k}$. Using n and b as defined in that proof, we say that for a given nonnegative integer m, there are at most $2(b-1)$ pairs j, k where $1 \leq j, k \leq \lceil \frac{n}{3} \rceil$ such that $a_{j,k} = m$. In simpler terms, out of all intervals between red points on the main diagonal (not just consecutive points), there can be at most $2(b-1)$ intervals of a given size.

An immediate problem is how to sequence the intervals in a diagonal of given length. As an example, take $n = 1803$, the largest number under our current bound. For this given n, we have $b = 61$, and so there can be at most 120 intervals of any given size. In this instance, we have at least $\lceil \frac{1803}{3} \rceil = 601$ red points and thus at least 600 intervals. If we make 120 intervals each of length 0, 1, 2, 3, and 4, then we have a total of $120 \cdot (0 + 1 + 2 + 3 + 4) + 601 = 1801$ minimum spaces taken up on the main diagonal. We wish to find a sequence of these intervals such that there are no more than $2(b-1)$ intervals of any given size.

We note that if $a_j = 0$, $a_{j+1} \leq 3$, then $a_{j,j+2} \leq 4$. Since we have exactly $2(b-1)$ intervals of length $0, 1, 2, 3$, and 4, this scenario would create another interval of length 4, resulting in a contradiction. Thus, an interval of length 0 can only have an interval of length 4 on its right. By the same logic, an interval of length 0 can only have an interval of length 4 on its left hand side, so a 0 can only be adjacent to a 4. Using this line of reasoning, an interval of length 1 can only be adjacent to an interval of length 3 or 4, and an interval of length 2 can only be adjacent to an interval of length 2, 3, or 4. Since we have the same number of 0s and 4s, if the sequence $0, 4, 0, 4, \ldots, 0, 4$ exists, the first 0 must be a_1 (or else the first interval of 0 will necessarily border an interval less than length 4 to its left). Similarly, if the subsequence $4, 0, 4, 0, \ldots, 4, 0$ exists, the last interval of length 0 must be a_{600} in the main sequence.

Since all the 0s "pair up" with the 4s, the 1s can only be adjacent to 3s or the 4s at the ends of the subsequences listed above. Since 2s can neighbor 2s, 3s, and 4s, we can gather some sequences such that all intervals $a_{j,j+2}$ are greater than 4:

$$0, 4, 0, 4, \ldots, 0, 4, 1, 3, 1, 3, \ldots, 1, 3, 2, 2, \ldots, 2, 3, 1, 3, 1, \ldots, 3, 1.$$

$$1, 3, 1, 3, \ldots, 1, 3, 2, 2, \ldots, 2, 3, 1, 3, 1, \ldots, 3, 1, 4, 0, 4, 0, \ldots.$$

$$0, 4, 0, 4, \ldots 1, 3, 1, 3, \ldots, 1, 3, 2, 2, \ldots, 2, 3, 1, 3, 1, \ldots, 3, 1, 4, 0, 4, 0, \ldots.$$

Note that if we group all adjacent intervals into pairs such that $\{a_1, a_2\}$, $\{a_3, a_4\}$ and so on are pairs, then nearly all pairs add up to 4. Recall that $a_{j,j+2} = a_j + a_{j+1} + 1$, and so from this we generate around 300 intervals of size 5. This is a contradiction since we can have at most 120 intervals of any given size. From this we gather that we cannot use only the intervals $0, 1, 2, 3,$ and 4: we must change some of these to a different size.

While we do not work further with this sequencing of intervals in this paper, the idea to include intervals between nonconsecutive red points motivates another lowering of our upper bound as detailed in the proof below.

