COMPLEX-TYPE CATALAN TRANSFORM OF k-JACOBSTHAL AND FOURTH-ORDER JACOBSTHAL RECURRENCES

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Abstract

This study explores the application of the complex-type Catalan transform to two key recurrence sequences: the k-Jacobsthal and the fourth-order Jacobsthal sequence J_n^F. We investigate the transformed sequences to reveal new structural properties and relations. Additionally, we analyze the Hankel transforms of these sequences, and the bounds for the norms of the matrices.

1. Introduction

Recurrence relations and their generalizations are important to the study of number theory and combinatorics. The *Jacobsthal sequence*, defined recursively by

$$J_n = J_{n-1} + 2J_{n-2}, \text{ for } n \geq 2,$$

has been generalized to the k-Jacobsthal sequence, where the recurrence involves higher powers of the k, and the fourth-order Jacobsthal sequence, a more complex relation that introduces higher-order terms. In this work, we apply a complex-type Catalan transform (CTCT) to the k-Jacobsthal and fourth-order Jacobsthal sequences. The *complex-type Catalan transform (CTCT) of the k-Jacobsthal sequence* $\mathcal{C}^{(i)}[J_{k,n}]$ is given by

$$\mathcal{C}^{(i)}[J_{k,n}] = \sum_{j=1}^{\infty} \frac{j}{2n-j} \binom{2n-j}{n-j} i^j J_{k,j},$$

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with $\mathcal{C}^{(i)}[J_{k,0}] = 0$. The generating function of the k-Jacobsthal sequence $(J_{k,n})_{n\in N}$ and the complex Catalan sequence $(C_n^{(i)})_{n\in N}$ are given by

$$g(x) = \frac{x}{1 - kx - 2x^2},$$

and

$$\mathcal{C}^{(i)}(x) = \frac{1 - \sqrt{1 - 4ix}}{2ix},$$

respectively. The motivation for this study stems from the need to uncover deeper combinatorial and algebraic connections between the Jacobsthal family of sequences and well-known combinatorial constructs, particularly the Catalan numbers. Additionally, these results have potential applications in areas like digital communication and cryptographic systems. A comprehensive collection of these sequences can be found in the work of Sloane [12]. In recent years, the Catalan transform has attracted significant interest as an effective method for uncovering new properties and relationships among various integer sequences. This paper aims to provide a detailed analysis of the utilization of the Catalan transform in the analysis of a variety of sequences, including the k-Pell, k-Pell–Lucas, k-Jacobsthal, Padavon and Fibonacci sequences [1], [2], [7], [10], [13], [15], and [21]. Properties of k-Jacobsthal numbers and its identities are studied by Uygun and Jhala et al. [8] and [9]. From the pioneering work by Hilton and Pederson in 1991 [5], which laid the foundation for understanding the applications of Catalan numbers.

The work by Cvetković et al. [14] successfully applied the Catalan and Hankel transforms to well-known sequences, such as the Fibonacci numbers, providing valuable insights into their algebraic properties and generating functions. Building on this foundation, this paper aims to extend the analysis by applying the complex Catalan transform to other sequence families, specifically the k-Jacobsthal and fourth-order Jacobsthal sequences. Additionally, we explore the norms of the corresponding matrices and investigate the Hankel transform of these sequences, further contributing to the understanding of their structural characteristics. The Catalan transform of integer sequences and k-balancing sequence were studied by Barry [3] as well as by Patra and Kaabar [6], and data hiding techniques were also analyzed [32].

Furthermore, [17], [18], [29], and [31] highlight the significance of the Hankel transform, as elucidated in the works of Rajkovic, Petkovic, and Barry, in the context of the sum of consecutive generalized Catalan numbers. The Catalan transform in different contexts [19] and [20], ranging from combinatorial analysis to applications in network impedance [11] and incomplete generalized Jacobsthal polynomials [16] have also been studied. New classes of Catalan-type numbers are discussed by Kucukoglu [4]. A study on generalized fourth-order Jacobsthal sequences by Soykan [22] and Merikoski et al. [23] discussed the spectral and Frobenius norms of a generalized Fibonacci r-circulant matrix. Solaki and et al. [24], [25], [26], [27], and [28] analyzed the bounds for the norms of circulant matrices. Mukhopadhyay et al. [30] studied the Catalan transform in image steganography.

