ON CERTAIN QUATERNARY QUADRATIC FORMS

K. R. Vasuki
Dept. of Studies in Mathematics, University of Mysore, Mysuru, Karnataka, India
vasuki_kr@hotmail.com

P. Nagendra
Dept. of Studies in Mathematics, University of Mysore, Mysuru, Karnataka, India
nagp149@gmail.com

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Abstract

Alaca and Altiary established an explicit formula for the number of representations of a positive integer by the forms $x^2+y^2+10z^2+10w^2$ and $2x^2+2y^2+5z^2+5w^2$. Also, Alaca established an explicit formula for the number of representations of a positive integer by the forms $3x^2+3y^2+5z^2+5w^2$, $x^2+y^2+15z^2+15w^2$ and $x^2+3y^2+5z^2+15w^2$. The theory of modular forms was used to obtain these formulas. This article aims to give an elementary proof of these five formulas which is completely free from the theory of modular forms.

1. Introduction

Fermat made the statement that "Every integer is a square number or a sum of two or three or four square numbers". This statement of Fermat motivated Euler, Gauss, Lagrange, Lorentz, Dirichlet, Jacobi, Ramanujan, Mordell, Hardy, Siegel, and many other mathematicians to work on the representation of integers as sums of squares. Jacobi was the first mathematician who gave the following explicit formula to find the number of representations of integers as a sum of two squares and four squares respectively:

$$r_2(n) = 4[d_{1,4}(n) - d_{3,4}(n)]$$

and

$$r_4(n) = 8\sum_{\substack{d|n \\ 4\nmid d}} d \ ,$$

where $r_k(n)$ denotes the number of representations of a positive integer as a sum of k square numbers and $d_{i,4}(n)$ denotes the number of positive divisors of n which

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are congruent to i modulo 4. Simple proofs of Jacobi's results can be found in [8] and [9]. Let $n, c_1, c_2, c_3, c_4 \in \mathbb{N}$. Then $N(c_1, c_2, c_3, c_4; n)$ and $\sigma(n)$ are defined by

$$\begin{aligned} &N(c_1, c_2, c_3, c_4; n) \\ &\qquad = card \left\{(x_1, x_2, x_3, x_4)/n = c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_4^2, x_i \in \mathbb{Z}, 1 \le i \le 4\right\}, \end{aligned}$$

and

$$\sigma(n) = \begin{cases} \text{sum of divisors of } n & \text{if } n \in \mathbb{Z}^+ \\ 0 & \text{otherwise.} \end{cases}$$

Recently, Alaca and Altiary [2] established the following explicit formulas for the number of representations of an integer by the quaternary quadratic forms $x^2 + y^2 + 10z^2 + 10w^2$ and $2x^2 + 2y^2 + 5z^2 + 5w^2$.

Theorem 1.1 (Alaca and Altiary [2])**.** *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} (i)\ N(1, 1, 10, 10; n) =& \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) + \frac{4}{3}\sigma(n/4) + \frac{10}{3}\sigma(n/5) - \frac{16}{3}\sigma(n/8) \\ & - \frac{10}{3}\sigma(n/10) + \frac{20}{3}\sigma(n/20) - \frac{80}{3}\sigma(n/40) + \frac{10}{3}b_1(n) \\ & + \frac{8}{3}b_2(n) + 4b_3(n) \end{aligned}$$

and

$$\begin{aligned} (ii)\ N(2, 2, 5, 5; n) =& \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) + \frac{4}{3}\sigma(n/4) + \frac{10}{3}\sigma(n/5) - \frac{16}{3}\sigma(n/8) \\ & - \frac{10}{3}\sigma(n/10) + \frac{20}{3}\sigma(n/20) - \frac{80}{3}\sigma(n/40) - \frac{2}{3}b_1(n) \\ & + \frac{8}{3}b_2(n) - 4b_3(n), \end{aligned}$$

where

$$\begin{aligned} \eta_k &:= \eta(k\tau) = q^{\frac{k}{24}} \prod_{n=1}^{\infty}(1 - q^{kn}), \quad |q| < 1, \\ \eta_2^2\eta_{10}^2 &= \sum_{n=1}^{\infty} b_1(n)q^n, \\ \eta_4^2\eta_{20}^2 &= \sum_{n=1}^{\infty} b_2(n)q^n, \end{aligned}$$

and

$$\frac{\eta_4^5\eta_{10}\eta_{40}^2}{\eta_2\eta_8^2\eta_{20}} = \sum_{n=1}^{\infty} b_3(n)q^n.$$

They employed the theory of modular forms to prove the formulas of Theorem 1.1. Also, Alaca [1] established the following explicit formulas for the number of representations of an integer by the quaternary quadratic forms $3x^2+3y^2+5z^2+5w^2$, $x^2+y^2+15z^2+15w^2$, and $x^2+3y^2+5z^2+15w^2$.