Theorem 4. *The value of $R_3(L)$ is at most* 1573*.*

Proof. Assume there is a 3-coloring of the $n \times n$ grid. We show that a 1573×1573 integer grid must contain a monochromatic L using intervals between non-adjacent red points on our main diagonal. We can see that the minimum space taken up by the red points on the main diagonal of length n is equivalent to $n^* + a_1 + a_2 + a_3 + \cdots + a_{n^*-1}$, where $n^* = \lceil \frac{n}{3} \rceil$. A space calculated using similar methodology to the one in the proof of Theorem 3 can be measured by taking the intervals between red points 1 and 3, 3 and 5, and so on until we run out of points. That is, for $t \in \mathbb{N}$ such that t is the greatest odd number such that $t \leq n^*$, the sum of the intervals $a_{1,3}, a_{3,5}, \ldots a_{t-2,t}$ plus the unaccounted points between these intervals will be the same space taken up by the sum of $a_1, a_2, \ldots, a_{t-1}$, as well as all the unaccounted points in this interval. In mathematical terms, we have the following inequality:

$$\begin{aligned} a_{1,3} + a_{3,5} + \cdots + a_{t-2,t} + \left\lfloor \frac{n^* - 1}{2} \right\rfloor + 1 &= a_1 + a_2 + \cdots + a_{t-1} + t \\ &\leq a_1 + a_2 + \cdots + a_{n^*-1} + n^* = s \leq n. \end{aligned}$$

Combining this equation with our first gives us the following:

$$a_1 + a_2 + \cdots + a_{n^*-1} + a_{1,3} + a_{3,5} + \cdots + a_{t-2,t} + n^* + \left\lfloor \frac{n^*-1}{2} \right\rfloor + 1 \leq 2n.$$

In this formula, we are counting both the space taken up by the red points and the intervals between them as well as every other red point and the space between these points. Let us call this combined space s^*. As before, at most $2(b-1)$ of these intervals can have length m where m is a nonnegative integer. Assume the intervals in the above equation are as small as possible in order to minimize space (we will call the minimum space s^*_{min}). Let $q, r \in \mathbb{Z}$ such that q is the quotient upon dividing the number of intervals $\left((n^*-1) + \left\lfloor \frac{n^*-1}{2} \right\rfloor\right)$ by $2(b-1)$ and r is the remainder. This gives us the following:

$$\begin{aligned} s^*_{\text{min}} &= n^* + \left\lfloor \frac{n^*-1}{2} \right\rfloor + 1 + 2(b-1)(0 + 1 + \cdots + (q-1)) + qr \\ &= n^* + \left\lfloor \frac{n^*-1}{2} \right\rfloor + 1 + 2(b-1)\binom{q}{2} + qr. \end{aligned}$$

We use a Python script to generate the first number n such that this condition does not hold. Similar to the last program, we iterate through each $n \in \mathbb{N}$ in increasing order, setting $b, q, r,$ and s^*_{min} for the current n, and checking to see whether $s^*_{\text{min}} > 2n$. This program gives us $n = 1573$. □

The pseudocode for Theorem 4 is included as Algorithm 2 below.

Algorithm 2 Proving $R_3(L) \leq 1573$

$n \leftarrow 100$
while $s_{\text{min}} \leq 2n$ **do**
  $n \leftarrow n + 1$
  $n^* \leftarrow \left\lceil \frac{n}{3} \right\rceil$
  $\text{intvls} \leftarrow (n^*-1) + \left\lfloor \frac{n^*-1}{2} \right\rfloor$
  $b \leftarrow \left\lceil \sqrt{2n - \frac{15}{4}} + \frac{1}{2} \right\rceil$
  $q \leftarrow \left\lfloor \frac{\text{intvls}}{2(b-1)} \right\rfloor$
  $r \leftarrow (\text{intvls}) \pmod{2(b-1)}$
  $s_{\text{min}} = \text{intvls} + 2 + 2(b-1)\binom{q}{2} + qr$
end while
return n

2.3. Further Improvements

In this section, we discuss further major improvements made to the upper bound of $R_3(L)$.

Theorem 5. *The value of* $R_3(L)$ *is at most* 772*.*

Proof. Recall that if there are $2b-1$ pairs of red points, each some fixed distance k apart, then there will be $2b-1$ points forced to be either blue or green in subdiagonal S_k. This implies there will be b points in S_k of some fixed color, say blue, thus forcing 2 green points in the same subdiagonal which in turn force a monochromatic L.