Sheppard and et al. [33] discussed the applications of the Hankel transform in optical propagation to improve security by encoding secret data or waveforms through a structured sequence, improving robustness and concealment. In recent studies, the construction of general forms of ordinary generating functions for various families of numbers and multivariable polynomials has been extensively explored by Simsek [34]. Additionally, generating functions for reciprocal Catalan-type sums have been developed, providing insights into linear differentiation equations and p-adic integral equations by Gun and Simsek [35]. Incorporating Fibonacci sequences into classical cryptography and steganography techniques, such as Least Significant Bit (LSB) encoding, has been shown to achieve high Peak Signal-to-Noise Ratio (PSNR), enhancing data security and image quality in encryption methods given by Sari et al. [36].

2. Preliminaries

The *k-Jacobsthal sequence* $(J_{k,n})_{n\in N}$ is given by

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \text{ for all } n \geq 1,$$

with initial conditions $J_{k,0} = 0$ and $J_{k,1} = 1$. The first few members of $(J_{k,n})_{n\in N}$ are $\{0, 1, k, 2 + k^2, 4k + k^3, k^4 + 6k^2 + 4, k^5 + 8k^3 + 12k, k^6 + 10k^4 + 24k^2 + 8, \ldots\}$. The *fourth-order Jacobsthal sequence* (J_n^F) is defined as

$$J_n^F = J_{n-1}^F + J_{n-2}^F + J_{n-3}^F + 2J_{n-4}^F, \text{ for all } n \geq 4,$$

with initial conditions $J_0^F = 0, J_1^F = 1, J_2^F = 1, J_3^F = 2$. The first few numbers are $\{0, 1, 1, 2, 4, 9, 17, 34, 68, 137, 273, 546, 1092, 2185, \ldots\}$. The *Catalan number* is defined by

$$\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n},$$

and its generating function is

$$g(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

The first few Catalan numbers are $\{1, 2, 5, 14, 42, 132, 429, \ldots\}$. The matrix form of the Catalan transform of the k-Jacobsthal sequence $(J_{k,n})_{n\in N}$ is given by

$$\begin{bmatrix} CJ_{k,1} \\ CJ_{k,2} \\ CJ_{k,3} \\ CJ_{k,4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 \\ 14 & 14 & 9 & 4 & 1 \end{bmatrix} \begin{bmatrix} J_{k,1} \\ J_{k,2} \\ J_{k,3} \\ J_{k,4} \end{bmatrix}.$$

The *complex-type Catalan numbers* is given by

$$\mathcal{C}_n^{(i)} = \frac{1}{n+1}\binom{2n}{n} i^n.$$

The first few complex-type Catalan numbers are $\{i, -2, -5i, 14, 42i, -132, -429i, \ldots\}$. The *Hankel matrix* connected with the given sequence of real numbers is as follows:

$$H_n = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and its Hankel matrix transform is given by

$$\mathcal{H}_{a_n} = \det(a_{i+j-2}),$$

where $1 \leq i, j \leq n+1$.

In addition, the *k-Jacobsthal polynomial* is given by

$$J_{k,n+1}(x) = kJ_{k,n}(x) + 2J_{k,n-1}(x), \text{ for all } n \geq 2,$$

with initial conditions $J_{k,0}(x) = 0; J_{k,1}(x) = 1$. The first few polynomials are $\{0, 1, k, k^2 + 2x, k^3 + kx, k^4 + 6k^2x + 4x^2, \ldots\}$. If we substitute $P_1(x) = -k, P_2(x) = -2$, and $Q_0(x) = 0, Q_1(x) = 1$ in *Theorem* 13 of [34], we get the *Binet formula for the k-Jacobsthal polynomial.*

