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DISTRIBUTION OF ZECKENDORF EXPRESSIONS

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Abstract

By Zeckendorf's Theorem, every positive integer can be uniquely written as a sum of distinct non-adjacent Fibonacci terms. In this paper, we investigate the asymptotic formula of the number of binary expansions that are less than x and have no adjacent terms, and generalize the result to the setting of general linear recurrences with non-negative integer coefficients.

1. Introduction

Zeckendorf's Theorem [21] states that each positive integer can be expressed uniquely as a sum of distinct non-adjacent terms of the Fibonacci sequence $(1, 2, 3, 5, \dots)$ where we reset $(F_1, F_2) = (1, 2)$, and the expression is called *the Zeckendorf expansion of a positive integer*. Zeckendorf expansions share the simplicity of representation with the binary expansion, but also they are quite curious in terms of the arithmetic operations, determining the 0th digits, the partitions in Fibonacci terms, the minimal summand property of Zeckendorf expansions, and its converse; see [4, 5, 7, 11, 15, 9, 20].

Zeckendorf's Theorem consists of three components: the sequence, the condition on expressions, and the set of numbers represented by the sequence. By varying each component and shifting the focus to a particular component, we encounter many interesting questions. For example, we may vary the Fibonacci sequence slightly by changing its initial values, but maintaining the recurrence relation, and ask ourselves how many non-negative integers less than x are sums of distinct non-adjacent terms of the new sequence–this question is investigated in [3].

In this paper, we fix the condition on expressions, and change the sequence more than slightly. The first example we considered was the sequence $\{2^{k-1}\}_{k=1}^{\infty}$ and

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Zeckendorf's expressions for the Fibonacci sequence, i.e., we asked ourselves how many non-negative integers less than x are expressed as a sum of distinct non-adjacent powers of 2. For example, the binary expansion $165 = 1 + 2^2 + 2^5 + 2^7$ satisfies the non-adjacency condition while the binary expansions of 166 and 167 do not. For a generalization, we reformulate the task as follows.

Definition 1. Let $\mathbb{N}_0$ be the set of non-negative integers. The infinite tuples μ in $\prod_{k=1}^{\infty} \mathbb{N}_0$ are called *coefficient functions*, and let μ_k for $k \in \mathbb{N}$ denote the kth entry of μ, i.e., $\mu = (\mu_1, \mu_2, \dots)$. A set $\widetilde{\mathcal{E}}$ of coefficient functions μ is said to be *for positive integers* if only finitely many entries of μ are positive for each $\mu \in \widetilde{\mathcal{E}}$, and a set of coefficient functions is also called a *collection*. Given a collection $\widetilde{\mathcal{E}}$ (of coefficient functions) for positive integers, an increasing sequence $\{\widetilde{H}_k\}_{k=1}^{\infty}$ of positive integers is called *a fundamental sequence of* $\widetilde{\mathcal{E}}$ if for each $n \in \mathbb{N}$, there is unique $\mu \in \widetilde{\mathcal{E}}$ such that $n = \sum_{k=1}^{\infty} \mu_k \widetilde{H}_k$.

The following is immediate from [4, Theorem 16, , Lemma 3 & 37], which is also stated in Theorem 5 below.

Lemma 1. *Let $\widetilde{\mathcal{E}}$ be the collection of coefficient functions μ for positive integers such that $\mu_k \le 1$ for all $k \in \mathbb{N}$, and let $\mathcal{E}$ be the subcollection of $\widetilde{\mathcal{E}}$ consisting of μ such that $\mu_k = 1$ implies that $\mu_{k+1} = 0$ for all $k \in \mathbb{N}$. Then, $\{2^{k-1}\}_{k=1}^{\infty}$ and the Fibonacci sequence are the only fundamental sequences of $\widetilde{\mathcal{E}}$ and $\mathcal{E}$, respectively.*

Definition 2. Let $\mathcal{E}$ be the collection defined Lemma 1. If $\mu \in \mathcal{E}$, then the binary expansion $\sum_{k=1}^{\infty} \mu_k 2^{k-1}$ is said to be *Zeckendorf*.

Then, the earlier task is equivalent to finding an asymptotic formula of the number of positive integers less than x that have Zeckendorf binary expansions.

The main goal of this paper is to investigate the asymptotic formula of a function that counts the number of positive integers n up to x in the setting where the collections $\widetilde{\mathcal{E}}$ and $\mathcal{E}$ are replaced with *periodic Zeckendorf collections*; see Definition 11. In [4], *generalized Zeckendorf expansions* are introduced in terms of a lexicographical order, which further generalized the expansions introduced in [17]; see Definition 9. The expansions introduced in [17] are for general linear recurrences with constant non-negative integer coefficients, and in the viewpoint of [4], the expansions are called *periodic*. The main result of this paper is for the generalized Zeckendorf expansions that are periodic. However, several results remain valid for non-periodic ones, and for this reason, we use the language introduced in [4] to present our work. Appealing to the reader's intuition, we formulate the first main result below without properly defining terms. The terms will be properly introduced in later sections, and its technical version is stated in Theorem 13 and 14.

Theorem 1. *Let $\widetilde{\mathcal{E}}$ be a periodic Zeckendorf collection for positive integers, and let $\{\widetilde{H}_k\}_{k=1}^{\infty}$ be the unique fundamental sequence of $\widetilde{\mathcal{E}}$. Let $\mathcal{E}$ be a periodic Zeckendorf* *subcollection of* $\widetilde{\mathcal{E}}$*. Let* $z(x)$ *be the number of non-negative integers* $n < x$ *such that* $n = \sum_{k=1}^{\infty} \mu_k \widetilde{H}_k$ *for some* $\mu \in \mathcal{E}$*. Then, (1) there are positive real numbers* $\gamma < 1$*,* a*, and* b *such that* $a < z(x)/x^\gamma < b$ *for all sufficiently large* x*; (2) there are finitely many explicit and computable sequences* $\{x_k\}_{k=1}^{\infty}$ *from which the* $\limsup$ *and* $\liminf$ *of* $z(x)/x^\gamma$ *can be determined.*

For the Zeckendorf binary expansion, using Theorem 13, we prove

$$\limsup_x \frac{z(x)}{x^\gamma} = \frac{\phi+2}{5} 3^\gamma \approx 1.55, \quad \liminf_x \frac{z(x)}{x^\gamma} = \frac{3\phi+1}{5} \approx 1.17 \tag{1}$$

where $\gamma := \log_2 \phi$ and ϕ is the golden ratio; see Section 5.1. This is interesting since the values still bear the golden ratio, which must have come from the expressions in $\mathcal{E}$; see Section 5.2 and Theorem 16 for more examples of explicit calculations of the bounds.

Let us demonstrate the behavior of $z(x)/x^\gamma$ for the Zeckendorf binary expansions. Shown in the first figure of Figure 1 is the graph of $z(x)/x^\gamma$. The fluctuating behavior of the graph suggests that $z(x)$ may not be asymptotic to an "elementary" increasing function. However, it reveals certain self-similarities as in some dynamic systems, and in particular, the distribution of the values are far from being random.

Shown in the second figure of Figure 1 is the frequency chart of the values of $z(x)/x^\gamma$ for $262144 \le x < 349525$, which is one of the maximal intervals in the figure on which the values are not (always) decreasing–the binary expansions of these boundary values will be explained later. The values of the ratio range approximately from 1.17 to 1.55, which are represented in the horizontal axis, and we partitioned it into 200 intervals of equal length, in order to count the number of the values that fall into each of the 200 intervals, which is represented vertically. The third figure is the probability distribution of the values, i.e., it is the graph of

$$\text{Prob}\{262144 \le x < 349525 : z(x)/x^\gamma \le r\} \quad \text{as a function of } r.$$

As observed in the first figure of Figure 1, there are values of x where the ratios are locally extremal over a relatively large interval, and we prove that their limits are as identified in Equation (1). The local minima are obtained at $x_1 = 2^{n-1}$ for each $n \ge 2$, and the local maxima are obtained at $x_2 = \sum_{k=0}^{t} 2^{n-2k-1}$ where

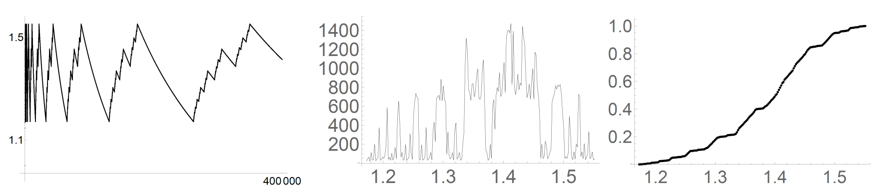

Figure 1: Graph, frequency, and distribution of $z(x)/x^\gamma$

$t = \lfloor (n-1)/2 \rfloor$. The sample interval $262144 \le x < 349525$ used in the second figure of Figure 1 is obtained at these values where $n = 19$.

The topic has a purely combinatorial interpretation as well. Consider the sets $\widetilde{\mathcal{E}}$ defined in Lemma 1, and notice that $\widetilde{\mathcal{E}}$ is lexicographically ordered; see Definition 6. Given a tuple $\tau \in \widetilde{\mathcal{E}}$, we may ask ourselves how many tuples in $\widetilde{\mathcal{E}}$ that are "less than" τ have no consecutive 1s. The fluctuating behavior of the distribution of such tuples in $\widetilde{\mathcal{E}}$ is precisely as presented in Figure 1.

In [1], the notion of *regular sequences* is introduced, and proved in [12] is an asymptotic formula of $\sum_{k=1}^{x} \chi(k)$ in full generality where $\chi(k)$ is a regular sequence. It turns out that the fluctuation behavior is a common feature of the summatory function of a regular sequence, and there are many examples of summatory functions in the literature that have similar fluctuation behaviors; see [6, 8, 19, 16]. This setup in [12] applies to some cases of ours. For the case of Zeckendorf binary expansions, the counting function $z(x)$ is equal to $\sum_{k=0}^{x-1} \chi(k)$ where χ is the regular sequence defined by $\chi(k) = 1$ if the binary expansion of k is Zeckendorf, and $\chi(k) = 0$ otherwise. The asymptotic formula involves a fluctuation factor as a continuous periodic function on the real numbers, and we follow their formulation of the asymptotic formula, which has been standard in the literature; see Section 6.

The authors of [12] describe the fluctuation factor as a Fourier expansion. In the context of the Mellin-Perron summation formula, the description of the Fourier coefficients is given in terms of the coefficients of Laurent expansions of the Dirichlet series $\frac{1}{s}\left(\chi(0) + \sum_{k=1}^{\infty} \chi(k)/k^s\right)$ at special points that line up in a vertical line in the complex plane. One of our main goals is to obtain exact bounds on the fluctuation factor. However, it does not seem feasible for us to use the Fourier expansion formulation and obtain exact bounds. Moreover, the current notion of regular sequences in the literature is formulated for base-N expansions, and hence, the asymptotic formula in [12] does not apply to the expansions in terms of the fundamental sequences of periodic Zeckendorf collections. We shall introduce an example in Section 5 where we count certain expressions under non-base-N expansions.

Introduced in Theorem 8 is a formula for the counting function $z(x)$ in full generality, and we call it *the duality formula*. We use the duality formula to obtain exact bounds on the fluctuation factor of $z(x)$ rather than the Fourier expansions described in [12]. The duality formula is a manifestation of the phenomenon that the coefficient functions of $\mathcal{E}$ themselves bring out their own fundamental sequence $\{H_k\}_{k=1}^{\infty}$ into the formula of $z(x)$. For example, as stated in Lemma 1, the Fibonacci sequence is the fundamental sequence of the coefficient functions that do not have consecutive 1s, and the number of Zeckendorf binary expansions less than x is written as a sum of Fibonacci terms in a fashion similar to the binary expansion of x; see Lemma 2 below. A subset of $\mathbb{N}$ is called an index subset in the context of series expansions, and an element of the subset is called *an index*.

Lemma 2 (Duality formula)**.** *Let A be a finite subset of indices. If A contains no* *adjacent indices, define* $\bar{A} := A$. *If there is a largest index* $j \in A$ *such that* $j+1 \in A$, *then define* $\bar{A} := \{k \in A : k \geq j\}$. *Then,* $z(\sum_{k\in A} 2^{k-1}) = \sum_{k\in \bar{A}} F_k$.

For example, $100 = 2^2 + 2^5 + 2^6 = \widetilde{H}_3 + \widetilde{H}_6 + \widetilde{H}_7$, and the indices of non-zero coefficients are $A = \{3, 6, 7\}$. Thus, $\bar{A} = \{6, 7\}$, and $z(100) = F_6 + F_7 = F_8 = 34$.

Let us formulate the counting function $z(x)$ using a continuous real-valued function as done in [12]. Recall the example of the Zeckendorf binary expansions and the formulas (1). By [12, Theorem A], there is a continuous real-valued function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $z(x) \sim x^\gamma \Phi(\{\log_2(x)\})$ where $\{\log_2(x)\}$ denotes the fractional part of $\log_2(x)$, and the authors of [12] describe $\Phi(t)$ as a Fourier series. As explained earlier, we use the duality formula, rather than the Fourier series description, and we also use *the generalized Zeckendorf expansions for the real numbers in the interval* $(0, 1)$, which is developed in [4]; see Section 2.3. In this section, we state our results on the properties of Φ in Theorem 2 below, and its technical versions are found in Theorem 10, 11, and 12.

Theorem 2. *Let* $z(x)$ *be the counting function defined in Theorem 1. Then,* $z(x) \sim x^\gamma \Phi(\{\log_{\widetilde{\phi}}(x)\})$ *for some positive real numbers* $\gamma < 1$ *and* $\widetilde{\phi} > 1$ *and a continuous function* Φ. *Moreover,* Φ *is differentiable almost everywhere with respect to Lebesgue measure, and there are explicit criteria in terms of the generalized Zeckendorf expansion of a real number* y *for determining whether* Φ *is differentiable at* y *or not, and whether* Φ *has a local maximum, a local minimum, or neither at* y.

When the quotient $z(x)/x^\gamma$ is calculated, we noticed that its values are naturally transitioned to the quotient of two real numbers in the interval $(0, 1)$, e.g., for the Zeckendorf binary expansions,

$$\frac{F_9 + F_6 + F_2}{(2^8 + 2^5 + 2)^\gamma} = \frac{\alpha\omega + \alpha\omega^2 + \alpha\omega^6 + E}{\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^6}\right)^\gamma} = \alpha\, \frac{\omega + \omega^2 + \omega^6 + E/\alpha}{\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^6}\right)^\gamma} \tag{2}$$

where $\omega = 1/\phi$, $\alpha = (3\phi + 1)/5$, and E is a relatively smaller quantity. Notice that properly estimating the value of the last expression in Equation (2) for general cases may rely on the uniqueness of the expansions of the real numbers in the numerator and the denominator, and it leads us naturally to the generalized Zeckendorf expansions of real numbers. Motivated from the non-constant quotient factor of the last expression in Equation (2), we define a function on the interval $(0, 1)$ to represent the factor, and denote it by $\delta^* : (0, 1) \to \mathbb{R}$; see Definition 23. Theorem 10, 11, and 12 are stated for δ^*. For the case of the Zeckendorf binary expansions, the relationship between the continuous functions can be

$$\Phi(\{\log_2(x)\}) \sim \alpha\delta^*(2^{\{\log_2 x\}-1})$$

where Φ is the function appearing in [12].