Theorem 1.2 (Alaca [1])**.** *Let* $n \in \mathbb{N}$. *Then*

$$\begin{aligned}(i)\ N(1,1,15,15;n) = &\frac{2}{3}\sigma(n) - \frac{4}{3}\sigma(n/2) + \frac{8}{3}\sigma(n/4) - 2\sigma(n/3)\\ &+ 4\sigma(n/6) - 8\sigma(n/12) + \frac{10}{3}\sigma(n/5) - \frac{20}{3}\sigma(n/10)\\ &+ \frac{40}{3}\sigma(n/20) - 10\sigma(n/15) + 20\sigma(n/30) - 40\sigma(n/60)\\ &+ \frac{2}{3}\left(a_1(n) - 2a_2(n) + 4a_3(n) + 4a_4(n) + 8a_5(n)\right),\end{aligned}$$

$$\begin{aligned}(ii)\ N(3,3,5,5;n) = &\frac{2}{3}\sigma(n) - \frac{4}{3}\sigma(n/2) + \frac{8}{3}\sigma(n/4) - 2\sigma(n/3)\\ &+ 4\sigma(n/6) - 8\sigma(n/12) + \frac{10}{3}\sigma(n/5) - \frac{20}{3}\sigma(n/10)\\ &+ \frac{40}{3}\sigma(n/20) - 10\sigma(n/15) + 20\sigma(n/30) - 40\sigma(n/60)\\ &+ \frac{2}{3}\left(-5a_1(n) - 14a_2(n) - 20a_3(n) + 4a_4(n) + 8a_5(n)\right),\end{aligned}$$

and

$$\begin{aligned}(iii)\ N(1,3,5,15;n) = &\frac{1}{2}(\sigma(n) - 2\sigma(n/2) + 4\sigma(n/4) + 3\sigma(n/3)\\ &- 6\sigma(n/6) + 12\sigma(n/12) - 5\sigma(n/5) + 10\sigma(n/10)\\ &- 20\sigma(n/20) - 15\sigma(n/15) + 30\sigma(n/30) - 60\sigma(n/60))\\ &+ \frac{3}{2}a_1(n) + a_2(n) + 6a_3(n),\end{aligned}$$

where

$$\eta_1\eta_3\eta_5\eta_{15} = \sum_{n=1}^{\infty} a_1(n)q^n,$$

$$\eta_2\eta_6\eta_{10}\eta_{30} = \sum_{n=1}^{\infty} a_2(n)q^n,$$

$$\eta_4\eta_{12}\eta_{20}\eta_{60} = \sum_{n=1}^{\infty} a_3(n)q^n,$$

$$\eta_3\eta_5\eta_6\eta_{10} = \sum_{n=1}^{\infty} a_4(n)q^n,$$

and

$$\eta_6\eta_{10}\eta_{12}\eta_{20} = \sum_{n=1}^{\infty} a_5(n)q^n.$$

She employed the theory of modular forms to prove Theorem 1.2. The aim of this article is to prove Theorem 1.1 and Theorem 1.2 by using classical techniques which are completely free from the theory of modular forms. We organize this article in four sections. In the next section, we recall the necessary preliminary results that are required to prove our main results. In Section 3, we prove Theorem 1.1, and in Section 4, we prove Theorem 1.2.

2. Preliminaries

The *complete elliptic integral of the first kind* is denoted by $K(k)$ and is defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \qquad |k| < 1.$$

We call k the *modulus* and $k' = \sqrt{1-k^2}$ the *complementary modulus* of $K(k)$. Set $\alpha = k^2$, $\beta = l_1^2$, $\gamma = l_2^2$, and $\delta = l_3^2$. Let $k' = \sqrt{1-k^2}$, $l_1' = \sqrt{1-l_1^2}$, $l_2' = \sqrt{1-l_2^2}$, and $l_3' = \sqrt{1-l_3^2}$. Suppose that the equality

$$n\frac{K(k')}{K(k)} = \frac{K(l_1')}{K(l_1)} \tag{2.1}$$

holds for some positive integer n. The relation between α and β induced by the Equation (2.1) is called a modular equation of degree n. The multiplier connecting α and β, denoted by m is defined as

$$m = \frac{z_1}{z_n} = \frac{K(\sqrt{\alpha})}{K(\sqrt{\beta})}.$$

and we also say that β has degree n over α. Also, suppose that the equalities

$$n\ \frac{K(k')}{K(k)} = \frac{K(l_1')}{K(l_l)},\ n_1\ \frac{K(k')}{K(k)} = \frac{K(l_2')}{K(l_2)},\ \text{and } nn_1\ \frac{K(k')}{K(k)} = \frac{K(l_3')}{K(l_3)},$$

hold for positive integers n and n_1. Then, the relation induced among α, β, γ and δ by the above is called a modular equation of composite degree nn_1. In Chapter 16 of his second notebook [12, p. 197], Ramanujan defined the following theta functions.

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{2.2}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.3}$$

$$f(-q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}=(q;q)_\infty,$$

and

$$\chi(q)=(-q;q^2)_\infty,$$

where $(a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n)$, with $|q|<1$.