3. The CTCT of the k-Jacobsthal Sequence $(J_{k,n})$

Theorem 1. *The generating function of the CTC sequence* $(C^{(i)}[J_{k,n}])$ *is given by*

$$g(xC(ix)) = \frac{1 - \sqrt{1-4ix}}{4(i-x) - k + (k-2i)\sqrt{1-4ix}}.$$

Proof. The generating function of the k-Jacobsthal sequence $(J_{k,n})_{n\in N}$ is

$$g(x) = \frac{x}{1 - kx - 2x^2},$$

and the CTC sequence is

$$\mathcal{C}^{(i)}(x) = C(ix) = \frac{1 - \sqrt{1-4ix}}{2ix}.$$

Then

$$g(xC(ix)) = \frac{xC(ix)}{1 - kxC(ix) - 2(xC(ix))^2} = \frac{\frac{1-\sqrt{1-4ix}}{2i}}{1 - k\left(\frac{1-\sqrt{1-4ix}}{2i}\right) - 2\left(\frac{1-\sqrt{1-4ix}}{2i}\right)^2}. \tag{1}$$

Simplifying the above equation we get that, $(C^{(i)}[J_{k,n}])$ is given by

$$g(xC(ix)) = \frac{1-\sqrt{1-4ix}}{4(i-x) - k + (k-2i)\sqrt{1-4ix}}.$$

□

Theorem 2. *For $n \geq 1$ and $k \geq 2$, we have*

$$[J_{k,n+1}]^2 - J_{k,n}.J_{k,n+2} = (-2)^n \ .$$

Proof. By definition of the k-Jacobsthal sequence, $(J_{k,n})_{n\in N}$,

$$\begin{aligned}[J_{k,n+1}]^2 - J_{k,n}.J_{k,n+2} &= (kJ_{k,n} + 2J_{k,n-1})^2 - J_{k,n}.(kJ_{k,n+1} + 2J_{k,n}) \\ &= (k^2-2)[J_{k,n}]^2 + 4J_{k,n-1}(kJ_{k,n} + J_{k,n-1}) - kJ_{k,n}.J_{k,n+1}.\end{aligned}$$

For different values of $n \geq 1$ and $k \geq 2$, we get $(-2)^n$. □

The first few terms of the sequence $\mathcal{C}^{(i)}[J_{k,n}]$ consist of polynomials in k:

$$\mathcal{C}^{(i)}[J_{k,1}] = \sum_{j=1}^{1} \frac{j}{2-j}\binom{2-j}{1-j} i^j J_{k,j} = i,$$

$$\mathcal{C}^{(i)}[J_{k,2}] = \sum_{j=1}^{2} \frac{j}{4-j}\binom{4-j}{2-j} i^j J_{k,j} = i - k,$$

$$\mathcal{C}^{(i)}[J_{k,3}] = \sum_{j=1}^{3} \frac{j}{6-j}\binom{6-j}{3-j} i^j J_{k,j} = -ik^2 - 2k,$$

$$\mathcal{C}^{(i)}[J_{k,4}] = \sum_{j=1}^{4} \frac{j}{8-j}\binom{8-j}{4-j} i^j J_{k,j} = k^3 - 3ik^2 - k - i,$$

$$\mathcal{C}^{(i)}[J_{k,5}] = \sum_{j=1}^{5} \frac{j}{10-j}\binom{10-j}{5-j} i^j J_{k,j} = ik^4 + 4k^3 - 3ik^2 + 2k,$$

$$\mathcal{C}^{(i)}[J_{k,6}] = \sum_{j=1}^{6} \frac{j}{12-j}\binom{12-j}{6-j} i^j J_{k,j} = 8i + 2k + 3ik^2 + 6k^3 + 5ik^4 - k^5.$$

Using the coefficients of the complex-type Catalan transform of the k-Jacobsthal sequence $(J_{k,n})_{n\in N}$, we can generate an infinite triangle. The first six rows of this triangle are given in Table 1.