As demonstrated in the case of Zeckendorf binary expansions, the factor δ^* is infinitesimally fluctuating for the general cases as well, and the main tools for analyzing the fluctuations were the generalized Zeckendorf expansions for the real numbers (see the work toward Definition 23) and Lemma 18, which describes a sufficient condition for δ^* having a higher value.

The remainder of the paper is organized as follows. In Section 2, we review the generalized Zeckendorf expansions for the positive integers and the real numbers in $(0, 1)$. These contents are also available in [4], but for the readability of our work, we review the contents in this paper. In Section 3, the setup of two generalized Zeckendorf expressions is introduced for the positive integers and for the interval $(0, 1)$, and in this setup, the duality formula and the transition from the integers to the real numbers are introduced. In Section 4, the main results Theorem 1 and 2 are proved. In Section 5, calculations for the Zeckendorf binary expansions and two more pairs of collections are demonstrated, one of which is a pair with non-base-N expansions. In Section 6, we conclude the paper discussing the generalization of regular sequences and their summatory functions.

2. Generalized Zeckendorf Expansions

We shall review the definitions and results related to the generalized Zeckendorf expansions for positive integers that are periodic, and they are also available in [4]. By [4, Theorem 7], the definition introduced in this paper is equivalent to the definition introduced in [17, Definition 1.1]. We also review the definitions and results related to the generalized Zeckendorf expansions for the real numbers in the interval $\mathbf{I} := (0, 1)$ that are periodic; see [4] for non-periodic ones for $\mathbf{I}$.

2.1. Notation and Definitions

We identify a sequence of numbers with an infinite tuple, and denote its terms using subscripts. For example, the sequence of positive odd integers is denoted by $Q = (1, 3, 5, 7, \dots)$, and we use subscripts to denote its values, e.g., $Q_3 = 5$. We may define a sequence by describing Q_k where k is assumed to be an index ≥ 1, e.g., the earlier example is the sequence given by $Q_k = 2k - 1$. Recall coefficient functions from Definition 1. Given a coefficient function ϵ and a sequence Q, we denote by $\sum \epsilon Q$ the formal sum $\sum_{k=1}^{\infty} \epsilon_k Q_k$.

Given two coefficient functions ϵ and μ, we define $\epsilon + \mu$ to be the coefficient function such that $(\epsilon + \mu)_k = \epsilon_k + \mu_k$ for $k \geq 1$, and $\epsilon - \mu$ to be the coefficient function such that $(\epsilon - \mu)_k = \epsilon_k - \mu_k$ for $k \geq 1$ if $\epsilon_k \geq \mu_k$ for all $k \geq 1$. Given $c \in \mathbb{N}_0$, we define $c\epsilon := (c\epsilon_1, c\epsilon_2, \dots)$.

Definition 3. Let β^i be the coefficient function such that $\beta^i_k = 0$ for all $k \neq i$ and $\beta^i_i = 1$, and call it *the ith basis coefficient function*; in particular, β^i_k does not denote the ith power of an integer.

Definition 4. For coefficient functions ϵ, we use the bar notation $\bar{a}$ to represent the repeating entries. For example, $\epsilon = (1, 2, 3, \bar{0})$ is the coefficient function such that $\epsilon_k = 0$ for all $k > 3$, and we denote it by $\epsilon = (1, 2, 3)$ as well. If $\epsilon = (\bar{0}, 1, 2, 3)$, then it means that there is an index n such that $\epsilon_k = 0$ for $1 \leq k \leq n$, and $\epsilon_{n+j} = j$ for $1 \leq j \leq 3$. We use the simpler notation 0 for the zero coefficient function $(\bar{0})$. If there is an index M such that $\epsilon_k = 0$ for all $k > M$, then ϵ is said to *have finite support*.

Definition 5. Let ϵ be a coefficient function, and let m and n be two positive integers. Define $\operatorname{rev}_n(\epsilon)$ to be the coefficient function $(\epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_1)$. Define $\operatorname{res}_m(\epsilon)$ and $\operatorname{res}^m(\epsilon)$ to be the coefficient functions such that $\operatorname{res}_m(\epsilon)_k = 0$ for all $k > m$, $\operatorname{res}_m(\epsilon)_k = \epsilon_k$ for all $1 \leq k \leq m$, $\operatorname{res}^m(\epsilon)_k = \epsilon_k$ for all $k \geq m$, and $\operatorname{res}^m(\epsilon)_k = 0$ for all $1 \leq k < m$. Also we define $\operatorname{res}^0(\epsilon) := \epsilon$, $\operatorname{res}_0(\epsilon) = 0$, and $\operatorname{res}^m_n(\epsilon)$ to be the coefficient function such that $\operatorname{res}^m_n(\epsilon)_k = \epsilon_k$ for all $m \leq k \leq n$ and $\operatorname{res}^m_n(\epsilon)_k = 0$ for other indices k.

2.2. Generalized Zeckendorf Expansions for Positive Integers

We shall use *the ascending lexicographical order* to define generalized Zeckendorf expansions for positive integers, and it is defined as follows.

Definition 6. Given two coefficient functions μ and μ' with finite support, if there is a largest positive integer k such that $\mu_j = \mu'_j$ for all $j > k$ and $\mu_k < \mu'_k$, then we denote the property by $\mu <_{\mathrm{a}} \mu'$.

For example, if $\mu = (1, 2, 10, 3, 7)$ and $\mu' = (1, 3, 1, 4, 7)$, then $\mu <_{\mathrm{a}} \mu'$ since $\mu_4 < \mu'_4$ and $\mu_5 = \mu'_5$.

Definition 7. We define *an ascendingly-ordered collection $\mathcal{E}$ of coefficient functions* to be a set of coefficient functions with finite support ordered by the ascending lexicographical order that contains the zero coefficient function and all basis coefficient functions β^i. Let $\mathcal{E}$ be an ascendingly-ordered collection, and let $\mu \in \mathcal{E}$. The smallest coefficient function in $\mathcal{E}$ that is greater than μ, if (uniquely) exists, is called *the immediate successor of μ in $\mathcal{E}$*, and we denote it by $\widetilde{\mu}$. The largest coefficient function in $\mathcal{E}$ that is less than μ, if (uniquely) exists, is called *the immediate predecessor of μ in $\mathcal{E}$*, and we denote it by $\widehat{\mu}$. In particular, $\hat{\beta}^n$ shall denote the immediate predecessor of the nth basis coefficient function β^n for each $n \geq 2$, if (uniquely) exists.

Definition 8. Let μ be an element of an ascendingly-ordered collection. If $\mu \neq 0$, the largest index n such that $\mu_n \neq 0$ is called the order of μ, denoted by $\operatorname{ord}(\mu)$. If $\mu = 0$, we define $\operatorname{ord}(\mu) = 0$.

By definition, an ascendingly-ordered collection $\mathcal{E}$ contains all the basis coefficient functions, i.e., $\beta^{n-1} \in \mathcal{E}$ for $n \geq 2$, and hence, the immediate predecessor $\hat{\beta}^n$, if exists, has a non-zero value at index $n-1$, i.e., $\text{ord}(\hat{\beta}^n) = n-1$.

Definition 9. Let $\mathcal{E}$ be an ascendingly-ordered collection of coefficient functions. The collection is called *Zeckendorf* if it satisfies the following:

1. For each $\mu \in \mathcal{E}$, there are at most finitely many coefficient functions that are less than μ.

2. Given $\mu \in \mathcal{E}$, if its immediate successor $\widetilde{\mu}$ is not $\beta^1 + \mu$, then there is an index $n \geq 2$ such that $\text{res}_{n-1}(\mu) = \hat{\beta}^n$ and $\widetilde{\mu} = \beta^n + \text{res}^n(\mu)$.

We also call $\mathcal{E}$ *a Zeckendorf collection for positive integers.*

Surprisingly enough, it turns out that a Zeckendorf collection for positive integers is completely determined by the subset $\{\hat{\beta}^n : n \geq 2\}$, which is the meaning of Theorem 3 below.

Theorem 3 ([4], Definition 8 & Corollary 9)**.** *Given coefficient functions θ^n of order $n-1$ for $n \geq 2$, there is a unique Zeckendorf collection for positive integers such that $\hat{\beta}^n = \theta^n$ for each $n \geq 2$.*

Let us demonstrate the Zeckendorf collection for positive integers determined by θ^n, which have a common periodic structure.

Definition 10. Let $L = (e_1, \ldots, e_N)$ be a list of non-negative integers where $N \geq 2$ and $e_1 \neq 0$, and let β^* be the coefficient function such that $\beta^*_k = e_j$ where $1 \leq j \leq N$ and $k \equiv j \pmod N$. Given $n \geq 2$, define $\theta^n := \text{rev}_{n-1}(\beta^*)$. For an integer $n \geq 2$, a coefficient function ζ is called *a proper L-block at index $n-1$* if there is an index $k \leq n-1$ such that $\text{res}^{k+1}(\zeta) = \text{res}^{k+1}(\theta^n)$, $\zeta_k < \theta^n_k$, and $\text{res}_{k-1}(\zeta) = 0$. The index interval $[k, n-1]$ is called *the support interval* of the proper L-block ζ at index $n-1$. The coefficient functions θ^n for $n \geq 2$ are called *maximal L-blocks*, and the index interval $[1, n-1]$ is called *the support interval* of the maximal L-block at index $n-1$. Both proper and maximal L-blocks are called *L-blocks*, and two L-blocks ζ and ξ are said to be *disjoint* if their support intervals are disjoint sets of indices.

Theorem 4 ([4], Theorem 7)**.** *Let L and θ^n for integers $n \geq 2$ be as defined in Definition 10. Then, the collection $\mathcal{E}$ consisting of the sums of finitely many disjoint L-blocks is a Zeckendorf collection for positive integers such that $\hat{\beta}^n = \theta^n$ for $n \geq 2$.*

Definition 11. Let L and $\mathcal{E}$ be as defined in Theorem 4. The collection $\mathcal{E}$ is called *the periodic Zeckendorf collection for positive integers determined by L.*

Definition 12. Let L be the list defined in Definition 10. Let $\epsilon = \sum_{m=1}^{n} \zeta^m$ where ζ^m are disjoint L-blocks at index i_m such that $i_m < i_{m+1}$ for $1 \leq m < n$. This expression is called *an L-block decomposition*, and if $\epsilon \neq 0$, the support intervals of ζ^m form a partition of $[1, i_n]$, and $\zeta^n \neq 0$, then the summation is called *the (full) L-block decomposition of ϵ*. If $\epsilon \neq 0$ and ζ^m are non-zero disjoint L-blocks, then the summation is called *the non-zero L-block decomposition of ϵ*. The expression $\mu + \tau$ is called an *L-decomposition* if $\sum_{m=1}^n \zeta^m$ is *an L-block decomposition*, $\mu = \sum_{m=1}^s \zeta^m$, and $\tau = \sum_{m=t}^n \zeta^m$ where $1 \leq s < t \leq n$.

Example 1. Let $L = (2, 3, 0)$, and let $\mathcal{E}$ be the periodic Zeckendorf collection for positive integers determined by L. Then, $\beta^* = (2, 3, 0, 2, 3, 0, 2, 3, 0, \dots)$, and the following are examples of the immediate predecessors:

$$\hat{\beta}^2 = (2),\ \hat{\beta}^3 = (3, 2),\ \hat{\beta}^4 = (0, 3, 2),\ \hat{\beta}^5 = (2, 0, 3, 2),\ \hat{\beta}^9 = (3, 2, 0, 3, 2, 0, 3, 2).$$

Listed below are examples of proper L-blocks at index 5:

$$\zeta^1 = (0, 0, 0, 0, 2),\ \zeta^2 = (0, 0, 0, 2, 2),\ \zeta^3 = (0, 0, 0, 3, 2).$$

The support intervals of ζ^k for $k = 1, 2, 3$ are $[4, 5]$, $[4, 5]$, and $[2, 5]$, respectively. Stated below are two examples of the sum of finitely many disjoint L-blocks where a semicolon is inserted instead of a comma to indicate the end of the support interval of a block, and their immediate successors are stated below as well:

$$\epsilon = (2, 0, 3, 2; 0; 0, 0, 3, 2; 2, 2; 1),\ \tau = (1, 0, 3, 2; 0; 0, 0, 3, 2; 2, 2; 1)$$
$$\widetilde{\epsilon} = (0; 0; 0; 0; 1; 0, 0, 3, 2; 2, 2; 1),\ \widetilde{\tau} = (2, 0, 3, 2; 0; 0, 0, 3, 2; 2, 2; 1).$$

Given a collection $\mathcal{E}$ of coefficient functions with finite support and a sequence G of positive integers, let $\mathrm{eval}_G : \mathcal{E} \to \mathbb{N}_0$ be the evaluation map given by $\epsilon \mapsto \sum \epsilon G$. The weak converse of Zeckendorf's theorem for Zeckendorf collections for positive integers is stated below. Recall Definition 6.

Theorem 5 ([4], Theorem 16, Lemma 3 & 37)**.** *Let $\mathcal{E}$ be a Zeckendorf collection for positive integers with the immediate predecessors $\hat{\beta}^n$ for $n \geq 2$. Then, there is a unique increasing sequence H of positive integers such that $\mathrm{eval}_H : \mathcal{E} \to \mathbb{N}_0$ is bijective. Moreover, for each $n \geq 2$,*

$$H_n = H_1 + \sum \hat{\beta}^n H, \tag{3}$$

and the map eval_H is increasing, i.e., $\epsilon <_{\mathrm{a}} \delta$ if and only if $\mathrm{eval}_H(\epsilon) < \mathrm{eval}_H(\delta)$.

Recall Definition 1. Then, by Theorem 5, given a Zeckendorf collection $\mathcal{E}$ for positive integers, there is one and only one fundamental sequence of $\mathcal{E}$.

Definition 13. Given a Zeckendorf collection $\mathcal{E}$ for positive integers, the expansion $n = \sum \epsilon H$ is called the $\mathcal{E}$-expansion of n where H is the fundamental sequence of $\mathcal{E}$.

For the collection in Example 1, the fundamental sequence H is given by the linear recurrence $H_{n+3} = 2H_{n+2} + 3H_{n+1} + H_n$ with $(H_1, H_2, H_3) = (1, 3, 10)$. In general, the fundamental sequence of a periodic Zeckendorf collection determined by $L = (e_1, \ldots, e_N)$ is given by

$$H_n = \sum_{k=1}^{N-1} e_k H_{n-k} + (1 + e_N)H_{n-N}, \quad n > N \tag{4}$$

with initial values $(H_1, H_2, \ldots, H_N)$ determined by Equation (3).