For convenience, we set

$$f_n:=f(-q^n)=(q^n;q^n)_\infty, \tag{2.4}$$

where n is any positive integer. It is easy to see that

$$\begin{aligned}&\varphi(q)=\frac{f_2^5}{f_1^2f_4^2},\ \varphi(-q)=\frac{f_1^2}{f_2},\ \psi(q)=\frac{f_2^2}{f_1},\ \psi(-q)=\frac{f_1f_4}{f_2},\\ &\chi(-q)=\frac{f_1}{f_2},\quad\text{and}\quad\chi(q)=\frac{f_2^2}{f_1f_4}.\end{aligned} \tag{2.5}$$

The *Eisenstein series* P_n of weight-2 is defined by

$$P_n=P(q^n):=1-24\sum_{k=1}^{\infty}\frac{kq^{nk}}{1-q^{nk}}=1-24\sum_{k=1}^{\infty}\sigma(k)q^k. \tag{2.6}$$

Setting $n=1$ and then changing q to $-q$ in Equation (2.6), one can easily see that

$$P(-q)=-P_1+6P_2-4P_4. \tag{2.7}$$

On pages 230 and 247 of his second notebook [12], Ramanujan recorded the following modular equations of degree 3 and 15, respectively,

$$\{\alpha\beta\}^{\frac14}+\{(1-\alpha)(1-\beta)\}^{\frac14}=1 \tag{2.8}$$

and

$$\begin{aligned}&\{\alpha\beta\gamma\delta\}^{\frac18}+\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac18}\\ &\qquad+2^{\frac13}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}=1.\end{aligned} \tag{2.9}$$

Many years before Ramanujan, the modular equation (2.8) was discovered by Legendre [11]. For proofs of the Equations (2.8) and (2.9), one may refer to [5, pp. 232, 394]. On page 245 of his second notebook Ramanujan recorded certain theta function identities, two among them being

$$\varphi(q)\varphi(q^{15})-\varphi(q^3)\varphi(q^5)=2qf_2f_{30}\chi(q^3)\chi(q^5) \tag{2.10}$$

and

$$\varphi(q)\varphi(q^{15})+\varphi(q^3)\varphi(q^5)=2f_6f_{10}\chi(q)\chi(q^{15}). \tag{2.11}$$

Proofs of the identities (2.10) and (2.11) can be found in [5, pp. 395-396] and [4].

On page 211 of his second notebook [12], Ramanujan has recorded the following Eisenstein series identities.

$$P_1 - 2P_2 = -16q\psi^4(q^2) - \varphi^4(q) \tag{2.12}$$

and

$$P_1 - 4P_4 = -3\varphi^4(q). \tag{2.13}$$

For proofs of the Eisenstein series identities (2.12) and (2.13), one may refer to [5, pp. 127-128] and [13]. From the Eisenstein series identities (2.12) and (2.13), we obtain

$$-P_1 + 3P_2 - 2P_4 = 24q\psi^4(q^2). \tag{2.14}$$

Changing q to $-q$ in Equation (2.13) and then using Equation (2.7), we find that

$$-P_1 + 6P_2 - 8P_4 = -3\varphi^4(-q). \tag{2.15}$$

From Entries 3(iii) and 3(iv) in Chapter 17 of Ramanujan's second notebook [12], we have

$$-P_1 + P_2 + 3P_3 - 3P_6 = 24\psi^2(q)\psi^2(q^3) \tag{2.16}$$

and

$$P_1 - 4P_2 - 3P_3 + 12P_6 = 6\varphi^2(-q)\varphi^2(-q^3). \tag{2.17}$$

Proofs of the identities (2.16) and (2.17) can be found in [5, pp. 223-226] and [14]. Cooper and Ye [7] established the following Eisenstein series identity using the theory of modular forms:

$$P_1 + 3P_3 - 5P_5 - 15P_{15} = -16\frac{f_3^4 f_5^4}{f_1^2 f_{15}^2} + 16\frac{f_1^4 f_{15}^4}{f_3^2 f_5^2} + 8qf_1f_3f_5f_{15}. \tag{2.18}$$

An elementary proof of Equation (2.18) which is completely free from the theory of modular forms is found in [6]. In Chapter 16 of his second notebook [12, p. 198, Entry 24], Ramanujan recorded several theta function identities which have been proved by Berndt [5, pp. 40-41]. Two among them are

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4) \tag{2.19}$$

and

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2). \tag{2.20}$$

From Equations (2.19) and (2.20), we obtain

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{2.21}$$

Kang [10] has proved the following three identities using theta function identities, the first two of which are due to Ramanujan [12, p. 234].

$$\varphi^2(q) - \varphi^2(q^5) = 4q\chi(q)\chi(q^5)\psi^2(-q^5), \tag{2.22}$$

$$\varphi^2(q) - 5\varphi^2(q^5) = -4f_2^2\frac{\chi(q^5)}{\chi(q)}, \tag{2.23}$$

and

$$\psi^2(q) - 5q\psi^2(q^5) = f_1^2\frac{\chi(-q)}{\chi(-q^5)}. \tag{2.24}$$

We make use of the following theorem due to Berndt [5, pp. 385-389] for our proof.