$\mathcal{C}^{(i)}[J_{k,1}]$	i					
$\mathcal{C}^{(i)}[J_{k,2}]$	-1	i				
$\mathcal{C}^{(i)}[J_{k,3}]$	$-i$	-2	0			
$\mathcal{C}^{(i)}[J_{k,4}]$	1	$-3i$	-1	$-i$		
$\mathcal{C}^{(i)}[J_{k,5}]$	i	4	$-3i$	2	0	
$\mathcal{C}^{(i)}[J_{k,6}]$	-1	$5i$	6	$3i$	2	$8i$

Table 1: Triangle form in first iteration for CTCT of the k-Jacobsthal sequence

We observe that the sequence along the first diagonal, given as $(i, i, 0, -i, 0, \ldots)$, represents the complex-type Catalan transform of the sequence $(1, 0, 2, 0, 4, \ldots)$, and the second diagonal sequence $(-1, -2, -1, 2, \ldots)$, represents the complex-type Catalan transform of the sequence $(i, 1, 0, 3, \ldots)$, etc., which can be obtained from the transform

$$a_n = \sum_{k=0}^{n} \binom{k}{n-k} (-1)^k i^n C^i[b_k].$$

The second iteration of the CTCT of the first five k-Jacobsthal numbers, i.e., $(C^{(i)})^2[J_{k,n}]$, are the polynomials in k.
For example,

$$(C^{(i)})^2[J_{k,1}] = \sum_{n=1}^{1} \frac{j}{2-j} \binom{2-j}{1-j} i^j C^i[J_{k,j}] = -1,$$

$$(C^{(i)})^2[J_{k,2}] = \sum_{n=1}^{2} \frac{j}{4-j} \binom{4-j}{2-j} i^j C^i[J_{k,j}] = -1 - i + k,$$

$$(C^{(i)})^2[J_{k,3}] = \sum_{n=1}^{3} \frac{j}{6-j} \binom{6-j}{3-j} i^j C^i[J_{k,j}] = -2 - 2i + (2+2i)k - k^2,$$

$$(C^{(i)})^2[J_{k,4}] = -5 - 6i + (4+6i)k - (3+3i)k^2 + k^3,$$

$$(C^{(i)})^2[J_{k,5}] = -15i - 14 + (13+20i)k - (6+12i)k^2 + (4+4i)k^3 - k^4.$$

Using the coefficients of the second iteration of the complex-type Catalan transform of the k-Jacobsthal sequence $(J_{k,n})_{n\in N}$, we can generate an infinite triangle. The first five rows of this triangle are given in Table 2.

We observe that the first diagonal sequence, $(-1, -1-i, -2-2i, -6i-5, -14-15i, \ldots)$, is identified as the complex-type Catalan transform (CTCT) of the sequence $(i, i, 0, -i, 7-32i, \ldots)$. Similarly, the second diagonal sequence, $(1, 2+2i, 4-6i, 13+20i, \ldots)$, is the CTCT of the sequence $(-i, -1-2i, 10+8i, 3+40i, \ldots)$. We will now examine the Hankel matrix transform of the sequence $\mathcal{C}^{(i)}[J_{k,n}]$.

$(C^{(i)})^2[J_{k,1}]$	-1				
$(C^{(i)})^2[J_{k,2}]$	1	$-1-i$			
$(C^{(i)})^2[J_{k,3}]$	-1	$2+2i$	$-2-2i$		
$(C^{(i)})^2[J_{k,4}]$	1	$-3-3i$	$4+6i$	$-6i-5$	
$(C^{(i)})^2[J_{k,5}]$	-1	$4+4i$	$-6-12i$	$13+20i$	$-14-15i$

Table 2: Triangle form in second iteration for CTCT of the k-Jacobsthal sequence

4. Hankel Transform of CTCT of k-Jacobsthal Sequence

Let the Hankel determinant of the CTCT up to the n^{th} term of the k-Jacobsthal sequence $(J_{k,n})_{n\in N}$ be represented by $\mathcal{H}C^{(i)}[J_{k,n}]$. Then we get the following determinant values:

$$\mathcal{H}C^{(i)}[J_{k,0}] = \det(0) = 0,$$

$$\mathcal{H}C^{(i)}[J_{k,1}] = \begin{vmatrix} 0 & i \\ i & i-k \end{vmatrix} = 1,$$

$$\mathcal{H}C^{(i)}[J_{k,2}] = \begin{vmatrix} 0 & i & i-k \\ i & i-k & -2k-ik^2 \\ i-k & -2k-ik^2 & -i-k-3ik^2+k^3 \end{vmatrix} = 0,$$

$$\mathcal{H}C^{(i)}[J_{k,3}] = 1,$$

$$\mathcal{H}C^{(i)}[J_{k,4}] = 0.$$

We also establish bounds for the norms of the circulant matrices derived from the k-Jacobsthal sequence.