Definition 14. Let $L = (e_1, \ldots, e_N)$ be the list defined in Definition 10. The following is called *the characteristic polynomial of L for positive integers:*

$$x^N - \sum_{k=1}^{N-1} e_k x^{N-k} - (1 + e_N). \tag{5}$$

By Descartes' Rule of Signs, the polynomial (5) has one positive simple real root ϕ, and it is greater than 1. By [4, Section 5.3.2], it is the only root with the largest modulus in the complex plane. Thus, by Binet's formula, $H_n = \alpha\phi^n + O(\phi^{nr})$ where α and $r < 1$ are positive real constants. Notice that the periodic Zeckendorf collection for positive integers determined by $L = (e_1, \ldots, e_N, e_1, \ldots, e_N)$ is equal to the collection determined by $(e_1, \ldots, e_N)$ since they have the same immediate predecessor of β^n for each $n \geq 2$.

2.3. Generalized Zeckendorf Expansions for Real Numbers

Let us review the definitions and results for the periodic Zeckendorf collections for the real numbers in the interval $\mathbf{I} := (0, 1)$. The general definition is given in [4, Definition 10], and in this paper, we review the definition for the periodic Zeckendorf collections.

Definition 15. Given two coefficient functions ϵ and ϵ', we define *the descending lexicographical order* as follows. If there is a smallest positive integer k such that $\epsilon_j = \epsilon'_j$ for all $j < k$ and $\epsilon_k < \epsilon'_k$, then we denote the property by $\epsilon <_{\mathrm{d}} \epsilon'$.

For example, if $\epsilon = (1, 2, 10, 5, \ldots)$ and $\epsilon' = (1, 3, 1, 10, \ldots)$, then $\epsilon <_{\mathrm{d}} \epsilon'$ since $\epsilon_2 < \epsilon'_2$ and $\epsilon_1 = \epsilon'_1$.

Definition 16. If $\epsilon \neq 0$ is a coefficient function, the smallest index n such that $\epsilon_n \neq 0$ is denoted by $\mathrm{ord}^*(\epsilon)$. If $\epsilon = 0$, we define $\mathrm{ord}^*(\epsilon) = \infty$.

Definition 17. Let $L = (e_1, \ldots, e_N)$ be a list of non-negative integers where $N \geq 2$ and $e_1 \neq 0$. Given $n \geq 1$, let $\bar{\beta}^n$ be the coefficient function such that $\mathrm{res}_{n-1}(\bar{\beta}^n) = 0$ and $\bar{\beta}^n_k = e_j$ for $k \geq n$ where $1 \leq j \leq N$ and $k - n + 1 \equiv j \pmod{N}$, and we call $\bar{\beta}^n$ *the maximal* L^**-block at index* n. For an integer $n \geq 1$, a coefficient function ζ is called *a proper* L^**-block at index* n if there is an index $k \geq n$ such that $\operatorname{res}_{k-1}(\zeta) = \operatorname{res}_{k-1}(\bar{\beta}^n)$, $\zeta_k < \bar{\beta}^n_k$, and $\operatorname{res}^{k+1}(\zeta) = 0$. The index interval $[n, k]$ is called *the support interval* of the proper L^*-block ζ at index n. Two proper L^*-blocks ζ and ξ are said to be *disjoint* if their support intervals are disjoint sets of indices. The descendingly ordered collection of non-zero coefficient functions consisting of the finite or infinite sums of disjoint proper L^*-blocks is called *the periodic Zeckendorf collection for* $\mathbf{I}$ *determined by* L. If $\mathcal{E}$ denotes the periodic Zeckendorf collection for positive integers determined by L, the collection for $\mathbf{I}$ is denoted by $\mathcal{E}^*$. Let $\epsilon = \sum_{m=1}^{\infty} \zeta^m$ where ζ^m are disjoint proper L^*-blocks at index i_m such that $i_m < i_{m+1}$. This expression is called *an* L^**-block decomposition of* ϵ, and if the support intervals of ζ^m form a partition of $[1, \infty)$, the summation is called *the (full)* L^**-block decomposition of* ϵ. If ζ^m are non-zero disjoint proper L^*-blocks, then the summation is called *the non-zero* L^**-block decomposition of* ϵ. The expression $\mu + \tau$ is called *an* L^**-decomposition* if $\sum_{m=1}^{\infty} \zeta^m$ is an L^*-block decomposition, $\mu = \sum_{m=1}^{s} \zeta^m$, and $\tau = \sum_{m=t}^{\infty} \zeta^m$ where $1 \leq s < t$.

Thus, given a list $L = (e_1, \ldots, e_N)$, we have the periodic Zeckendorf collection $\mathcal{E}$ for positive integers and the periodic Zeckendorf collection $\mathcal{E}^*$ for $\mathbf{I}$. The collection $\mathcal{E}$ is ascendingly ordered, and $\mathcal{E}^*$ is descendingly ordered. Also notice that the zero coefficient function and the maximal blocks $\bar{\beta}^n$ for $n \geq 1$ are not members of $\mathcal{E}^*$, but the zero coefficient function can be a proper L^*-block at any index n.

Example 2. Let $L = (2, 3, 0)$, and let $\mathcal{E}^*$ be the periodic Zeckendorf collection for $\mathbf{I}$ determined by L. The following are examples of maximal L^*-blocks:

$$\bar{\beta}^1 = (2, 3, 0, 2, 3, 0, \ldots),\ \bar{\beta}^2 = (0, 2, 3, 0, 2, 3, 0, \ldots),\ \bar{\beta}^3 = (0, 0, 2, 3, 0, 2, 3, 0, \ldots).$$

Listed below are examples of proper L^*-blocks at index 4, and semicolons are inserted to indicate the ends of the support intervals. See Definition 4 for the bar notation:

$$\zeta^1 = (0; 0; 0; 2, 0; \bar{0}),\ \zeta^2 = (0; 0; 0; 2, 2; \bar{0}),\ \zeta^3 = (0; 0; 0; 2, 3, 0, 0; \bar{0}).$$

The support intervals of ζ^k for $k = 1, 2, 3$ are $[4, 5]$, $[4, 5]$, and $[4, 7]$, respectively. Stated below are two examples of coefficient functions where $\epsilon \notin \mathcal{E}^*$ and $\tau \in \mathcal{E}^*$:

$$\epsilon = (1, 2, 2, \bar{0}) + \bar{\beta}^4 = (1, 2, 2, \overline{2, 3, 0}), \qquad \tau = (1, 2, 2, \overline{2, 3, 0, 0}).$$

Lemma 3 ([4], Definition 10, Theorem 13)**.** *Let* ϵ *be a coefficient function. Then,* $\operatorname{rev}_n(\epsilon) \in \mathcal{E}$ *for all* $n \geq 1$ *if and only if* $\epsilon \in \mathcal{E}^*$ *or* $\epsilon = \sum_{m=1}^{n} \zeta^m + \bar{\beta}^{b+1}$ *where* $\sum_{m=1}^{n} \zeta^m$ *is an* L^**-block decomposition and* $[i, b]$ *is the support interval of* ζ^n*.*

Recall Definition 15.

Theorem 6 ([4], Theorem 24)**.** *Let $\mathcal{E}^*$ be the periodic Zeckendorf collection for* **I** *determined by $L = (e_1, \ldots, e_N)$. Then, there is a unique decreasing sequence Q of real numbers in* **I** *such that the map* $\mathrm{eval}_Q : \mathcal{E}^* \to \mathbf{I}$ *given by $\epsilon \mapsto \sum \epsilon Q$ is bijective, and the map* eval_Q *is increasing, i.e., $\epsilon <_{\mathrm{d}} \delta$ if and only if* $\mathrm{eval}_Q(\epsilon) < \mathrm{eval}_Q(\delta)$*. Moreover, for each $n \geq 1$,*

$$Q_n = \sum \bar{\beta}^{n+1} Q, \text{ and } Q_n = \omega^n \tag{6}$$

where ω is the (only) positive real zero of the polynomial

$$-1 + \sum_{k=1}^{N-1} e_k x^k + (1 + e_N) x^N. \tag{7}$$

Definition 18. Let $\mathcal{E}^*$ be a periodic Zeckendorf collection for **I**. The sequence Q defined in Theorem 6 is called *the fundamental sequence of* $\mathcal{E}^*$, and the polynomial (7) is called *the characteristic polynomial of L for* **I**.

For the collection in Example 2, the fundamental sequence Q is given by the linear recurrence $Q_n = 2Q_{n+1} + 3Q_{n+2} + Q_{n+3}$, and its initial values are given by the formula (6). In general, the fundamental sequence of the periodic Zeckendorf collection for **I** determined by $L = (e_1, \ldots, e_N)$ is given by

$$Q_n = \sum_{k=1}^{N-1} e_k Q_{n+k} + (1 + e_N) Q_{n+N}, \quad n \geq 1. \tag{8}$$

The polynomial (7) is a reciprocal version of the polynomial (5), i.e., if $f(x)$ is the polynomial, then the reciprocal version is equal to $-x^N f(1/x)$. Thus, it has a positive simple real zero $\omega < 1$, and it is the only zero with the smallest modulus.

For the remainder of Section 2.3, let $\mathcal{E}^*$ be the periodic Zeckendorf collection for **I** determined by L, and let Q denote the fundamental sequence of $\mathcal{E}^*$. The results introduced below will be used in Section 4, but they are introduced here in order to provide the reader with an opportunity to become more familiar with generalized Zeckendorf expansions of real numbers.

Definition 19. If $\epsilon \in \mathcal{E}^*$, we define $\inf_{\mathcal{E}^*}(\epsilon)$ as follows. If ϵ does not have finite support, then $\inf_{\mathcal{E}^*}(\epsilon) := \epsilon$. If c is the largest index such that $\epsilon_c \geq 1$, then $\inf_{\mathcal{E}^*}(\epsilon) = \mathrm{res}_c(\epsilon) - \beta^c + \bar{\beta}^{c+1}$, which is not a member of $\mathcal{E}^*$. If there is no confusion, let inf denote $\inf_{\mathcal{E}^*}$.

The long recursion (6) implies the following lemma, and we leave the proof to the reader.

Lemma 4. *If $\epsilon \in \mathcal{E}^*$, then $\sum \epsilon Q = \sum \inf(\epsilon) Q$.*

Recall the restriction notation from Definition 5 and L^*-decompositions from Definition 17. Throughout the proof of Lemma 5, we use the increasing property of eval_Q defined in Theorem 6.

Lemma 5. *Let $\epsilon \in \mathcal{E}^*$, let $x = \sum \epsilon Q > 0$, and let Δx be a sufficiently small positive real number such that $x + \Delta x < 1$ and $x - \Delta x > 0$. If $x + \Delta x = \sum \epsilon^+ Q$ for $\epsilon^+ \in \mathcal{E}^*$, and n is the largest integer ≥ 0 such that $\mathrm{res}_n(\epsilon) = \mathrm{res}_n(\epsilon^+)$, then $n \to \infty$ as $\Delta x \to 0$. If $x - \Delta x = \sum \epsilon^- Q$ for $\epsilon^- \in \mathcal{E}^*$, and n is the largest integer ≥ 0 such that $\mathrm{res}_n(\inf(\epsilon)) = \mathrm{res}_n(\epsilon^-)$, then $n \to \infty$ as $\Delta x \to 0$.*

Proof. Let $\epsilon = \sum_{m=1}^{\infty} \zeta^m$ be an L^*-block decomposition, and let $[i_m, b_m]$ be the support interval of ζ^m. Notice that given $m \geq 1$, there is a smallest index $k_m > b_m$ such that $\bar{\beta}^{i_m}_{k_m} > 0$, and that b_m and k_m approach ∞ as $m \to \infty$.

Let $\Delta x = \sum \epsilon^0 Q$ where $\mathrm{ord}^*(\epsilon^0) = t$, and let T be the largest index such that $k_T < t$ where t is sufficiently large. Then, $\Delta x < Q_{k_T}$. Recall from Definition 17 the support intervals of proper L^*-blocks. Then, $\xi := \zeta^T + \beta^{b_T}$ is a proper L^*-block, and its support interval is either $[i_T, b_T]$ or $[i_T, k_T]$, and hence, $\sum_{m=1}^{T-1} \zeta^m + \xi + \beta^{k_T} \in \mathcal{E}^*$. Thus,

$$\begin{aligned} x = \sum \left(\textstyle\sum_{m=1}^{\infty} \zeta^m\right) Q < x + \Delta x &< \sum \left(\textstyle\sum_{m=1}^{T} \zeta^m\right) Q + Q_{b_T} + \Delta x \\ &< \sum \left(\textstyle\sum_{m=1}^{T-1} \zeta^m\right) Q + \sum \xi Q + Q_{k_T}. \end{aligned}$$

By Theorem 6,

$$\textstyle\sum_{m=1}^{\infty} \zeta^m <_{\mathrm{d}} \epsilon^+ <_{\mathrm{d}} \sum_{m=1}^{T-1} \zeta^m + \xi + \beta^{k_T},$$

and by the definition of the lexicographical order, we have

$$\mathrm{res}_{b_{T-1}}(\epsilon^+) = \textstyle\sum_{m=1}^{T-1} \zeta^m = \mathrm{res}_{b_{T-1}}(\epsilon).$$

Notice that as $\Delta x \to 0$, we have $t \to \infty$ and $T \to \infty$, and hence, $b_{T-1} \to \infty$.

Let us prove the statement about $x - \Delta x$. Suppose that $\epsilon = \sum_{m=1}^{\infty} \zeta^m$ is the non-zero L^*-block decomposition. Then, $\inf(\epsilon) = \epsilon$, and

$$\begin{aligned} x &= \sum \left(\textstyle\sum_{m=1}^{\infty} \zeta^m\right) Q > x - \Delta x \\ &> \sum \left(\textstyle\sum_{m=1}^{T-1} \zeta^m\right) Q + \sum \zeta^T Q - \Delta x > \sum \left(\textstyle\sum_{m=1}^{T-1} \zeta^m\right) Q. \end{aligned}$$

As in the earlier case, we have $\mathrm{res}_{b_{T-1}}(\epsilon^-) = \sum_{m=1}^{T-1} \zeta^m = \mathrm{res}_{b_{T-1}}(\epsilon)$, and $b_{T-1} \to \infty$ as $t \to \infty$.