Theorem 2.1 (Berndt [5])**.** *Let* $A = (\alpha\delta)^{\frac{1}{8}}$*,* $A' = \{(1-\alpha)(1-\delta)\}^{\frac{1}{8}}$*,* $B = (\beta\gamma)^{\frac{1}{8}}$*,* $B' = \{(1-\beta)(1-\delta)\}^{\frac{1}{8}}$*, and* $t = \left(\frac{z_3z_5}{z_1z_{15}}\right)^{\frac{1}{2}}$*, where* β*,* γ*,* δ *have degree* $3, 5, 15,$ *respectively, over* α*. Then*

$$\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{\frac{1}{24}} = 2^{-\frac{2}{3}}\frac{(1+t)}{t},$$

$$\left(\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{\frac{1}{24}} = 2^{-\frac{2}{3}}(1-t),$$

$$\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}} = 2^{-\frac{4}{3}}\frac{(1-t^2)}{t},$$

$$BB' = \frac{1}{4}\frac{(1+t)(1-t^2)}{t^2},$$

$$AA' = \frac{(1-t)(1-t^2)}{4t},$$

$$\frac{B'^4}{A'^2} + \frac{B^4}{A^2} = \frac{t^3+5t^2+5t-1}{2t^3},$$

$$\frac{A'^4}{B'^2} + \frac{A^4}{B^2} = \frac{t^3+5t^2-5t+1}{2},$$

and

$$\frac{z_3}{z_{15}} + 5\frac{z_5}{z_1} = \frac{(1+t^2)(t^2+3t-1)}{t}.$$

The following theorem plays an important role in transforming a theta function into a modular equation and vice versa.

Theorem 2.2 (Ramanujan [5, 12])**.** *If* $|x| < 1$*,* $K = \int\limits_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-x\sin^2\phi}}$*,* $K' = \int\limits_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-(1-x)\sin^2\phi}}$*,* $q = e^{\frac{-\pi K'}{K}}$*, and* $z = \frac{2}{\pi}K$*, then*

$$\varphi(q) = \sqrt{z},$$
$$\varphi(-q) = \sqrt{z}(1-x)^{\frac{1}{4}},$$
$$\varphi(-q^2) = \sqrt{z}(1-x)^{\frac{1}{8}},$$
$$\psi(q) = \sqrt{\tfrac{1}{2}z}(xq)^{\frac{1}{8}},$$
$$\psi(-q) = \sqrt{\tfrac{1}{2}z}\left\{x(1-x)q\right\}^{\frac{1}{8}},$$
$$\psi(q^2) = \tfrac{1}{2}\sqrt{z}(xq)^{\frac{1}{4}},$$
$$f(q) = \sqrt{z}2^{-\frac{1}{6}}\left\{x(1-x)q\right\}^{\frac{1}{24}},$$
$$f(-q) = \sqrt{z}2^{-\frac{1}{6}}(1-x)^{\frac{1}{6}}(xq)^{\frac{1}{24}},$$
$$f(-q^2) = \sqrt{z}2^{-\frac{1}{3}}\left\{x(1-x)q\right\}^{\frac{1}{12}},$$
$$f(-q^4) = \sqrt{z}4^{-\frac{1}{3}}\left\{(1-x)\right\}^{\frac{1}{24}}(xq)^{\frac{1}{6}},$$
$$\chi(q) = 2^{\frac{1}{6}}\left\{x(1-x)q\right\}^{-\frac{1}{24}},$$
$$\chi(-q) = 2^{\frac{1}{6}}\left\{(1-x)\right\}^{\frac{1}{12}}(xq)^{-\frac{1}{24}},$$

and

$$\chi(-q^2) = 2^{\frac{1}{3}}\left\{(1-x)\right\}^{\frac{1}{24}}(xq)^{-\frac{1}{12}}.$$

3. Proof of Theorem 1.1

In this section, we first prove the following three lemmas, which play a key role in proving Theorem 1.1.

Lemma 3.1. *We have*

$$\varphi^2(q)\varphi^2(q^{10}) + 5\varphi^2(q^2)\varphi^2(q^5) = \varphi^4(q^2) + 5\varphi^4(q^{10}) + 4q\psi^4(q^2) + 20q^5\psi^4(q^{10}) + 16q^2 f_4^2 f_{20}^2 - 16q^2\frac{f_4^5 f_{10} f_{40}^2}{f_8^2 f_2 f_{20}}. \quad (3.1)$$

Proof. Multiplying both sides of Equation (2.21) by $\varphi^2(q^{10})$, we obtain

$$\varphi^2(q)\varphi^2(q^{10}) = \varphi^2(q^2)\varphi^2(q^{10}) + 4q\psi^2(q^4)\varphi^2(q^{10}). \quad (3.2)$$

Changing q to q^5 in Equation (2.21) and then multiplying throughout by $5\varphi^2(q^2)$, we obtain

$$5\varphi^2(q^2)\varphi^2(q^5) = 5\varphi^2(q^2)\varphi^2(q^{10}) + 20q^5\varphi^2(q^2)\psi^2(q^{20}). \quad (3.3)$$

Adding Equations (3.2) and (3.3), we obtain

$$\varphi^2(q)\varphi^2(q^{10}) + 5\varphi^2(q^2)\varphi^2(q^5) = 6\varphi^2(q^2)\varphi^2(q^{10}) + 4q\psi^2(q^4)\varphi^2(q^{10}) + 20q^5\varphi^2(q^2)\psi^2(q^{20}). \tag{3.4}$$

Changing q to q^2 in Equation (2.22) and q to q^4 in Equation (2.24) and then multiplying the resulting identities, we find that

$$q\psi^2(q^4)\varphi^2(q^{10}) + 5q^5\varphi^2(q^2)\psi^2(q^{20}) = q\psi^4(q^2) + 5q^5\psi^4(q^{10}) - 4q^2\frac{f_4^5 f_{10} f_{40}^2}{f_8^2 f_2 f_{20}}. \tag{3.5}$$