Theorem 3. *If $A = [a_{i,j}]$ is an $n \times n$ matrix such that $a_{i,j} = J_{k,n(mod(j-i,n))}$, then*

$$\sqrt{\frac{J_{k,n} \cdot J_{k,n-1}}{k}} \leq \|A\|_2 \leq \frac{J_{k,n} \cdot J_{k,n-1}}{k} - 1,$$

where $n \geq 2$, $\|.\|_2$ is the spectral norm, and $k = 1$.

Proof. By our definition of A, it is of the form

$$A = \begin{bmatrix} J_{k,0} & J_{k,1} & J_{k,2} & \cdots & J_{k,n-1} \\ J_{k,n-1} & J_{k,0} & J_{k,1} & \cdots & J_{k,n-2} \\ J_{k,n-2} & J_{k,n-1} & J_{k,0} & \cdots & J_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{k,1} & J_{k,2} & J_{k,3} & \cdots & J_{k,0} \end{bmatrix}.$$

Clearly, we have

$$\|A\|_F^2 = n\sum_{s=0}^{n-1} J_{k,s}^2.$$

Let

$$D = (d_{i,j}) = \begin{cases} d_{i,j} = J_{k,n(mod(j-i,n))}, & i \geq j \\ d_{i,j} = 1, & i < j \end{cases},$$

and

$$E = (e_{i,j}) = \begin{cases} e_{i,j} = J_{k,n(mod(j-i,n))}, & i \leq j \\ e_{i,j} = 1, & i > j \end{cases},$$

provided $A = D \circ E$. Then

$$r_1(D) = \max_i \sqrt{\sum_j |d_{i,j}|^2} = \sqrt{\sum_{t=0}^{n-1} J_{k,t}^2} = \sqrt{1+k^2},$$

and

$$C_1(E) = \max_j \sqrt{\sum_i |e_{i,j}|^2} = \sqrt{\sum_{t=0}^{n-1} J_{k,t}^2} = \sqrt{1+k^2}.$$

By taking the Hadamard product, we get

$$\sqrt{\frac{J_{k,n} \cdot J_{k,n-1}}{k}} \leq \|A\|_2 \leq 1 + k^2 = \frac{J_{k,n} \cdot J_{k,n-1}}{k} - 1.$$

□

Corollary 1. *If $A = [a_{i,j}]$ is an $n \times n$ matrix such that $a_{i,j} = J_{k,n(mod(j+i,n))}$, then*

$$\sqrt{\frac{J_{k,n} \cdot J_{k,n-1}}{k}} \leq \|A\|_2 \leq \frac{J_{k,n} \cdot J_{k,n-1}}{k} - 1$$

where $n \geq 2$, $\|.\|_2$ is the spectral norm, and $k = 1$.

5. The CTCT of the Fourth-Order Jacobsthal Sequence J_n^F and its Hankel Transform

The first few terms of the CTCT of the fourth-order Jacobsthal sequence are

$$\begin{aligned}
(C^{(i)})[J_1^F] &= i,\\
(C^{(i)})[J_2^F] &= i-1,\\
(C^{(i)})[J_3^F] &= -2,\\
(C^{(i)})[J_4^F] &= -i-1,\\
(C^{(i)})[J_5^F] &= 5i+2,\\
(C^{(i)})[J_6^F] &= 31i-3.
\end{aligned}$$

The first few terms of the second iteration of the CTCT of the fourth-order Jacobsthal sequence J_n^F, i.e., $(C^{(i)})^2[J_k^F]$, are given by

$$\begin{aligned}
(C^{(i)})^2[J_1^F] &= 1,\\
(C^{(i)})^2[J_2^F] &= -i,\\
(C^{(i)})^2[J_3^F] &= 0,\\
(C^{(i)})^2[J_4^F] &= -1,\\
(C^{(i)})^2[J_5^F] &= -2i-9,\\
(C^{(i)})^2[J_6^F] &= -21i-36.
\end{aligned}$$