Suppose that $\epsilon = \sum_{m=1}^{\ell} \zeta^m$ is the non-zero L^*-block decomposition, and let c be the largest index such that $\zeta^\ell_c \geq 1$. Then, $\inf(\epsilon) = \sum_{m=1}^{\ell-1} \zeta^m + \xi + \bar{\beta}^{c+1}$ where $\xi = \zeta^\ell - \beta^c$. Then, there is a largest index p such that $p < t$ and $c + 1 \equiv p \pmod{N}$

where N is the length of L, and hence, by the periodic structure of the entries of $\bar\beta^{c+1}$, we have $\bar\beta^{c+1} = \mathrm{res}_{p-1}(\bar\beta^{c+1}) + \bar\beta^{p}$. Thus,

$$\begin{aligned} x &= \sum \inf(\epsilon) Q > x - \Delta x \\ &= \sum \left(\textstyle\sum_{m=1}^{\ell-1} \zeta^m\right) Q + \sum \xi Q + \sum \mathrm{res}_{p-1}(\bar\beta^{c+1}) Q + \sum \bar\beta^{p} Q - \Delta x \\ &> \sum \left(\textstyle\sum_{m=1}^{\ell-1} \zeta^m\right) Q + \sum \xi Q + \sum \mathrm{res}_{p-1}(\bar\beta^{c+1}) Q = \sum \mathrm{res}_{p-1}(\inf(\epsilon)) Q. \end{aligned}$$

Then, $\sum(\sum_{m=1}^{\ell-1} \zeta^m + \zeta^\ell) Q > \sum \epsilon^- Q > \sum \mathrm{res}_{p-1}(\inf(\epsilon)) Q$. It is clear that

$$\mathrm{res}_{c-1}(\epsilon) = \mathrm{res}_{c-1}(\epsilon^-) = \mathrm{res}_{c-1}(\inf(\epsilon))$$

and $\epsilon^-_c = \zeta^\ell_c - 1 = \inf(\epsilon)_c$. Thus, $\epsilon^- = \mathrm{res}_c(\epsilon^-) + \mathrm{res}^{c+1}(\epsilon^-)$ is an L^*-decomposition, and hence, $\mathrm{res}^{c+1}(\epsilon^-)$ is a member of $\mathcal{E}^*$. By Theorem 6,

$$\sum \epsilon^- Q > \sum \mathrm{res}_{p-1}(\inf(\epsilon)) Q$$

implies $\mathrm{res}_{p-1}(\inf(\epsilon)) <_{\mathrm{d}} \epsilon^-$, and hence,

$$\mathrm{res}^{c+1}_{p-1}(\bar\beta^{c+1}) = \mathrm{res}^{c+1}_{p-1}(\inf(\epsilon)) <_{\mathrm{d}} \mathrm{res}^{c+1}(\epsilon^-).$$

Thus, we have

$$\mathrm{res}^{c+1}_{p-1}(\bar\beta^{c+1}) <_{\mathrm{d}} \mathrm{res}^{c+1}(\epsilon^-) <_{\mathrm{d}} \bar\beta^{c+1},$$

and hence, $\mathrm{res}_{p-1}(\epsilon^-) = \mathrm{res}_{p-1}(\inf(\epsilon))$. Since $p \to \infty$ as $t \to \infty$, we prove the result. □

3. Generalized Zeckendorf Expressions

3.1. Expressions for Positive Integers

Throughout this section, let $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ be periodic Zeckendorf collections for positive integers determined by lists $L = (e_1, \dots, e_N)$ and $\widetilde{L} = (\widetilde{e}_1, \dots, \widetilde{e}_M)$, respectively, such that $\mathcal{E}$ is a proper subcollection of $\widetilde{\mathcal{E}}$. Let H and $\widetilde{H}$ be their fundamental sequences, respectively. Let $\hat\beta^n$ and $\hat\theta^n$ be the immediate predecessor of β^n in $\mathcal{E}$ and $\widetilde{\mathcal{E}}$, respectively. Recall from Definition 13 that $\sum \epsilon \widetilde{H}$ is called an $\widetilde{\mathcal{E}}$-expansion if $\epsilon \in \widetilde{\mathcal{E}}$, and if, in addition, $\epsilon \in \mathcal{E}$, then the summation, which is written in terms of $\widetilde{H}$, is called *an $\mathcal{E}$-expression*. The main object of this paper is the function that counts the number of positive integers less than x whose $\widetilde{\mathcal{E}}$-expansion is an $\mathcal{E}$-expression.

The proofs of Proposition 1, Lemma 7, and Theorem 8 given below in this section remain valid for non-periodic Zeckendorf collections. The main result of this paper is for periodic ones, and they are stated for periodic collections.

Proposition 1. *Let $\widetilde{\mathcal{E}}$ and $\mathcal{E}$ be Zeckendorf collections for positive integers. Then, the collection $\mathcal{E}$ is a subcollection of $\widetilde{\mathcal{E}}$ if and only if the immediate predecessors $\hat{\beta}^n$ in $\mathcal{E}$ are members of $\widetilde{\mathcal{E}}$ for $n \geq 2$.*

Proof. If $\mathcal{E}$ is a subcollection of $\widetilde{\mathcal{E}}$, then $\hat{\beta}^n \in \mathcal{E}$ for $n \geq 2$ are members of $\widetilde{\mathcal{E}}$. Suppose that $\hat{\beta}^n$ for $n \geq 2$ are members of $\widetilde{\mathcal{E}}$, and let us show that $\mathcal{E} \subset \widetilde{\mathcal{E}}$. First, let us show that the proper L-blocks are members of $\widetilde{\mathcal{E}}$. Let $n \geq 2$, and let $\hat{\beta}^n = \sum_{m=1}^{M} \zeta^m$ be the $\widetilde{L}$-block decomposition (including zero blocks); see Definition 10. Let ξ be a non-zero proper L-block with support interval $[a, n-1]$, i.e., $\xi_a < \hat{\beta}^n_a$ and $\xi_k = \hat{\beta}^n_k$ for all $a < k \leq n-1$, and let us show that $\xi \in \widetilde{\mathcal{E}}$.

Notice that a is contained in the support interval of ζ^{m_0} for some $1 \leq m_0 \leq M$, and let $[b, s]$ be the support interval where $b \leq a \leq s$. Then, ζ^{m_0} is a nonzero $\widetilde{L}$-block since $\xi_a < \hat{\beta}^n_a = \zeta^{m_0}_a$ implies that $\zeta^{m_0}_a > 0$. Let η be the coefficient function such that $\eta_a = \xi_a$, $\eta_k = \zeta^{m_0}_k$ for all $a < k \leq s = \mathrm{ord}(\zeta^{m_0})$, and $\eta_k = 0$ for other indices k. Then, η is a proper $\widetilde{L}$-block with support interval $[a, s]$ since

$$\eta_a = \xi_a < \hat{\beta}^n_a = \zeta^{m_0}_a \leq \hat{\theta}^{s+1}_a \text{ and } \eta_k = \zeta^{m_0}_k = \hat{\theta}^{s+1}_k$$

for $a < k \leq s$. Notice that $\xi = \eta + \sum_{m=m_0+1}^{M} \zeta^m$, which is the sum of disjoint $\widetilde{L}$-blocks, i.e., $\xi \in \widetilde{\mathcal{E}}$. Moreover, the support interval of the proper L-block ξ is $[a, n-1]$, and the (disjoint) union of the support intervals of the $\widetilde{L}$-blocks η and ζ^m for $m_0 + 1 \leq m \leq M$ is $[a, n-1]$ as well. Thus, in general, a disjoint sum of proper L-blocks is a member of $\widetilde{\mathcal{E}}$.

Let us consider the case where the sum involves a maximal L-block $\hat{\beta}^c$. Let ϵ be the sum of disjoint proper L-blocks ξ^m for $1 \leq m \leq T$, i.e., $\epsilon = \sum_{m=1}^{T} \xi^m$, and suppose that $[c, n]$ where $c > 1$ is the union of the support intervals of the proper L-blocks ξ^m. Then, as shown earlier, the union of the support intervals of the $\widetilde{L}$-block decompositions of ξ^m for $1 \leq m \leq T$ is $J_1 := [c, n]$ as well. Recall that $\hat{\beta}^c \in \widetilde{\mathcal{E}}$, and hence, it is the $\widetilde{L}$-block decomposition, the union of whose support intervals is $J_2 := [1, c-1]$. Since J_1 and J_2 are disjoint, the L-decomposition $\hat{\beta}^c + \epsilon$ is an $\widetilde{L}$-decomposition, and hence, it is a member of $\widetilde{\mathcal{E}}$ as well. This proves that all L-block decompositions are members of $\widetilde{\mathcal{E}}$. □

Example 3. Let us demonstrate examples of subcollections. For each of the following, we first check if $\hat{\beta}^n \leq_{\mathrm{a}} \hat{\theta}^n$ for $n = 2, \ldots, 7$, and check if $\hat{\beta}^n$ is a member of $\widetilde{\mathcal{E}}$. We use the semicolon to indicate the end of the support interval of an $\widetilde{L}$-block.

1. Let $\widetilde{L} = (1, 3)$ and $L = (1, 2, 1)$. Then,

$$\hat{\beta}^7 = (1, 2; 1, 1; 2, 1) <_{\mathrm{a}} \hat{\theta}^7 = (3, 1, 3, 1, 3, 1).$$

   However, $\hat{\beta}^7$ does not have an $\widetilde{L}$-block decomposition since $\mathrm{res}^3(\hat{\beta}^7)$ has an $\widetilde{L}$-block decomposition, but $\mathrm{res}_2(\hat{\beta}^7)$ does not have an $\widetilde{L}$-block decomposition. Thus, $\mathcal{E}$ determined by L is not a subcollection of $\widetilde{\mathcal{E}}$.

2. Let $\widetilde{L} = (3,2)$ and $L = (2,3,1)$. Then,

$$\hat{\beta}^7 = (;1,3;2;1,3;2) <_{\text{a}} \hat{\theta}^7 = (2,3,2,3,2,3),$$

and $\hat{\beta}^7$ has an $\widetilde{L}$-block decomposition. This example is sufficient to understand that $\hat{\beta}^n$ is a member of $\widetilde{\mathcal{E}}$ for all $n \geq 2$.

3. Let $\widetilde{L} = (3,2)$ and $L = (3,1,2)$. Then,

$$\hat{\beta}^7 = (;2;1,3;2;1,3) <_{\text{a}} \hat{\delta}^7 = (2,3,2,3,2,3),$$

and $\hat{\beta}^7$ has an $\widetilde{L}$-block decomposition. This example is sufficient to understand that $\hat{\beta}^n$ is a member of $\widetilde{\mathcal{E}}$ for all $n \geq 2$.

Let us compare the fundamental sequences of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ below. The values of α and $\widetilde{\alpha}$ that are mentioned in Theorem 7 will be identified later in Theorem 15.

Theorem 7. *There are positive real numbers α, $\widetilde{\alpha}$, ϕ, $\widetilde{\phi}$, and $r < 1$ such that*

$$H_n = \alpha\phi^{n-1} + O(\phi^{nr}), \quad \widetilde{H}_n = \widetilde{\alpha}\widetilde{\phi}^{n-1} + O(\widetilde{\phi}^{nr}), \quad \textit{and} \quad \lim_{n\to\infty} \frac{H_n}{\widetilde{H}_n^{\gamma}} = \frac{\alpha}{\widetilde{\alpha}^{\gamma}} > 0 \tag{9}$$

where $\gamma = \log_{\widetilde{\phi}} \phi$ and $1 < \phi < \widetilde{\phi}$.

Proof. Recall $L = (e_1, \ldots, e_N)$ and $\widetilde{L} = (\widetilde{e}_1, \ldots, \widetilde{e}_M)$. Let $\mathbf{b} = (e_N, \ldots, e_1)$ and $\widetilde{\mathbf{b}} = (\tilde{e}_M, \ldots, \tilde{e}_1)$, and let f and $\tilde{f}$ be the characteristic polynomials of L and $\widetilde{L}$, respectively. As explained in the paragraphs appearing below Equation (5), there are positive real numbers α, $\widetilde{\alpha}$, ϕ, $\widetilde{\phi}$, and r such that Equation (9) hold and $\phi > 1$, where ϕ and $\widetilde{\phi}$ are the dominant positive real zeros of f and $\widetilde{f}$, respectively.

Let us prove that $\widetilde{\phi} > \phi$. Since $\mathcal{E} \neq \widetilde{\mathcal{E}}$, if we duplicate the repeating blocks of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ to the length of NM as follows, Proposition 1 implies

$$\mathbf{b}' = (e_N, \ldots, e_N, \ldots, e_1, e_N, \ldots, e_1) <_{\text{a}} \widetilde{\mathbf{b}}' = (\tilde{e}_M, \ldots, \tilde{e}_M, \ldots, \tilde{e}_1, \tilde{e}_M, \ldots, \tilde{e}_1).$$

Notice that $\mathcal{E}$ is equal to the periodic Zeckendorf collection determined by the reverse of the list $\mathbf{b}'$, which is equal to $\hat{\beta}^{1+NM}$, and $\widetilde{\mathcal{E}}$ is equal to the periodic Zeckendorf collection determined by the reverse of the list $\widetilde{\mathbf{b}}'$, which is equal to $\hat{\theta}^{1+NM}$. By Proposition 1, there is a largest index $s \leq NM$ such that $\hat{\beta}^{1+NM}_s < \hat{\theta}^{1+NM}_s$. Let

$$f_*(x) := x^{NM} - (e_1 x^{NM-1} + \cdots + e_{N-1}x + (1 + e_N))$$

be the characteristic polynomial of $\text{rev}_{NM}(\mathbf{b}')$ for positive integers. Then, the following induction step shows that ϕ is a zero of f_*:

$$\phi^{kN} = e_1\phi^{kN-1} + \cdots + e_{N-1}\phi + (1 + e_N).$$

Then,

$$\phi^{kN+N} = e_1\phi^{kN-1+N} + \cdots + e_{N-1}\phi^{1+N} + (1+e_N)\phi^N,$$

which implies

$$\phi^{(k+1)N} = e_1\phi^{(k+1)N-1} + \cdots + e_{N-1}\phi^{1+N} + e_N\phi^N + e_1\phi^{N-1} + \cdots + (1+e_N).$$

Since ϕ is the only positive real root of f_*, it would be sufficient to prove that $f_*(\widetilde{\phi}) > 0$.

Notice that for any integer $t > 3$, we have $\hat{\beta}^{1+tNM}_{s+(t-1)NM} < \hat{\theta}^{1+tNM}_{s+(t-1)NM}$. For convenience, let

$$\hat{\beta} := \hat{\beta}^{1+tNM},\ \hat{\theta} := \hat{\theta}^{1+tNM},\ s_* := s + (t-1)NM, \text{ and } s_1 := 1 + (t-1)NM.$$

Let $y := \sum \mathrm{res}^{s_1}(\hat{\beta})\widetilde{H}$. Below we shall establish that $\widetilde{H}_{1+tNM} - (y + \widetilde{H}_{s_1})$ is greater than a term $\widetilde{H}_{t_*}$ where $t_* < 1 + tNM$ is sufficiently close to $1 + tNM$. Then, the asymptotic version of the inequality will imply that $f_*(\phi)$ is positive.