Employing Equation (3.5) in the right side of Equation (3.4), we obtain

$$\varphi^2(q)\varphi^2(q^{10}) + 5\varphi^2(q^2)\varphi^2(q^5) = 6\varphi^2(q^2)\varphi^2(q^{10}) + 4q\psi^4(q^2) + 20q^5\psi^4(q^{10}) - 16q^2\frac{f_4^5 f_{10} f_{40}^2}{f_8^2 f_2 f_{20}}. \tag{3.6}$$

Multiplying Equation (2.22) with Equation (2.23), and then rearranging the terms in the resulting identity, we obtain

$$6\varphi^2(q)\varphi^2(q^5) = \varphi^4(q) + 5\varphi^4(q^5) + 16qf_2^2 f_{10}^2. \tag{3.7}$$

The identity (3.7) can also found in [3]. Replacing q by q^2 in Equation (3.7), and then using the resulting equation in the right side of Equation (3.6), we obtain the required result. □

Lemma 3.2. *We have*

$$12\varphi^2(q)\varphi^2(q^{10}) + 12\varphi^2(q^2)\varphi^2(q^5) = \varphi^4(q) + 5\varphi^4(q^5) + 4\varphi^4(q^2) + 20\varphi^4(q^{10}) - \varphi^4(-q) - 5\varphi^4(-q^5) + 32qf_2^2 f_{10}^2 + 64q^2 f_4^2 f_{20}^2. \tag{3.8}$$

Proof. From Equation (2.20), it follows that

$$\varphi^2(-q^5) = 2\varphi^2(q^{10}) - \varphi^2(q^5) \tag{3.9}$$

and

$$\varphi^2(-q) = 2\varphi^2(q^2) - \varphi^2(q). \tag{3.10}$$

Multiplying Equations (3.9) and (3.10), we obtain

$$\varphi^2(-q)\varphi^2(-q^5) = 4\varphi^2(q^2)\varphi^2(q^{10}) + \varphi^2(q)\varphi^2(q^5) - 2\varphi^2(q)\varphi^2(q^{10}) - 2\varphi^2(q^2)\varphi^2(q^5). \tag{3.11}$$

Multiplying the Equation (3.11) throughout by six and rearranging the terms in the resulting identity, we obtain

$$12\varphi^2(q)\varphi^2(q^{10})+12\varphi^2(q^2)\varphi^2(q^5)=24\varphi^2(q^2)\varphi^2(q^{10})+6\varphi^2(q)\varphi^2(q^5)$$
$$-6\varphi^2(-q)\varphi^2(-q^5). \quad (3.12)$$

Changing q to $-q$ and q^2 in Equation (3.7), and then using the resulting identities and Equation (3.7) in the right side of Equation (3.12), we obtain Equation (3.8). □

Lemma 3.3. *We have*

$$\varphi^2(q)\varphi^2(q^{10})=\frac{-1}{36}\Big[P_1-P_2+2P_4+5P_5-8P_8-5P_{10}+10P_{20}-40P_{40}\Big]$$
$$+\frac{10}{3}qf_2^2f_{10}^2+\frac{8}{3}q^2f_4^2f_{20}^2+4q^2\frac{f_4^5f_{10}f_{40}^2}{f_8^2f_2f_{20}} \quad (3.13)$$

and

$$\varphi^2(q^2)\varphi^2(q^5)=\frac{-1}{36}\Big[P_1-P_2+2P_4+5P_5-8P_8-5P_{10}+10P_{20}-40P_{40}\Big]$$
$$-\frac{2}{3}qf_2^2f_{10}^2+\frac{8}{3}q^2f_4^2f_{20}^2-4q^2\frac{f_4^5f_{10}f_{40}^2}{f_8^2f_2f_{20}}. \quad (3.14)$$

Proof. Multiplying Equation (3.1) throughout by twelve, and then subtracting the resulting identity from Equation (3.8), we obtain

$$\varphi^2(q^2)\varphi^2(q^5)=\frac{1}{48}\big(\varphi^4(-q)+5\varphi^4(-q^5)-\varphi^4(q)-5\varphi^4(q^5)+8\varphi^4(q^2)+40\varphi^4(q^{10})$$
$$+48q\psi^4(q^2)+240q^5\psi^4(q^{10})\big)-\frac{2}{3}qf_2^2f_{10}^2+\frac{8}{3}q^2f_4^2f_{20}^2-4q^2\frac{f_4^5f_{10}f_{40}^2}{f_8^2f_2f_{20}}.$$
$$(3.15)$$

Employing Equations (2.13), (2.14), and (2.15) in the left side of Equation (3.15), we obtain Equation (3.14). Using Equation (3.15) in Equation (3.1), we obtain

$$\varphi^2(q)\varphi^2(q^{10})=\frac{1}{48}\big(5\varphi^4(q)+25\varphi^4(q^5)-5\varphi^4(-q)-25\varphi^4(-q^5)+8\varphi^4(q^2)$$
$$+40\varphi^4(q^{10})-48q\psi^4(q^2)-240q^5\psi^4(q^{10})\big)+\frac{10}{3}qf_2^2f_{10}^2$$
$$+\frac{8}{3}q^2f_4^2f_{20}^2+4q^2\frac{f_4^5f_{10}f_{40}^2}{f_8^2f_2f_{20}}. \quad (3.16)$$

Employing Equations (2.13), (2.14), and (2.15) in the left side of Equation (3.16), we obtain Equation (3.13). □

Proof of Theorem 1.1 (i). Equating the coefficients of q^n in Equation (3.13), we obtain the required result. □

Proof of Theorem 1.1 (ii). Equating the coefficients of q^n in Equation (3.14), we obtain the required result. □

4. Proof of Theorem 1.2

In this section, we first prove the following six lemmas, which play an important role in proving Theorem 1.2.