Let the Hankel determinant of the CTCT up to the n^{th} term of the fourth-order Jacobsthal sequence $(J_n^F)_{n\in N}$ be represented by $\mathcal{H}C^{(i)}[J_n^F]$. Then we get the following determinant values:

$$\begin{aligned}
\mathcal{H}C^{(i)}[J_1^F] &= \det(i) = 1,\\
\mathcal{H}C^{(i)}[J_2^F] &= \begin{vmatrix} i & i-1 \\ i-1 & -2 \end{vmatrix} = 0,\\
\mathcal{H}C^{(i)}[J_3^F] &= \begin{vmatrix} i & i-1 & -2 \\ i-1 & -2 & -i-1 \\ -2 & -i-1 & 5i+2 \end{vmatrix} = -18-15i.
\end{aligned}$$

We also establish bounds for the norms of the circulant matrices derived from the fourth-order Jacobsthal sequence.

Theorem 4. *If $A = [a_{i,j}]$ is an $n \times n$ matrix such that $a_{i,j} = J^F_{k(mod(j-i,n))}$, then*

$$\sqrt{\frac{J^F_k \cdot J^F_{k-1}}{k}} \leq \|A\|_2 \leq \frac{J^F_k \cdot J^F_{k-1}}{k}, \quad \text{for all } k \geq 2.$$

where $\|.\|_2$ is the spectral norm.

Corollary 2. *If $A = [a_{i,j}]$ is an $n \times n$ matrix such that $a_{i,j} = J^F_{k(mod(j+i,n))}$, then*

$$\sqrt{\frac{J^F_k \cdot J^F_{k-1}}{k}} \leq \|A\|_2 \leq \frac{J^F_k \cdot J^F_{k-1}}{k}, \quad \text{for all } k \geq 2.$$

where $\|.\|_2$ is the spectral norm.

To compare the growth rates of the sequences as n increases, one can analyze the ratio of consecutive terms. The recurrence relation is given by

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \text{ for all } n \geq 2,$$

with initial conditions $J_{k,0} = 0$ and $J_{k,1} = 1$. For the fourth-order Jacobsthal sequence, this involves examining ratios such as $J_{k,n+1}/J_{k,n}$ and J^F_n/J^F_{n+1}. In computational contexts, such as algorithm performance, sequences with slower growth rates may offer advantages.

6. Applications

The k-Jacobsthal sequence, when incorporated into the Q-matrix framework of size $p \times p$, provides a robust foundation for cryptographic systems. Embedding variable shifts governed by a non-linear progression significantly reduces the predictability of ciphertext patterns. Constructing the Q-matrix of size $p \times p$ with entries from the k-Jacobsthal sequence

$$\mathcal{Q}^s_p = \begin{pmatrix} J_{k,s+p-1} & J_{k,s+p-2} + \cdots + J_{k,s} & \cdots & J_{k,s+p-2} \\ J_{k,s+p-2} & J_{k,s+p-3} + \cdots + J_{k,s-1} & \cdots & J_{k,s+p-3} \\ \vdots & \vdots & \ddots & \vdots \\ J_{k,s+1} & J_{k,s} + \cdots + J_{k,s-(p-2)} & \cdots & J_{k,s} \\ J_{k,s} & J_{k,s-1} + \cdots + J_{k,s-(p-1)} & \cdots & J_{k,s-1} \end{pmatrix},$$

strengthens encryption significantly. This makes it harder for attackers to predict shifts or deduce patterns from the ciphertext and provides enhanced cryptographic strength compared to conventional approaches, such as those using Fibonacci sequences [36]. Similarly, the CTCT of the k-Jacobsthal sequence in the Q-matrix encryption and decryption process demonstrates superior performance compared to the Fibonacci sequence.

7. Conclusion

We investigated the complex-type Catalan transform of k-Jacobsthal sequences and fourth-order Jacobsthal sequences, deriving their generating functions and norms. Furthermore, we defined and analyzed their Hankel transforms, revealing important properties that contribute to the understanding of these sequences. Our results provide new insights into their algebraic and combinatorial structures, with potential applications in number theory, cryptography, and signal processing.

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