Notice below that if $s = 1$, i.e., $s_* = 1 + (t-1)NM = s_1$, then $\sum \mathrm{res}^{s_1}_{s_*-1}(\hat{\beta})\widetilde{H}$ is interpreted as 0;

$$\begin{aligned}
&\widetilde{H}_{(t-2)NM} + \widetilde{H}_{s_1} + y \\
&\qquad = \widetilde{H}_{(t-2)NM} + \widetilde{H}_{s_1} + \sum \mathrm{res}^{s_1}_{s_*-1}(\hat{\beta})\widetilde{H} + \sum \mathrm{res}^{s_*}(\hat{\beta})\widetilde{H} \\
&\qquad = \widetilde{H}_{(t-2)NM} + \widetilde{H}_{s_1} + \sum \mathrm{res}_{s_*-1}(\hat{\beta})\widetilde{H} - \sum \hat{\beta}^{s_1}\widetilde{H} + \sum \mathrm{res}^{s_*}(\hat{\beta})\widetilde{H}.
\end{aligned}$$

Notice that by the definition of s, the coefficient function $\mathrm{res}^{s_*}(\hat{\beta})$ is a proper $\widetilde{L}$-block, which implies that $\mathrm{res}_{s_*-1}(\hat{\beta}) \in \widetilde{\mathcal{E}}$. Thus,

$$\begin{aligned}
\widetilde{H}_{(t-2)NM} + \widetilde{H}_{s_1} + y &< \widetilde{H}_{(t-2)NM} + \widetilde{H}_{s_1} + \widetilde{H}_{s_*} - \sum \hat{\beta}^{s_1}\widetilde{H} + \sum \mathrm{res}^{s_*}(\hat{\beta})\widetilde{H} \\
&\le \widetilde{H}_{(t-2)NM} - \sum \hat{\beta}^{s_1}\widetilde{H} + \widetilde{H}_{s_1} + \sum \mathrm{res}^{s_*}(\hat{\theta})\widetilde{H}.
\end{aligned}$$

Notice that $0 < y_1 := \widetilde{H}_{(t-2)NM} - \sum \hat{\beta}^{s_1}\widetilde{H} + \widetilde{H}_{s_1} < \widetilde{H}_{s_1}$ implies that there is $\epsilon^0 \in \widetilde{\mathcal{E}}$ such that $\mathrm{ord}(\epsilon^0) \le s_1 - 1$ and $y_1 = \sum \epsilon^0 \widetilde{H}$. Let $\epsilon^0 = \sum_{m=1}^{\ell} \zeta^m$ be an $\widetilde{L}$-block decomposition such that the support interval of ζ^ℓ is $[a, s_1 - 1]$, and let us claim that $\epsilon := \epsilon^0 + \mathrm{res}^{s_*}(\hat{\theta}) \in \widetilde{\mathcal{E}}$. Let $[s_0, (t-1)NM]$ be the support interval of the proper $\widetilde{L}$-block $\mathrm{res}^{s_*}(\hat{\theta})$. Notice that the support interval of the $\widetilde{L}$-block $\mathrm{res}^{s_*}(\hat{\theta})$ may not be $[s_*, (t-1)NM]$ or $[s_* - 1, (t-1)NM]$, e.g., if $\widetilde{L} = (1, 3, 0)$ and $s_* = 2 + (t-1)NM$, then the support interval is $[s_1 - 1, (t-1)NM]$. Since $\widetilde{e}_1 = \hat{\theta}_{s_1-1} > 0$ and $\mathrm{res}^{s_*}(\hat{\theta})_{s_1-1} = 0$, we have $s_1 - 1 \le s_0$. If $s_1 \le s_0$, then ϵ is a sum of two disjoint $\widetilde{L}$-blocks. If $s_1 - 1 = s_0$, then $\hat{\theta}_{s_1} = \widetilde{e}_M$ implies that $\zeta^\ell + \mathrm{res}^{s_1}(\hat{\theta})$ forms a single $\widetilde{L}$-block. Thus, $\epsilon^0 + \mathrm{res}^{s_*}(\hat{\theta}) \in \widetilde{\mathcal{E}}$, and

$$\widetilde{H}_{(t-2)NM} + \widetilde{H}_{s_1} + y \leq \sum \epsilon^0 \widetilde{H} + \sum \mathrm{res}^{s_*}(\hat{\theta})\widetilde{H} \leq \sum \hat{\theta}\widetilde{H} < \widetilde{H}_{1+tNM}.$$

Then,

$$\widetilde{H}_{(t-2)NM} < \widetilde{H}_{tNM+1} - (y + \widetilde{H}_{s_1}),$$

which implies

$$\widetilde{\phi}^{-NM-1} \leq \widetilde{\phi}^{NM} - (e_1 \widetilde{\phi}^{NM-1} + \cdots + (1 + e_N)) + o_t(1).$$

This proves that $0 < \widetilde{\phi}^{-NM-1} \leq f_*(\widetilde{\phi})$ as $t \to \infty$. Since $f_*(\phi) = 0$, we prove that $\widetilde{\phi} > \phi$. $\square$

3.2. The Duality Formula

Lemma 2 is the specialized version of the duality formula. In this section, we introduce the general version for Zeckendorf collections for positive integers, including non-periodic ones.

Definition 20. Given a non-zero coefficient function μ with finite support that is not necessarily in $\widetilde{\mathcal{E}}$, we denote by $\check{\mu}$ the largest coefficient function in $\mathcal{E}$ that is less than μ, and we denote the immediate successor of $\check{\mu}$ in $\mathcal{E}$ by $\bar{\mu}$.

Lemma 6 (Mixed decomposition)**.** *Given $\epsilon \in \widetilde{\mathcal{E}} - \mathcal{E}$, write $\epsilon = \zeta^0 + \mu$ as an $\widetilde{L}$-decomposition such that $\mu = \sum_{m=1}^{n} \zeta^m$ is an L-block decomposition with the maximal integer $n \geq 0$. If $n \geq 1$, then let $[a, i]$ be the support interval of ζ^1, and if $n = 0$, then let $a = \mathrm{ord}(\zeta^0) + 1$. Then, $\hat{\beta}^a <_{\mathrm{a}} \zeta^0$ and $\bar{\epsilon} = \beta^a + \sum_{m=1}^{n} \zeta^m$.*

Proof. Notice that $\epsilon \notin \mathcal{E}$ implies that $a \geq 2$. Suppose that $\zeta^0 <_{\mathrm{a}} \hat{\beta}^a$. There is an index $j \leq a-1$ such that $\zeta^0_j < \hat{\beta}^a_j$ and $\zeta^0_k = \hat{\beta}^a_k$ for $j+1 \leq k \leq a-1$. Thus, $\mathrm{res}^j_{a-1}(\zeta^0)$ is a proper L-block with support interval $[j, a-1]$, and this contradicts the maximality of n. Notice that $\zeta^0 = \hat{\beta}^a$ also contradicts the maximality of n. Thus, $\hat{\beta}^a <_{\mathrm{a}} \zeta^0$, and

$$\hat{\beta}^a + \sum_{m=1}^{n} \zeta^m <_{\mathrm{a}} \zeta^0 + \sum_{m=1}^{n} \zeta^m.$$

Notice that $\check{\epsilon} = \hat{\beta}^a + \sum_{m=1}^{n} \zeta^m$, and hence, $\bar{\epsilon} = \beta^a + \sum_{m=1}^{n} \zeta^m$. $\square$

Example 4. Let $L = (1, 0)$ and $\widetilde{L} = (1, 1)$. Suppose that $\epsilon = (1, 1, 1, 0, 1, 0, 1)$, which is a member of $\widetilde{\mathcal{E}} - \mathcal{E}$. Notice that there is no L-block at index 7 in ϵ, and hence, the maximal number n described in Lemma 6 is 0, and $a = 8$. So, $\hat{\beta}^8 <_{\mathrm{a}} \epsilon$ and $\bar{\epsilon} = \beta^8$.

Definition 21. Given a positive integer x, let $R(x)$ denote the subset of non-negative integers $\sum \epsilon\widetilde{H} < x$ where $\epsilon \in \mathcal{E}$, and let $z(x)$ denote $\#R(x)$.

Lemma 7. *For each $\mu \in \widetilde{\mathcal{E}}$, we have $R(\sum \mu\widetilde{H}) = R(\sum \bar{\mu}\widetilde{H})$.*

Proof. Let $y \in R(\sum \bar{\mu}\widetilde{H})$. Then, $y = \sum \epsilon\widetilde{H}$ where $\epsilon \in \mathcal{E}$ and $\sum \epsilon\widetilde{H} < \sum \bar{\mu}\widetilde{H}$. By Theorem 5 applied to $\widetilde{\mathcal{E}}$ and the fundamental sequence $\widetilde{H}$, we have $\epsilon <_{\mathrm{a}} \bar{\mu}$, and hence, $\epsilon \leq_{\mathrm{a}} \check{\mu} <_{\mathrm{a}} \bar{\mu}$. Since $\check{\mu} <_{\mathrm{a}} \mu$, we have $\epsilon <_{\mathrm{a}} \mu$. Thus, $y = \sum \epsilon\widetilde{H} < \sum \mu\widetilde{H}$ and $y \in R(\sum \mu\widetilde{H})$.

Let $y = \sum \epsilon\widetilde{H} \in R(\sum \mu\widetilde{H})$ for some $\epsilon \in \mathcal{E}$. Then, by Theorem 5, $\epsilon <_{\mathrm{a}} \mu$, and hence, $\epsilon \leq_{\mathrm{a}} \check{\mu} < \bar{\mu}$. Thus, $y = \sum \epsilon\widetilde{H} \leq \sum \check{\mu}\widetilde{H} < \sum \bar{\mu}\widetilde{H}$, i.e., $y \in R(\sum \bar{\mu}\widetilde{H})$. □

Theorem 8 (Duality Formula)**.** *For each $\mu \in \widetilde{\mathcal{E}}$, we have $z\left(\sum \mu\widetilde{H}\right) = \sum \bar{\mu}H$.*

Proof. By Lemma 7, we have $z(\sum \mu\widetilde{H}) = z(\sum \bar{\mu}\widetilde{H})$. If Theorem 5 is applied to $\mathcal{E}$ and H, then eval_H restricts to a bijective function from $S := \{\epsilon \in \mathcal{E} : \epsilon <_{\mathrm{a}} \bar{\mu}\}$ to $T := \{n \in \mathbb{N}_0 : n < \sum \bar{\mu}H\}$. Theorem 5 can be applied to $\widetilde{\mathcal{E}}$ and $\widetilde{H}$ as well. Since $\mathcal{E}$ is a subcollection of $\widetilde{\mathcal{E}}$, $\mathrm{eval}_{\widetilde{H}}$ restricts to a bijective function from $S := \{\epsilon \in \mathcal{E} : \epsilon <_{\mathrm{a}} \bar{\mu}\}$ to $\widetilde{T} := R(\sum \bar{\mu}\widetilde{H})$. In other words, the three subsets S, T, and $\widetilde{T}$ are bijective to each other. Thus,

$$z(\sum \bar{\mu}\widetilde{H}) = \#\widetilde{T} = \#S = \#T = \sum \bar{\mu}H,$$

which proves that $z(\sum \mu\widetilde{H}) = z(\sum \bar{\mu}\widetilde{H}) = \sum \bar{\mu}H$. □

Recall the setup in Example 4. Then, $H_k = F_k$ is the fundamental sequence of $\mathcal{E}$ determined by L. By the duality formula, $z(\sum \epsilon\widetilde{H}) = \sum \bar{\epsilon}F = F_8$. However, it turns out that using $\mu = (0,1,1,0,1,0,1)$ is also convenient for this case. Notice that the linear recurrence of F implies that $\sum \mu F = \sum \bar{\epsilon}F$. This observation is formulated in Lemma 2.

3.3. Transition to Expressions for the Interval I

By the limit value in Equation (9) and the Duality Formula, the magnitude of $z(x)$ is approximately x^{γ}, and for the case of $\widetilde{L} = (B-1, \ldots, B-1)$, the same magnitude is predicted in [12]. As in [12], we define $\delta(x) := z(x)/x^{\gamma}$, and transform this ratio as a function on real numbers.

Recall from Definition 17 and Theorem 6 periodic Zeckendorf collections for $\mathbf{I}$, their maximal blocks, and their fundamental sequences.

Notation 1. For $n \in \mathbb{N}$, let $\bar{\beta}^n$ and $\bar{\theta}^n$ be the maximal L^*-block and $\widetilde{L}^*$-block at index n, respectively, and let $\widetilde{\omega} = \widetilde{\phi}^{-1}$ and $\omega = \phi^{-1}$. Let Q and $\widetilde{Q}$ denote the fundamental sequences of $\mathcal{E}^*$ and $\widetilde{\mathcal{E}}^*$.

Recall Definition 20. For $\epsilon \in \widetilde{\mathcal{E}}$ of order $n \geq 1$, we have $\bar{\epsilon} \in \mathcal{E}$. Suppose that $\bar{\epsilon} \neq \beta^{n+1}$, and recall the function rev_n from Definition 5. Then, Equation (9) imply

$$\begin{aligned}\delta(x) &= \frac{\sum \bar{\epsilon} H}{(\sum \epsilon \widetilde{H})^\gamma} = \frac{\sum \bar{\epsilon}_k (\alpha \phi^{k-1} + O(\phi^{rk}))}{(\sum \epsilon_k (\widetilde{\alpha} \widetilde{\phi}^{k-1} + O(\widetilde{\phi}^{rk})))^\gamma} \quad \text{where } r < 1 \text{ and } \textstyle\sum = \sum_{k=1}^n \\ &= \frac{\left(\sum \bar{\epsilon}_k \alpha \phi^{k-1}\right) + O(\phi^{rn})}{\left(\left(\sum \epsilon_k \widetilde{\alpha} \widetilde{\phi}^{k-1}\right) + O(\widetilde{\phi}^{rn})\right)^\gamma} = \frac{\alpha \phi^n}{\widetilde{\alpha}^\gamma \widetilde{\phi}^{\gamma n}} \cdot \frac{\sum \bar{\epsilon}_k \phi^{k-n-1} + O(\phi^{rn-n})}{\left(\sum \epsilon_k \widetilde{\phi}^{k-n-1} + O(\widetilde{\phi}^{rn-n})\right)^\gamma} \\ &= \frac{\alpha}{\widetilde{\alpha}^\gamma} \cdot \frac{\sum_{t=1}^n \bar{\epsilon}_{n+1-t} \omega^t + O(\omega^{(1-r)n})}{\left(\sum_{t=1}^n \epsilon_{n+1-t} \widetilde{\omega}^t + O(\widetilde{\omega}^{(1-r)n})\right)^\gamma} = \frac{\alpha}{\widetilde{\alpha}^\gamma} \cdot \frac{\sum \mathrm{rev}_n(\bar{\epsilon}) Q + O(\omega^n)}{\left(\sum \mathrm{rev}_n(\epsilon) \widetilde{Q} + O(\widetilde{\omega}^n)\right)^\gamma}. \qquad (10)\end{aligned}$$

If $\bar{\epsilon} = \beta^{n+1}$, then

$$\delta(x) = \frac{\alpha}{\widetilde{\alpha}^\gamma} \cdot \frac{1 + O(\omega^{(1-r)n})}{\left(\sum_{t=1}^n \epsilon_{n+1-t} \widetilde{\omega}^t + O(\widetilde{\omega}^{(1-r)n})\right)^\gamma} = \frac{\alpha}{\widetilde{\alpha}^\gamma} \cdot \frac{1 + O(\omega^n)}{\left(\sum \mathrm{rev}_n(\epsilon) \widetilde{Q} + O(\widetilde{\omega}^n)\right)^\gamma}. \qquad (11)$$

Notice that $n \to \infty$ as $x \to \infty$, and hence, $O(\omega^n) = o(1) \to 0$. Thus, as $x \to \infty$, the ratio in Equation (10) approaches a value depending only on the real number $\mathrm{rev}_n(\epsilon)\widetilde{Q} \in \mathbf{I}$. This observation motivates us to define $\delta(x)$ as a function on $\mathbf{I}$, and it is introduced in Definition 23 and Proposition 2 below.