Lemma 4.1. *We have*

$$\varphi^2(q)\varphi^2(q^{15}) - \frac{2}{3}F(q) = \sqrt{z_1 z_3 z_5 z_{15}}\frac{(1+t^2)(t^2+3t-1)}{6t^2}, \tag{4.1}$$

where

$$F(q) = qf_1f_3f_5f_{15} - 2q^2f_2f_6f_{10}f_{30} + 4q^4f_4f_{12}f_{20}f_{60} + 4qf_3f_5f_6f_{10} + 8q^2f_6f_{10}f_{12}f_{20}, \tag{4.2}$$

z_1*,* z_3*,* z_5*,* z_{15}*, and* t *are as in Theorem 2.1.*

Proof. Employing Theorem 2.2 in the left side of Equation (4.1) and after some simplification, we find that

$$\begin{aligned}
&\varphi^2(q)\varphi^2(q^{15}) - \frac{2}{3}F(q)\\
&= z_1z_{15} - \frac{2}{3}\Big[2^{-\frac{2}{3}}\sqrt{z_1z_3z_5z_{15}}\,\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}\\
&\quad\times\Big\{\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{8}} + \{\alpha\beta\gamma\delta\}^{\frac{1}{8}}\Big\}\\
&\quad -2\times 2^{\frac{-4}{3}}\sqrt{z_1z_3z_5z_{15}}\,\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{12}}\\
&\quad +2z_3z_5\,\{\beta\gamma(1-\beta)(1-\gamma)\}^{\frac{1}{8}}\Big\{\{(1-\beta)(1-\gamma)\}^{\frac{1}{8}} + \{\beta\gamma\}^{\frac{1}{8}}\Big\}\Big].
\end{aligned} \tag{4.3}$$

Using Equation (2.9) in the right side of Equation (4.3), we obtain

$$\begin{aligned}
&\varphi^2(q)\varphi^2(q^{15}) - \frac{2}{3}F(q)\\
&= z_1z_{15} - \frac{2}{3}\Big[2^{-\frac{2}{3}}\sqrt{z_1z_3z_5z_{15}}\,\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}\\
&\quad\times\Big\{1 - 2^{\frac{1}{3}}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}\Big\}\\
&\quad -2\times 2^{\frac{-4}{3}}\sqrt{z_1z_3z_5z_{15}}\,\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{12}}\\
&\quad +2z_3z_5\,\{\beta\gamma(1-\beta)(1-\gamma)\}^{\frac{1}{8}}\Big\{\{(1-\beta)(1-\gamma)\}^{\frac{1}{8}} + \{\beta\gamma\}^{\frac{1}{8}}\Big\}\Big].
\end{aligned} \tag{4.4}$$

Applying Theorem 2.1 to express the right side of Equation (4.4) in terms of the parameter t, we find that

$$\begin{aligned}\varphi^2(q)\varphi^2(q^{15}) - \frac{2}{3}F(q)\\ &= z_1z_{15} - \frac{2}{3}\sqrt{z_1z_3z_5z_{15}}\left[\frac{(1-t^2)}{4t}\left\{1-\frac{(1-t^2)}{2t}\right\} - \frac{(1-t^2)^2}{8t^2}\right]\\ &\quad - \frac{1}{3}z_3z_5\frac{(1+t)(1-t^2)}{t^3}.\end{aligned}$$

This identity reduces to Equation (4.1), completing the proof. □

Lemma 4.2. *We have*

$$\begin{aligned}\varphi^2(q)\varphi^2(q^{15}) = -\frac{1}{36}\Big[&P_1 - 2P_2 - 3P_3 + 4P_4 + 5P_5 + 6P_6 - 10P_{10}\\ &-12P_{12} - 15P_{15} + 20P_{20} + 30P_{30} - 60P_{60}\Big]\\ &+\frac{2}{3}\Big(qf_1f_3f_5f_{15} - 2q^2f_2f_6f_{10}f_{30} + 4q^4f_4f_{12}f_{20}f_{60}\\ &+4qf_3f_5f_6f_{10} + 8q^2f_6f_{10}f_{12}f_{20}\Big). \qquad (4.5)\end{aligned}$$