For $c \leq n^* - 1$, there are $n^* - c$ intervals of the form $a_{j,j+c}$. We can partition these intervals into the following sums:

$$\begin{gathered} a_{1,1+c} + a_{1+c,1+2c} + \dots \\ a_{2,2+c} + a_{2+c,2+2c} + \dots \\ \vdots \\ a_{c,2c} + a_{2c,3c} + \dots . \end{gathered}$$

Since the intervals in each sum are consecutive on the main diagonal, each sum plus all the included red points can be at most n. Each red point is included once across all sums, so we get the following formula:

$$\left(\sum_{i=1}^{n^*-c} a_{i,i+c}\right) + n^* \leq cn.$$

Considering the space taken up by all intervals with length *at most* c, we get the following formula:

$$\left(\sum_{j=1}^{c}\sum_{i=1}^{n^*-j} a_{i,i+j}\right) + cn^* \leq n \cdot \frac{c(c+1)}{2}.$$

Moreover, $k_i \leq 2b-1$ for every i. For $n = 772$ and $c = 12$, both of these conditions cannot be true at the same time. Thus, the $n \times n$ grid must contain a monochromatic L. □

Before we discuss the next improvement, we introduce the concept of Golomb rulers to those not acquainted. A *Golomb ruler* is a set of integers such that no two pairs of integers are the same distance apart. The number of integers in a Golomb ruler is its *order*, and the distance between the largest and smallest integers in a Golomb ruler is its *length*. A Golomb ruler is *optimal* if for all Golomb rulers with the same order, there are none with smaller length.

Theorem 6. *The value of* $R_3(L)$ *is at most* 493*.*

Proof. Consider the points forced by red to be blue. In any given subdiagonal, if any two pairs of these blue points are the same distance apart, then two green points forced by blue lie on some subdiagonal below. This forces a monochromatic L. So by definition, these blue points must form a Golomb ruler. We can thus lower our bound b for the number of these blue points allowed to b_k, the largest order possible in a Golomb ruler of length $n-k-1$, or the largest number of forced blue points in S_k such that no two pairs are the same distance apart. Applying b_k instead of b to Theorem 5 gives us a contradiction when $n = 493$ and $c = 12$. □

The pseudocode for this algorithm is written in Algorithm 3 on the page below.

Algorithm 3 Proving $R_3(L) \leq 493$

```
Golomb ← {0, 1, 2, 4, 7, 12, 18, 26, 35, 45, 56, 73, 86, 107, 128, 152, 178, 200, 217, 247,
284, 334, 357, 373, 426, 481, 493, 554, 586}  ▷ 1 + the length of optimal Golomb rulers
for orders 0, 1, ..., 28
blue_array ← int[586]
i, j ← 0
while i < Golomb[length(Golomb)-1] do
    blue_array[i] ← j
    if Golomb[j] = i then
        j ← j + 1
    end if
    i ← i + 1
end while                                                  ▷ Set blue_array[k] = b_{n−k}
c ← 5
while c < 20 do   ▷ Testing different c values to find which one gives the lowest bound
    sum, space ← 0
    n ← 100
    while sum < space do                   ▷ Find least n such that we reach a contradiction
        n ← n + 1
        space ← nc(c+1)/2
        n* ← ⌈n/3⌉
        intvls ← cn* − c(c+1)/2
        sum ← c(c+1)(c+2)/6 + c(c−1)(c+1)/6 +ints
        k ← 1
        while intvls > 0 do                  ▷ Setting intervals to smallest lengths possible
            if 2(blue_array[n − k]) > intvls then
                sum ← sum + intvls(k − 1)
                intvls=0
            else
                sum ← sum + 2(blue_array[n − k])(k-1)
                intvls ← intvls-2(blue_array[n − k])
            end if
        end while
    end while
end while
```

3. Open Problems

We close this paper with some open problems regarding the "L" problem.

- Can interval sequencing (as detailed in Section 2.2 before Theorem 4) be used to further improve the upper bound?
- Can properties of diagonals below the main diagonal and subdiagonal of length $n-1$ be used to improve the upper bound?
- What are upper and lower bounds for $R_4(L)$? $R_k(L)$?
- Though not found, we speculate that a 3-coloring of $[22] \times [22]$ with no monochromatic L exists. Try to find one, perhaps by using SAT solvers or AI/ML techniques.

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