For that goal, we need to understand the relationship between $\mathrm{rev}_n(\epsilon)$ and $\mathrm{rev}_n(\bar{\epsilon})$ as coefficient functions in $\widetilde{\mathcal{E}}^*$, which appear in Equation (10). We begin with the version of Lemma 6 for $\widetilde{\mathcal{E}}^*$, which is the main content of Section 3.3.1, and the relationship between the coefficient functions is explained in Definition 23 in Section 3.3.1.

3.3.1. Mixed Decomposition

Recall from Notation 1 that $\bar{\beta}^n$ for $n \geq 1$ denotes the maximal L^*-blocks of $\mathcal{E}^*$.

Lemma 8. *Let ϵ be a coefficient function not necessarily in $\widetilde{\mathcal{E}}^*$ such that $\mathrm{res}_b(\epsilon) = 0$ for an integer $b \geq 0$. Then, $\epsilon <_{\mathrm{d}} \bar{\beta}^{b+1}$ if and only if $\mathrm{res}_n(\epsilon)$ is a proper L^*-block with support interval $[b+1, n]$ for some $n \geq b+1$.*

Proof. Suppose that $\epsilon <_{\mathrm{d}} \bar{\beta}^{b+1}$ for an integer $b \geq 0$. By the definition of the inequality, there is an index $n \geq b+1$ such that $\mathrm{res}_{n-1}(\epsilon) = \mathrm{res}_{n-1}(\bar{\beta}^{b+1})$ and $\epsilon_n < \bar{\beta}^{b+1}_n$, so $\mathrm{res}_n(\epsilon)$ is a proper L^*-block with support interval $[b+1, n]$.

Suppose that $\mathrm{res}_n(\epsilon)$ is a proper L^*-block with support interval $[b+1, n]$ for some $n \geq b+1$. By the definition of a proper L^*-block with support interval $[b+1, n]$, we have $\mathrm{res}_n(\epsilon) <_{\mathrm{d}} \bar{\beta}^{b+1}$, $\mathrm{res}_n(\epsilon)_n < \bar{\beta}^{b+1}_n$, and hence, $\epsilon <_{\mathrm{d}} \bar{\beta}^{b+1}$. □

For the remainder of the paper, we define $\bar{\beta}^\infty := 0$ for convenience.

Corollary 1. *Let $\epsilon \in \widetilde{\mathcal{E}}^*$. Then, there is a (unique) largest integer $\ell \geq 0$ or $\ell = \infty$ such that $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \mu$ and $\sum_{m=1}^{\ell} \zeta^m$ is a proper L^*-block decomposition where* *the support interval of* ζ^ℓ *is* $[i,b]$, $\mathrm{res}_b(\mu)=0$, *and* $\bar\beta^{b+1} \le_{\mathrm{d}} \mu$*; if* $\ell=0$*, then* $b=0$ *and* $\epsilon=\mu$*, and if* $\ell=\infty$*, then* $\mu=0$ *and* $\epsilon\in\mathcal{E}^*$*.*

Proof. Suppose that $\epsilon \in \mathcal{E}^*$. Then, we have the L^*-block decomposition $\epsilon = \sum_{m=1}^{\infty}\zeta^m$, and this corresponds to the case $\ell=\infty$.

Suppose that $\epsilon \notin \mathcal{E}^*$. Then, there must be a unique largest integer $\ell \ge 0$ such that $\sum_{m=1}^{\ell}\zeta^m$ is a proper L^*-block decomposition where $[i,b]$ is the support interval of ζ^ℓ, $\epsilon=\sum_{m=1}^{\ell}\zeta^m+\mu$, and $\mathrm{res}_b(\mu)=0$. As in Lemma 6, the maximality of ℓ implies that $\mathrm{res}_n(\mu)$ is not a proper L^*-block for any $n \ge b+1$, and hence, by Lemma 8, we find $\bar\beta^{b+1}\le_{\mathrm{d}}\mu$. □

Definition 22. Given $\epsilon\in\widetilde{\mathcal{E}}^*$, an expansion $\epsilon=\sum_{m=1}^{\ell}\zeta^m+\mu$ where ζ^m, ℓ, and μ are as described in Corollary 1 is called *an* $\overline{L^*}$*-block decomposition of a member of* $\widetilde{\mathcal{E}}^*$ where ζ^ℓ is uniquely determined, and if $[i,b]$ is the support interval of ζ^ℓ, the index b is called *the largest* $\overline{L^*}$*-support index of* ϵ; $b=\infty$ if $\epsilon\in\mathcal{E}^*$, and $b=0$ if $\epsilon=\mu$. Let $\overline{\mathcal{E}^*}$ denote the union of $\mathcal{E}^*$ and the set of coefficient functions $\sum_{m=1}^{\ell}\zeta^m+\bar\beta^{b+1}$ where ζ^m and $\mu=\bar\beta^{b+1}$ are as described in Corollary 1.

The following lemma and corollary show that an $\widetilde{L}^*$-block decomposition of an $\overline{L^*}$-block decomposition maintains the boundaries of the support intervals of the proper L^*-blocks.

Lemma 9. *Let* ζ *be a non-zero proper* L^**-block with support interval* $[i,b]$*. Then, there is an* $\widetilde{L}^*$*-block decomposition* $\zeta=\sum_{m=1}^{t}\eta^m$ *such that the support interval of* η^t *is* $[a,b]$*.*

Proof. Let ζ be a non-zero proper L^*-block with support interval $[i,b]$. Let $\bar\beta^i=\sum_{m=1}^{\infty}\eta^m$ be the full $\widetilde{L}^*$-block decomposition where $[i_m,c_m]$ is the support interval of η^m, so b is contained in a unique interval $[i_t,c_t]$. Notice that

$$\mathrm{res}_{b-1}(\zeta)=\mathrm{res}_{b-1}(\bar\beta^i) \text{ and } \zeta_b<\bar\beta^i_b=\eta^{i_t}_b\le\bar\theta^{i_t}_b.$$

It follows that $\zeta_b<\bar\theta^{i_t}_b$, and hence, $\eta^0:=\mathrm{res}^{i_t}_b(\zeta)$ is a proper $\widetilde{L}^*$-block with support interval $[i_t,b]$. Thus, $\zeta=\sum_{m=1}^{t-1}\eta^m+\eta^0$ is an $\widetilde{L}^*$-block decomposition. □

Corollary 2. *Let* $\epsilon=\sum_{m=1}^{\ell}\zeta^m+\mu$ *be an* $\overline{L^*}$*-block decomposition of* $\epsilon\in\widetilde{\mathcal{E}}^*$*. Then,* $\mu\in\widetilde{\mathcal{E}}^*$*.*

Proof. If $[i,b]$ is the support interval of ζ^ℓ, by Lemma 9, the (disjoint) union of the support intervals of an $\widetilde{L}^*$-block decomposition of $\sum_{m=1}^{\ell}\zeta^m$ is contained in $[1,b]$. Thus, μ has an $\widetilde{L}^*$-block decomposition, and hence, it is a member of $\widetilde{\mathcal{E}}^*$. □

3.3.2. Transition to a Function on I

Given $\epsilon \in \widetilde{\mathcal{E}}^*$ and an integer $n \geq 1$, notice that $\mathrm{rev}_n(\epsilon) \in \widetilde{\mathcal{E}}$ and $\overline{\mathrm{rev}_n(\epsilon)} \in \mathcal{E}$, which is defined in Definition 20.

Lemma 10. *Suppose that $\epsilon \in \widetilde{\mathcal{E}}^*$ and $\epsilon <_{\mathrm{d}} \bar{\beta}^1$. If $\epsilon \in \mathcal{E}^*$, then*

$$\mathrm{rev}_n\left(\overline{\mathrm{rev}_n(\epsilon)}\right) = \mathrm{res}_n(\epsilon).$$

If $\epsilon \in \widetilde{\mathcal{E}}^* - \mathcal{E}^*$, i.e., $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \mu$ is an $\overline{L^*}$-block decomposition with the largest $\overline{L^*}$-support index $b < \infty$ where $\ell \geq 1$ and $b \geq 1$, then for sufficiently large n,

$$\mathrm{rev}_n\left(\overline{\mathrm{rev}_n(\epsilon)}\right) = \begin{cases} \sum_{m=1}^{\ell} \zeta^m + \beta^b & \text{if } \bar{\beta}^{b+1} <_{\mathrm{d}} \mu, \\ \mathrm{res}_n(\epsilon) & \text{if } \bar{\beta}^{b+1} = \mu. \end{cases}$$

Proof. If $\epsilon \in \mathcal{E}^*$, then $\mathrm{rev}_n(\epsilon) \in \mathcal{E}$, so

$$\mathrm{rev}_n\left(\overline{\mathrm{rev}_n(\epsilon)}\right) = \mathrm{rev}_n\left(\mathrm{rev}_n(\epsilon)\right) = \mathrm{res}_n(\epsilon).$$

Suppose that $\epsilon \in \widetilde{\mathcal{E}}^* - \mathcal{E}^*$, so we have the $\overline{L^*}$-block decomposition as described in the lemma, and notice that, by Lemma 8, $\epsilon <_{\mathrm{d}} \bar{\beta}^1$ implies $b \geq 1$. Let $n \geq b+1$. If $\bar{\beta}^{b+1} <_{\mathrm{d}} \mu$, then there is a smallest index $p \geq b+1$ such that $\bar{\beta}^{b+1}_p < \mu_p$. So, if $n \geq p$, then

$$\mathrm{rev}_n(\sum_{m=1}^{\ell} \zeta^m + \bar{\beta}^{b+1}) <_{\mathrm{a}} \mathrm{rev}_n(\sum_{m=1}^{\ell} \zeta^m + \mu),$$

and by applying Lemma 6 to $\mathrm{rev}_n(\epsilon) = \mathrm{rev}_n(\sum_{m=1}^{\ell} \zeta^m + \mu)$, we have

$$\overline{\mathrm{rev}_n(\epsilon)} = \mathrm{rev}_n\left(\sum_{m=1}^{\ell} \zeta^m + \beta^b\right)$$

which implies that

$$\mathrm{rev}_n\left(\overline{\mathrm{rev}_n(\epsilon)}\right) = \mathrm{res}_n(\sum_{m=1}^{\ell} \zeta^m + \beta^b) = \sum_{m=1}^{\ell} \zeta^m + \beta^b.$$

If $\bar{\beta}^{b+1} = \mu$, then

$$\mathrm{rev}_n(\epsilon) = \mathrm{rev}_n(\sum_{m=1}^{\ell} \zeta^m + \bar{\beta}^{b+1}) \in \mathcal{E},$$

and hence, $\overline{\mathrm{rev}_n(\epsilon)} = \mathrm{rev}_n(\epsilon)$. Thus, $\mathrm{rev}_n\left(\overline{\mathrm{rev}_n(\epsilon)}\right) = \mathrm{rev}_n\left(\mathrm{rev}_n(\epsilon)\right) = \mathrm{res}_n(\epsilon)$. $\square$

Let us use Lemma 10 to define $\overline{\text{rev}}(\epsilon)$ for $\epsilon \in \widetilde{\mathcal{E}}^*$, which is a version of Lemma 6 for $\mathbf{I}$.

Definition 23. Given $\epsilon \in \widetilde{\mathcal{E}}^*$, if $\bar\beta^1 \le_{\text{d}} \epsilon$, then define $\overline{\text{rev}}(\epsilon) := \bar\beta^1$, and if $\epsilon <_{\text{d}} \bar\beta^1$, define

$$\overline{\text{rev}}(\epsilon) := \lim_{n\to\infty} \text{rev}_n\left(\overline{\text{rev}_n(\epsilon)}\right).$$

Let $\delta_1^* : [\widetilde\omega, 1) \to \mathbb{R}$ be the function given by

$$\delta_1^*(\textstyle\sum \epsilon\widetilde{Q}) := \dfrac{\sum \overline{\text{rev}}(\epsilon) Q}{\left(\sum \epsilon\widetilde{Q}\right)^\gamma} \quad \text{if } \widetilde\omega \le \sum \epsilon\widetilde{Q} < 1 \text{ for } \epsilon \in \widetilde{\mathcal{E}}^*.$$

Given a real number $y \in \mathbf{I}$, there is a unique positive integer n such that $y \in [\omega^n, \omega^{n-1})$. Let $\delta^* : \mathbf{I} \to \mathbb{R}$ be the function given by $\delta^*(y) = \delta_1^*(y/\omega^{n-1})$ where n is defined as above. We also denote $\delta^*(\sum \epsilon\widetilde{Q})$ simply by $\delta^*[\epsilon]$ for all $\epsilon \in \widetilde{\mathcal{E}}^*$.

If $\epsilon <_{\text{d}} \bar\beta^1$ and $\epsilon \in \widetilde{\mathcal{E}}^*$, then by Lemma 10, the limit considered in Definition 23 is well-defined. By Theorem 6, given a real number $y \in [\omega, 1)$, there is a unique $\epsilon \in \widetilde{\mathcal{E}}^*$ such that $y = \sum \epsilon\widetilde{Q}$, so δ_1^* is well-defined.

Recall that $\delta(x) = z(x)/x^\gamma$. In Proposition 2, we compare the values of δ with those of δ^*, and we begin with two preliminary lemmas:

Lemma 11. *Let $\epsilon \in \widetilde{\mathcal{E}}^*$. Then, $\epsilon \in \overline{\mathcal{E}^*}$ if and only if $\overline{\text{rev}}(\epsilon) = \epsilon$.*

Proof. Suppose that $\bar\beta^1 \le_{\text{d}} \epsilon$. Then, by Lemma 8 and Corollary 1, $\epsilon \in \overline{\mathcal{E}^*}$ if and only if $\epsilon = \bar\beta^1$. If $\epsilon \in \overline{\mathcal{E}^*}$, then $\epsilon = \bar\beta^1$, and by Definition 23, $\overline{\text{rev}}(\epsilon) = \epsilon$. If $\epsilon \notin \overline{\mathcal{E}^*}$, then $\bar\beta^1 <_{\text{d}} \epsilon$ by Corollary 1, and by Definition 23, $\overline{\text{rev}}(\epsilon) = \bar\beta^1 \ne \epsilon$.