Proof. From Equations (2.16) and (2.17), we have

$$\begin{aligned}P_1 - 2P_2 - 3P_3 + 4P_4 + 5P_5 + 6P_6 - 10P_{10} - 12P_{12} - 15P_{15} + 20P_{20} + 30P_{30} - 60P_{60}\\ &= (P_1 - P_2 - 3P_3 + 3P_6) - (P_2 - 4P_4 - 3P_6 + 12P_{12})\\ &\quad +5(P_5 - P_{10} - 3P_{15} + 3P_{30}) - 5(P_{10} - 4P_{20} - 3P_{30} + 12P_{60})\\ &= -24q\psi^2(q)\psi^2(q^3) - 6\varphi^2(-q^2)\varphi^2(-q^6) - 5\times 24q^5\psi^2(q^5)\psi^2(q^{15})\\ &\quad -5\times 6\phi^2(-q^{10})\phi^2(-q^{30}). \qquad (4.6)\end{aligned}$$

Applying Theorem 2.2 in Equation (4.6), we obtain

$$\begin{aligned}P_1 - 2P_2 - 3P_3 + 4P_4 + 5P_5 + 6P_6 - 10P_{10} - 12P_{12} - 15P_{15} + 20P_{20} + 30P_{30} - 60P_{60}\\ = -6\Big[z_1z_3\{\alpha\beta\}^{\frac{1}{4}} + z_1z_3\{(1-\alpha)(1-\beta)\}^{\frac{1}{4}} + 5z_5z_{15}\{\gamma\delta\}^{\frac{1}{4}}\\ +5z_5z_{15}\{(1-\gamma)(1-\delta)\}^{\frac{1}{4}}\Big]. \qquad (4.7)\end{aligned}$$

Employing Equation (2.8) in Equation (4.7), we find that

$$\begin{aligned}P_1 - 2P_2 - 3P_3 + 4P_4 + 5P_5 + 6P_6 - 10P_{10} - 12P_{12} - 15P_{15} + 20P_{20} + 30P_{30} - 60P_{60}\\ = -6z_1z_{15}\left(\frac{z_3}{z_{15}} + 5\frac{z_5}{z_1}\right). \qquad (4.8)\end{aligned}$$

Using Theorem 2.1 in the right side of Equation (4.8), we obtain

$$P_1 - 2P_2 - 3P_3 + 4P_4 + 5P_5 + 6P_6 - 10P_{10} - 12P_{12} - 15P_{15} + 20P_{20} + 30P_{30} - 60P_{60} = -36\sqrt{z_1z_3z_5z_{15}}\frac{(1+t^2)(t^2+3t-1)}{6t^2}. \tag{4.9}$$

Employing Lemma 4.1 in Equation (4.9), we have

$$\begin{aligned}&P_1 - 2P_2 - 3P_3 + 4P_4 + 5P_5 + 6P_6 - 10P_{10} - 12P_{12} - 15P_{15} + 20P_{20} + 30P_{30} - 60P_{60}\\ &= -36\Big[\varphi^2(q)\varphi^2(q^{15}) - \frac{2}{3}(qf_1f_3f_5f_{15} - 2q^2f_2f_6f_{10}f_{30}\\ &\quad + 4q^4f_4f_{12}f_{20}f_{60} + 4qf_3f_5f_6f_{10} + 8q^2f_6f_{10}f_{12}f_{20})\Big].\end{aligned}$$

This completes the proof. □

Lemma 4.3. *We have*

$$qf(q)f(q^3)f(q^5)f(q^{15}) = qf_1f_3f_5f_{15} + 2q^2f_2f_6f_{10}f_{30} + 4q^4f_4f_{12}f_{20}f_{60}.$$

Proof. Upon multiplying by $\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}$ throughout Equation (2.9), we obtain

$$\begin{aligned}&\{\alpha\beta\gamma\delta\}^{\frac{1}{24}}\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{6}} + \{\alpha\beta\gamma\delta\}^{\frac{1}{6}}\\ &\times\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}} + 2^{\frac{1}{3}}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{12}}\\ &= \{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}.\end{aligned} \tag{4.10}$$

Transforming the modular equation (4.10) in terms of theta functions using Theorem 2.2, we find that

$$qf(q)f(q^3)f(q^5)f(q^{15}) = qf_1f_3f_5f_{15} + 2q^2f_2f_6f_{10}f_{30} + 4q^4f_4f_{12}f_{20}f_{60}.$$

This completes the proof. □

Lemma 4.4. *We have*

$$\begin{aligned}&\varphi^2(q^3)\varphi^2(q^5)\\ &= -\frac{1}{36}\Big[P_1 - 2P_2 - 3P_3 + 4P_4 + 5P_5 + 6P_6 - 10P_{10}\\ &\quad -12P_{12} - 15P_{15} + 20P_{20} + 30P_{30} - 60P_{60}\Big]\\ &\quad -\frac{2}{3}\Big(5qf_1f_3f_5f_{15} + 14q^2f_2f_6f_{10}f_{30} + 20q^4f_4f_{12}f_{20}f_{60}\\ &\quad -4qf_3f_5f_6f_{10} - 8q^2f_6f_{10}f_{12}f_{20}\Big).\end{aligned} \tag{4.11}$$

Proof. From Equations (2.10) and (2.11), we have

$$\varphi^2(q)\varphi^2(q^{15}) - \varphi^2(q^3)\varphi^2(q^5) = 4qf(q)f(q^3)f(q^5)f(q^{15}). \tag{4.12}$$

Employing Lemmas 4.2 and 4.3 in Equation (4.12), we obtain the required result. □