Suppose that $\epsilon <_{\text{d}} \bar\beta^1$. If $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$, then by Corollary 1, we have an $\overline{L^*}$-block decomposition $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \bar\beta^{b+1}$ where b is the largest $\overline{L^*}$-support index of ϵ (see Definition 22), and by Lemma 8, $\epsilon <_{\text{d}} \bar\beta^1$ implies that $b \ge 1$. If $\epsilon \in \overline{\mathcal{E}^*}$, i.e., $\epsilon \in \mathcal{E}^*$ or $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \bar\beta^{b+1}$, then $1 \le \ell \le \infty$ by Corollary 1, and by Lemma 10, $\text{rev}_n\left(\overline{\text{rev}_n(\epsilon)}\right) = \text{res}_n(\epsilon)$ for sufficiently large n, which implies that $\overline{\text{rev}}(\epsilon) = \epsilon$. If $\epsilon \notin \overline{\mathcal{E}^*}$, i.e., $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \mu$ where $\bar\beta^{b+1} <_{\text{d}} \mu$, then by Lemma 10, $\text{rev}_n\left(\overline{\text{rev}_n(\epsilon)}\right) = \sum_{m=1}^{\ell} \zeta^m + \beta^b$ for sufficiently large n. Thus,

$$\overline{\text{rev}}(\epsilon) = \sum_{m=1}^{\ell} \zeta^m + \beta^b >_{\text{d}} \epsilon;$$

in particular, $\overline{\text{rev}}(\epsilon) \ne \epsilon$. □

In Lemma 12, we show that $\overline{\text{rev}}$ extends the map $\text{rev}_n(\epsilon) \mapsto \text{rev}_n(\bar\epsilon)$ appearing in Equation (10). Recall Definition 20.

Lemma 12. *Suppose that* $\mu \in \widetilde{\mathcal{E}}$*, and* $t := \operatorname{ord}(\mu) > 0$*. If* $\mu \leq_{\mathrm{a}} \hat{\beta}^{t+1}$*, then* $\overline{\operatorname{rev}}(\operatorname{rev}_t(\mu)) = \operatorname{rev}_t(\bar{\mu})$*. If* $\hat{\beta}^{t+1} <_{\mathrm{a}} \mu$*, then* $\overline{\operatorname{rev}}(\operatorname{rev}_t(\mu)) = \bar{\beta}^1$*.*

Proof. Suppose that $\mu \leq_{\mathrm{a}} \hat{\beta}^{t+1}$. If $\mu \in \mathcal{E}$, then $\bar{\mu} = \mu$, and by Lemma 3, $\operatorname{rev}_t(\mu) \in \mathcal{E}^*$. Thus, by Lemma 11, $\overline{\operatorname{rev}}(\operatorname{rev}_t(\mu)) = \operatorname{rev}_t(\mu) = \operatorname{rev}_t(\bar{\mu})$.

Suppose that $\mu \in \widetilde{\mathcal{E}} - \mathcal{E}$ and $\mu <_{\mathrm{a}} \hat{\beta}^{t+1}$. By Lemma 6, $\mu = \zeta^0 + \sum_{m=1}^n \zeta^m$ where ζ^0 and ζ^m for $1 \leq m \leq n$ are as defined in the lemma, and $[a, i]$ is the support interval of ζ^1, so that $\hat{\beta}^{a-1} <_{\mathrm{a}} \zeta^0$ and $n \geq 1$. Then, $\bar{\mu} = \beta^a + \sum_{m=1}^n \zeta^m$. Let $\epsilon := \operatorname{rev}_t(\mu) \in \widetilde{\mathcal{E}}^*$. Then, $\epsilon \in \widetilde{\mathcal{E}}^* - \mathcal{E}^*$ and $\epsilon <_{\mathrm{d}} \bar{\beta}^t \leq_{\mathrm{d}} \bar{\beta}^1$. By Lemma 10, we have

$$\operatorname{rev}_n\left(\overline{\operatorname{rev}_n(\epsilon)}\right) = \operatorname{rev}_t\left(\beta^a + \sum_{m=1}^{n} \zeta^m\right) = \operatorname{rev}_t(\bar{\mu})$$

for sufficiently large n. Thus, $\overline{\operatorname{rev}}(\epsilon) = \operatorname{rev}_t(\bar{\mu})$, i.e., $\overline{\operatorname{rev}}(\operatorname{rev}_t(\mu)) = \operatorname{rev}_t(\bar{\mu})$.

Suppose that $\hat{\beta}^{t+1} <_{\mathrm{a}} \mu$. Then, there is an index $1 \leq p \leq t$ such that $\hat{\beta}^{t+1}_p < \mu_p$ and $\operatorname{res}^{p+1}(\hat{\beta}^{t+1}) = \operatorname{res}^{p+1}(\mu)$. Thus, $\operatorname{res}_t(\bar{\beta}^1) <_{\mathrm{d}} \operatorname{rev}_t(\mu)$, and hence, $\bar{\beta}^1 <_{\mathrm{d}} \operatorname{rev}_t(\mu)$. By definition, $\overline{\operatorname{rev}}(\operatorname{rev}_t(\mu)) = \bar{\beta}^1$. □

Proposition 2. *Given a positive integer* x*, let* $x = \sum \mu \widetilde{H}$ *where* $\mu \in \widetilde{\mathcal{E}}$*, and let* $n := \operatorname{ord}(\mu)$*. Then,* $\delta(x) \sim \delta^*(\sum \operatorname{rev}_n(\mu)\widetilde{Q})\ \alpha/\widetilde{\alpha}^\gamma$ *as* $x \to \infty$*.*

Proof. If $\mu \leq_{\mathrm{a}} \hat{\beta}^{n+1}$, then by Equation (10) and Lemma 12,

$$\begin{aligned}\delta(x) &= \frac{\alpha}{\widetilde{\alpha}^\gamma} \cdot \frac{\sum \operatorname{rev}_n(\bar{\mu})Q + O(\omega^n)}{\left(\sum \operatorname{rev}_n(\mu)\widetilde{Q} + O(\widetilde{\omega}^n)\right)^\gamma} \\ &\sim \frac{\alpha}{\widetilde{\alpha}^\gamma} \cdot \frac{\sum \overline{\operatorname{rev}}(\operatorname{rev}_n(\mu))Q}{\left(\sum \operatorname{rev}_n(\mu)\widetilde{Q}\right)^\gamma} = \delta^*(\sum \operatorname{rev}_n(\mu)\widetilde{Q})\, \frac{\alpha}{\widetilde{\alpha}^\gamma}.\end{aligned}$$

If $\hat{\beta}^{n+1} <_{\mathrm{a}} \mu$, then $\bar{\mu} = \beta^{n+1}$, and by Lemma 12 and Equation (6),

$$\delta^*(\sum \operatorname{rev}_n(\mu)\widetilde{Q}) = \frac{\sum \overline{\operatorname{rev}}(\operatorname{rev}_n(\mu))Q}{\left(\sum \operatorname{rev}_n(\mu)\widetilde{Q}\right)^\gamma} = \frac{\sum \bar{\beta}^1 Q}{\left(\sum \operatorname{rev}_n(\mu)\widetilde{Q}\right)^\gamma} = \frac{1}{\left(\sum \operatorname{rev}_n(\mu)\widetilde{Q}\right)^\gamma}.$$

By Equation (11), the quotient of $\delta(x)$ and $\delta^*(\sum \operatorname{rev}_n(\mu)\widetilde{Q})\ \alpha/\widetilde{\alpha}^\gamma$ approaches 1. □

Notice that for large positive integers $x = \sum \mu \widetilde{H}$ and $\operatorname{ord}(\mu) = n$, i.e.,

$$x = \widetilde{\alpha} \sum_{k=1}^{n} \mu_k \widetilde{\phi}^{k-1} + O(\widetilde{\phi}^{nr})$$

for $0 < r < 1$, we have $\sum \operatorname{rev}_n(\mu)\widetilde{Q} \sim \widetilde{\phi}^{\{\log_{\widetilde{\phi}}(x/\widetilde{\alpha})\}-1}$. So, we may state the asymptotic relation in Proposition 2 as follows:

$$\delta(x) \sim \frac{\alpha}{\widetilde{\alpha}^\gamma}\, \delta^*\left(\widetilde{\phi}^{\{\log_{\widetilde{\phi}}(x/\widetilde{\alpha})\}-1}\right). \tag{12}$$

4. Proof of the Main Results

We shall prove the main results in this section. Assume the notation and context of Section 3. Recall from Theorem 7 that $\widetilde{\omega} < \omega$ and $\gamma = \log_{\widetilde{\phi}} \phi < 1$, and from Theorem 6, recall the evaluation map $\operatorname{eval}_{\widetilde{Q}} : \widetilde{\mathcal{E}}^* \to \mathbf{I}$, which is increasing and bijective. Also recall the subcollection $\overline{\mathcal{E}^*}$ of $\widetilde{\mathcal{E}}^*$ from Definition 22. For the remainder of the paper, we use the following notation.

Notation 2. Let $\mathcal{E}^*_\circ$ denote the subset of $\mathcal{E}^*$ consisting of coefficient functions that have finite support, and let U denote the image $\operatorname{eval}_{\widetilde{Q}}(\widetilde{\mathcal{E}}^* - \overline{\mathcal{E}^*})$ in $\mathbf{I}$.

4.1. Continuity and Differentiability

For the case of $\widetilde{L} = (B-1, \ldots, B-1)$, the asymptotic formula introduced in [12] implies that δ^* is continuous. For the general cases considered in this paper, δ^* is continuous as well, and differentiable precisely on U which has Lesbegue measure 1. Let us prove these facts.

4.1.1. Continuity

Let us first understand the behavior of $\overline{\operatorname{rev}}$ in the "neighborhood" of $\epsilon \in \widetilde{\mathcal{E}}^*$. Recall $\inf_{\widetilde{\mathcal{E}}^*}$ from Definition 19, and let $\inf := \inf_{\widetilde{\mathcal{E}}^*}$. For example, $\inf(\beta^1) = \bar{\theta}^2$, and by Equation (6), we have

$$\sum \beta^1 \widetilde{Q} = \sum \inf(\beta^1) \widetilde{Q} = \sum \bar{\theta}^2 \widetilde{Q}.$$

Lemma 13. *Let $x = \sum \epsilon \widetilde{Q} > 0$ for $\epsilon \in \widetilde{\mathcal{E}}^*$, and let Δx be a sufficiently small positive real number. Let $x + \Delta x = \sum \epsilon^+ \widetilde{Q}$ and $x - \Delta x = \sum \epsilon^- \widetilde{Q}$ where $\{\epsilon^+, \epsilon^-\} \subset \widetilde{\mathcal{E}}^*$. For the following statements, n represents an index determined by Δx such that $n \to \infty$ as $\Delta x \to 0$. If $\epsilon \in \widetilde{\mathcal{E}}^* - \overline{\mathcal{E}^*}$, then*

$$\overline{\operatorname{rev}}(\epsilon) = \overline{\operatorname{rev}}(\epsilon^+) = \overline{\operatorname{rev}}(\epsilon^-).$$

If $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$, then

$$\overline{\operatorname{rev}}(\epsilon) = \inf_{\mathcal{E}^*}(\overline{\operatorname{rev}}(\epsilon^+)), \text{ and } \operatorname{res}_n(\overline{\operatorname{rev}}(\epsilon)) = \operatorname{res}_n(\overline{\operatorname{rev}}(\epsilon^-)).$$

If $\epsilon \in \mathcal{E}^*$, then

$$\operatorname{res}_n(\overline{\operatorname{rev}}(\epsilon)) = \operatorname{res}_n(\overline{\operatorname{rev}}(\epsilon^+)) = \operatorname{res}_n(\overline{\operatorname{rev}}(\epsilon^-)).$$

Proof. Let $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \mu$ be an $\overline{L^*}$-block decomposition with the largest L^*-block support index b. Recall that, by Lemma 5, there is $n \in \mathbb{N}$ determined by the sufficiently small Δx such that $\operatorname{res}_n(\epsilon) = \operatorname{res}_n(\epsilon^+)$ and $\operatorname{res}_n(\inf(\epsilon)) = \operatorname{res}_n(\epsilon^-)$, and that $n \to \infty$ as $\Delta x \to 0$. Also recall the meaning of L^*- or $\widetilde{L}^*$-decompositions from Definition 17. For the remainder of the proof, all the expressions of ϵ, ϵ^+, ϵ^-, $\inf(\epsilon)$, and $\inf(\epsilon^+)$ are $\widetilde{L}^*$-decompositions unless specified differently.

Suppose that $\epsilon \in \widetilde{\mathcal{E}}^* - \overline{\mathcal{E}^*}$, i.e., $\bar{\beta}^{b+1} <_{\mathrm{d}} \mu$. By Theorem 6,

$$\sum_{m=1}^{\ell} \zeta^m + \bar{\beta}^{b+1} <_{\mathrm{d}} \epsilon^- <_{\mathrm{d}} \epsilon <_{\mathrm{d}} \epsilon^+$$

for sufficiently small Δx. By Lemma 5, we have $\epsilon^+ = \sum_{m=1}^{\ell} \zeta^m + \tau^+$. Then, the earlier inequalities and the lexicographical order imply $\bar{\beta}^{b+1} <_{\mathrm{d}} \tau^+$, and also $\epsilon^- = \sum_{m=1}^{\ell} \zeta^m + \tau^-$ where $\bar{\beta}^{b+1} <_{\mathrm{d}} \tau^-$. Thus, both expressions are $\overline{L^*}$-block decompositions, and by Lemma 10,

$$\overline{\mathrm{rev}}(\epsilon^+) = \overline{\mathrm{rev}}(\epsilon^-) = \overline{\mathrm{rev}}(\epsilon) = \sum_{m=1}^{\ell} \zeta^m + \beta^b.$$

Suppose that $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$, i.e., $\mu = \bar{\beta}^{b+1}$. By Lemma 5, $\epsilon^+ = \sum_{m=1}^{\ell} \zeta^m + \tau^+$ where $\bar{\beta}^{b+1} <_{\mathrm{d}} \tau^+$. Since ϵ does not have finite support, we have $\inf(\epsilon) = \epsilon$, and by Lemma 5, $\epsilon^- = \sum_{m=1}^{\ell} \zeta^m + \tau^-$ where $\tau^- <_{\mathrm{d}} \bar{\beta}^{b+1}$. Since $\bar{\beta}^{b+1} <_{\mathrm{d}} \tau^+$, as in the earlier case,

$$\overline{\mathrm{rev}}(\epsilon^+) = \sum_{m=1}^{\ell} \zeta^m + \beta^b, \text{ and } \inf_{\mathcal{E}^*}(\overline{\mathrm{rev}}(\epsilon^+)) = \epsilon = \overline{\mathrm{rev}}(\epsilon).$$