Lemma 4.5. *We have*

$$\varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) - \frac{3}{2}qf_1f_3f_5f_{15} - 2q^2f_2f_6f_{10}f_{30} - 6q^4f_4f_{12}f_{20}f_{60} = \frac{1}{8}\frac{(t^4+3t^3+6t^2-3t+1)}{t^2}. \tag{4.13}$$

Proof. Employing Theorem 2.2 in the left side of Equation (4.13) and after simplification, we find that

$$\begin{aligned}
&\varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) - \frac{3}{2}qf_1f_3f_5f_{15} - 2q^2f_2f_6f_{10}f_{30} - 6q^4f_4f_{12}f_{20}f_{60}\\
&= \sqrt{z_1z_3z_5z_{15}} - \frac{3}{2}\times 2^{-\frac{2}{3}}\sqrt{z_1z_3z_5z_{15}}\,\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}\\
&\quad\times\left(\{\alpha\beta\gamma\delta\}^{\frac{1}{8}} + \{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{8}}\right)\\
&\quad - 2^{-\frac{1}{3}}\sqrt{z_1z_3z_5z_{15}}\,\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{12}}.
\end{aligned} \tag{4.14}$$

Employing Equation (2.9) in the right side of Equation (4.14), we find that

$$\begin{aligned}
&\varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) - \frac{3}{2}qf_1f_3f_5f_{15} - 2q^2f_2f_6f_{10}f_{30} - 6q^4f_4f_{12}f_{20}f_{60}\\
&= \sqrt{z_1z_3z_5z_{15}} - \frac{3}{2}\times 2^{-\frac{2}{3}}\sqrt{z_1z_3z_5z_{15}}\,\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}\\
&\qquad\times\left(1 - 2^{\frac{1}{3}}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}\right)\\
&\qquad -2^{-\frac{1}{3}}\sqrt{z_1z_3z_5z_{15}}\,\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{12}}.
\end{aligned} \tag{4.15}$$

Applying Theorem 2.1 in the right side of Equation (4.15), we obtain

$$\begin{aligned}
&\varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) - \frac{3}{2}qf_1f_3f_5f_{15} - 2q^2f_2f_6f_{10}f_{30} - 6q^4f_4f_{12}f_{20}f_{60}\\
&= \frac{1}{8}\frac{(t^4+3t^3+6t^2-3t+1)}{t^2}.
\end{aligned}$$

This completes the proof. □

Lemma 4.6. *We have*

$$\begin{aligned}\varphi(q)\varphi(q^3)\varphi(q^5)\varphi(q^{15}) = -\frac{1}{48}\Big[&P_1 - 2P_2 + 3P_3 + 4P_4 - 5P_5 - 6P_6 + 10P_{10} \\ &+12P_{12} - 15P_{15} - 20P_{20} + 30P_{30} - 60P_{60}\Big] \\ &+\frac{3}{2}qf_1f_3f_5f_{15} + q^2f_2f_6f_{10}f_{30} + 6q^4f_4f_{12}f_{20}f_{60}. \qquad (4.16)\end{aligned}$$

Proof. From Equation (2.18), we have

$$\begin{aligned}&P_1 - 2P_2 + 3P_3 + 4P_4 - 5P_5 - 6P_6 + 10P_{10} + 12P_{12} - 15P_{15} - 20P_{20} \\ &+30P_{30} - 60P_{60} \\ &= -16\frac{f_3^4f_5^4}{f_1^2f_{15}^2} + 16\frac{f_1^4f_{15}^4}{f_3^2f_5^2} + 8qf_1f_3f_5f_{15} + 32\frac{f_6^4f_{10}^4}{f_2^2f_{30}^2} - 32\frac{f_2^4f_{30}^4}{f_6^2f_{10}^2} \\ &\quad -16q^2f_2f_6f_{10}f_{30} - 64\frac{f_{12}^4f_{20}^4}{f_4^2f_{60}^2} + 64\frac{f_4^4f_{60}^4}{f_{12}^2f_{20}^2} + 32q^4f_4f_{12}f_{20}f_{60}. \qquad (4.17)\end{aligned}$$

Transforming the right side of Equation (4.17) in terms of the variables α, β, γ, δ,z_1,z_3,z_5, and z_{15} using Theorem 2.2, then employing Equation (2.9), and finally transforming in terms of t using Theorem 2.1, we find that

$$\begin{aligned}P_1 + 3P_3 - 5P_5 - 15P_{15} - 2(P_2 + 3P_6 - 5P_{10} - 15P_{30}) + 4(P_4 + 3P_{12} - 5P_{20} - 15P_{60}) \\ = -6\sqrt{z_1z_3z_5z_{15}}\frac{(t^4 + 3t^3 + 6t^2 - 3t + 1)}{t^2}. \qquad (4.18)\end{aligned}$$

Employing Lemma 4.5 in the right side of Equation (4.18), we obtain the required result. □

Proof of Theorem 1.2 (i). Equating coefficients of q^n in Equation (4.5), we obtain the required result. □

Proof of Theorem 1.2 (ii). Equating the coefficients of q^n in Equation (4.11), we obtain the required result. □

Proof of Theorem 1.2 (iii). Equating the coefficient of q^n in Equation (4.16), we obtain the required result. □

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