Notice that, by Lemma 8, $\tau^- <_{\mathrm{d}} \bar{\beta}^{b+1}$ implies that τ^- has an $\overline{L^*}$-block decomposition $\sum_{m=\ell+1}^{s} \zeta^m + \mu^-$ where $\ell+1 \le s \le \infty$, and let $[b+1, c]$ be the support interval of the proper L^*-block $\zeta^{\ell+1}$. From Lemma 5, we have

$$\mathrm{res}_n(\inf(\epsilon)) = \mathrm{res}_n(\epsilon) = \mathrm{res}_n(\epsilon^-)$$

for $n \to \infty$, which implies that $\mathrm{res}_n(\bar{\beta}^{b+1}) = \mathrm{res}_n(\tau^-)$. Since we have $\zeta_c^{\ell+1} < \bar{\beta}_c^{b+1}$ from $\zeta^{\ell+1}$ being a proper L^*-block at index $b+1$, we have $c > n$, and hence, $c \to \infty$ as $n \to \infty$. By considering all cases of μ^- for Lemma 10, we have

$$\mathrm{res}_{c-1}(\overline{\mathrm{rev}}(\epsilon^-)) = \mathrm{res}_{c-1}(\epsilon) = \mathrm{res}_{c-1}(\overline{\mathrm{rev}}(\epsilon)).$$

Suppose that $\epsilon \in \mathcal{E}^* - \mathcal{E}^*_\circ$, i.e., $\mu = 0$ and $\ell = \infty$. We have $\inf(\epsilon) = \epsilon$, and by Lemma 5,

$$\epsilon^+ = \sum_{m=1}^{T} \zeta^m + \tau^+ \text{ and } \epsilon^- = \sum_{m=1}^{T} \zeta^m + \tau^-$$

where $T \to \infty$ as $\Delta x \to 0$. For all cases of τ^+ and τ^- for Lemma 10, Corollary 2 implies

$$\overline{\mathrm{rev}}(\epsilon^+) = \sum_{m=1}^{T-1} \zeta^m + \eta^+ \text{ and } \overline{\mathrm{rev}}(\epsilon^-) = \sum_{m=1}^{T-1} \zeta^m + \eta^-$$

are $\widetilde{L}^*$-decompositions. Since $T \to \infty$, we have

$$\mathrm{res}_n(\overline{\mathrm{rev}}(\epsilon)) = \mathrm{res}_n(\overline{\mathrm{rev}}(\epsilon^+)) = \mathrm{res}_n(\overline{\mathrm{rev}}(\epsilon^-)).$$

Suppose that $\epsilon \in \mathcal{E}^*_\circ$, i.e., $\mu = 0$, $\ell = \infty$, and $\zeta^m = 0$ for all sufficiently large m. Let T be the largest index such that $\zeta^T \neq 0$, and let c be the largest index such that $\zeta^T_c \geq 1$. By Lemma 5, we have

$$\epsilon^+ = \sum_{m=1}^{T} \zeta^m + \tau^+ \text{ and } \epsilon^- = \sum_{m=1}^{T} \zeta^m - \beta^c + \tau^-$$

where there is a sufficiently large $n \in \mathbb{N}$ such that $\mathrm{res}_n(\tau^+) = 0$, and $\mathrm{res}_n(\tau^-) = \mathrm{res}_n(\bar{\theta}^{c+1})$, which implies $\bar{\beta}^{c+1} <_{\mathrm{d}} \tau^- <_{\mathrm{d}} \beta^c$. For all cases of τ^+ and τ^-, by Corollary 2 and Lemma 10,

$$\overline{\mathrm{rev}}(\epsilon^+) = \sum_{m=1}^{T} \zeta^m + \overline{\mathrm{rev}}(\tau^+) \text{ and } \overline{\mathrm{rev}}(\epsilon^-) = \sum_{m=1}^{T} \zeta^m - \beta^c + \beta^c = \epsilon.$$

By Lemma 10 applied to all cases of τ^+, there is a sufficiently large $n \in \mathbb{N}$ such that

$$\mathrm{res}_n(\overline{\mathrm{rev}}(\epsilon^+)) = \sum_{m=1}^{T} \zeta^m = \epsilon = \overline{\mathrm{rev}}(\epsilon) = \mathrm{res}_n(\overline{\mathrm{rev}}(\epsilon))$$

and $\overline{\mathrm{rev}}(\epsilon^-) = \epsilon = \mathrm{res}_n(\overline{\mathrm{rev}}(\epsilon))$. This proves the assertion. □

Recall the subset U from Notation 2.

Theorem 9. *The function δ^* is continuous, and locally decreasing and differentiable on U.*

Proof. Let us prove that δ^*_1, defined in Definition 23, is continuous. Let $\epsilon \in \widetilde{\mathcal{E}}^*$ such that $\beta^1 <_{\mathrm{d}} \epsilon$, and let $x = \sum \epsilon \widetilde{Q}$ and $y = \sum \overline{\mathrm{rev}}(\epsilon) Q$. Let Δx, ϵ^+, and ϵ^- be as defined in Lemma 13.

Suppose that $\epsilon \in \widetilde{\mathcal{E}}^* - \overline{\mathcal{E}^*}$. Then, by Lemma 13,

$$\delta^*(x + \Delta x) = \sum \overline{\mathrm{rev}}(\epsilon) \widetilde{Q} / (x + \Delta x)^\gamma$$

and $\delta^*(x - \Delta x) = \sum \overline{\mathrm{rev}}(\epsilon) \widetilde{Q} / (x - \Delta x)^\gamma$. Since the numerator remains constant, δ^* is decreasing and differentiable on an open neighborhood of x; in particular, it is continuous at x. This proves the assertion about U.

Suppose that $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$. Notice that given $\tau \in \overline{\mathcal{E}^*}$, by Equation (6), we have $\sum \tau Q = \sum \inf_{\mathcal{E}^*}(\tau) Q$. Since $\overline{\mathrm{rev}}(\epsilon^+) \in \overline{\mathcal{E}^*}$, by Lemma 13,

$$\sum \overline{\mathrm{rev}}(\epsilon^+) Q = \sum \inf_{\mathcal{E}^*}(\overline{\mathrm{rev}}(\epsilon^+)) Q = \sum \overline{\mathrm{rev}}(\epsilon) Q.$$

Thus,

$$\sum \overline{\text{rev}}(\epsilon^+)Q/(x+\Delta x)^\gamma = \sum \overline{\text{rev}}(\epsilon)Q/(x+\Delta x)^\gamma \to \sum \overline{\text{rev}}(\epsilon)Q/x^\gamma$$

as $\Delta x \to 0$. By Lemma 13, there is a sufficiently large n such that

$$\text{res}_n(\epsilon) = \text{res}_n(\overline{\text{rev}}(\epsilon)) = \text{res}_n(\overline{\text{rev}}(\epsilon^-)),$$

and let $\tau := \text{res}^{n+1}(\overline{\text{rev}}(\epsilon^-))$, so that $\overline{\text{rev}}(\epsilon^-) = \text{res}_n(\overline{\text{rev}}(\epsilon^-)) + \tau$. By Lemma 13,

$$\sum \overline{\text{rev}}(\epsilon^-)Q = \sum (\text{res}_n(\epsilon^-) + \tau)Q = \sum \text{res}_n(\overline{\text{rev}}(\epsilon))Q + \sum \tau Q.$$

Notice that $\overline{\text{rev}}(\epsilon^-) \in \overline{\mathcal{E}^*}$ implies $\overline{\text{rev}}(\epsilon^-)_k \le M$ for all $k \in \mathbb{N}$ where M is the largest entry in L, and hence, $\sum \tau Q = \sum_{k=n+1}^{\infty} \tau_k \omega^k \le \sum_{k=n+1}^{\infty} M\omega^k \to 0$ as $n \to \infty$. By Lemma 13,

$$\sum \overline{\text{rev}}(\epsilon^-)Q = \sum (\text{res}_n(\epsilon^-) + \tau)Q = \sum \text{res}_n(\overline{\text{rev}}(\epsilon))Q + \sum \tau Q.$$

Thus, $\sum \overline{\text{rev}}(\epsilon^-)Q \to \sum \overline{\text{rev}}(\epsilon)Q$ as $\Delta x \to 0$, which proves that

$$\sum \overline{\text{rev}}(\epsilon^-)Q/(x-\Delta x)^\gamma \to \sum \overline{\text{rev}}(\epsilon)Q/x^\gamma$$

as $\Delta x \to 0$.

It remains to show that $\delta_1^*[\epsilon^\pm] \to \delta_1^*[\epsilon]$ where $\epsilon \in \mathcal{E}^*$ with $\beta^1 <_{\text{d}} \epsilon$ and that $\delta_1^*[\epsilon^+] \to \delta_1^*[\epsilon]$ where $\epsilon = \beta^1$. The proofs of these cases are similar to the earlier cases, and we leave it to the reader. This concludes the proof of δ_1 being continuous on $[\widetilde{\omega}, 1)$.

Let us show $\delta_1^*(\widetilde{\omega}) = \lim_{x\to 1^-} \delta_1^*(x) = 1$, so it demonstrates that when we define δ^* using δ_1^*, the function values of those end points coincide. Notice that $\delta^*(\widetilde{\omega}) = \omega/\widetilde{\omega}^\gamma = 1$. Recall that $\bar{\theta}^1$ is the maximal $\widetilde{L}^*$-block at index 1, and $\sum \bar{\theta}^1 \widetilde{Q} = 1$. If $\sum \text{res}_n(\bar{\theta}^1)\widetilde{Q} < x := \sum \epsilon \widetilde{Q} < 1$ where n is sufficiently large and $\epsilon \in \widetilde{\mathcal{E}}^*$, then by Theorem 6, $\text{res}_n(\bar{\theta}^1) = \text{res}_n(\epsilon)$. By Equation (6),

$$\left|1 - \sum \epsilon \widetilde{Q}\right| = \left|\sum \bar{\theta}^1 \widetilde{Q} - \sum \epsilon \widetilde{Q}\right| = \left|\sum_{k=n+1}^{\infty} (\bar{\theta}^1_k - \epsilon_k)\widetilde{\omega}^k\right| = O(\widetilde{\omega}^n) \to 0$$

as $n \to \infty$. Thus, δ is continuous on $(0,1)$. □

4.1.2. Differentiability

Let us prove the differentiability of δ^* on $\text{eval}_{\widetilde{Q}}(\overline{\mathcal{E}^*})$.

Lemma 14. *Let $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*_\circ$. Then, there are infinitely many indices r such that $\epsilon - \beta^r \in \overline{\mathcal{E}^*}$.*

Proof. Let $\epsilon = \sum_{m=1}^{\infty} \zeta^m \in \mathcal{E}^* - \mathcal{E}^*_\circ$ be a non-zero L^*-block decomposition. Then, for each m, there is a largest index r such that $[i, b]$ is the support interval of ζ^m and $\zeta^m_r \neq 0$. Thus, $\zeta^m - \beta^r$ is a proper L^*-block with support interval $[i, r]$, and $\sum_{m=1}^{\infty} \zeta^m - \beta^r \in \mathcal{E}^*$.

Let $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \bar{\beta}^{b+1} \in \overline{\mathcal{E}^*} - \mathcal{E}^*$ be an $\overline{L^*}$-block decomposition. Recall that $L = (e_1, \ldots, e_N)$ is the list, by which the period Zeckendorf collection $\mathcal{E}^*$ is determined, and notice that there is the largest index $r_0 \leq N$ such that $e_{r_0} > 0$. Then, for all large $r \equiv b + r_0 \pmod N$,

$$\begin{aligned}
\epsilon - \beta^r &= \sum_{m=1}^{\ell} \zeta^m + \left(\mathrm{res}_r(\bar{\beta}^{b+1}) - \beta^r\right) + \mathrm{res}^{r+1}(\bar{\beta}^{b+1}) \\
&= \sum_{m=1}^{\ell} \zeta^m + \left(\mathrm{res}_r(\bar{\beta}^{b+1}) - \beta^r\right) + \bar{\beta}^{r+N-r_0+1}.
\end{aligned}$$

Since $\mathrm{res}_r(\bar{\beta}^{b+1}) - \beta^r$ is a proper L^*-block with support interval $[b+1, r]$, the above decomposition implies that $\epsilon - \beta^r \in \overline{\mathcal{E}^*}$. $\square$

Theorem 10. *The function δ^* is not differentiable on $\mathrm{eval}_{\widetilde{Q}}(\overline{\mathcal{E}^*})$.*

Proof. Let $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*_\circ$, and let $x := \sum \epsilon \widetilde{Q}$ and $y := \sum \epsilon Q$. By Lemma 14, there is a sufficiently large r such that $\epsilon - \beta^r \in \overline{\mathcal{E}^*}$. Let $\Delta x := \widetilde{Q}_r = \widetilde{\omega}^r$, and let $\Delta y := Q_r = \omega^r$. Then, by Lemma 10 and the linear approximation of $1/x^\gamma$,

$$\begin{aligned}
\frac{1}{-\Delta x}\left(\delta^*(x - \Delta x) - \delta^*(x)\right) &= \frac{1}{-\widetilde{\omega}^r}\left(\frac{y - \omega^r}{(x - \widetilde{\omega}^r)^\gamma} - \frac{y}{x^\gamma}\right) \\
&= \frac{1}{-\widetilde{\omega}^r}\left((y - \omega^r)\left(\frac{1}{x^\gamma} + \frac{\gamma}{x^{\gamma+1}}\widetilde{\omega}^r + O(\widetilde{\omega}^{2r})\right) - \frac{y}{x^\gamma}\right) \\
&= -\frac{y\gamma}{x^{\gamma+1}} + O(\widetilde{\omega}^r) + \frac{\omega^r}{\widetilde{\omega}^r}\left(\frac{1}{x^\gamma} + O(\widetilde{\omega}^r)\right).
\end{aligned}$$

As $r \to \infty$, the ratio $(\omega/\widetilde{\omega})^r \to \infty$, and hence, δ^* is not differentiable at $x \in \mathrm{eval}_{\widetilde{Q}}(\overline{\mathcal{E}^*} - \mathcal{E}^*_\circ)$. If $\epsilon = \sum_{m=1}^{\ell} \zeta^m \in \mathcal{E}^*_\circ$ where ℓ is a positive integer, a similar calculation for $\epsilon + \beta^r \in \mathcal{E}^*_\circ$ for sufficiently large r shows that the difference quotient approaches $+\infty$ as well, and hence, δ^* is not differentiable at $x \in \mathrm{eval}_{\widetilde{Q}}(\mathcal{E}^*_\circ)$. $\square$

Thus, δ^* is differentiable precisely on the subset U defined in Notation 2.

4.1.3. Lesbegue Measure

Let us prove that U is a Lesbegue measurable subset of measure 1.

Proposition 3. *Given an $\overline{L^*}$-block decomposition $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \bar\beta^{b+1} \in \overline{\mathcal{E}^*} - \mathcal{E}^*$, let $x_\epsilon := \sum \epsilon\widetilde{Q}$ and $x'_\epsilon := \sum(\sum_{m=1}^{\ell} \zeta^m + \beta^b)\widetilde{Q} \in \mathcal{E}^*_\circ$. Then, the subset U is the disjoint union of the open intervals $(x_\epsilon, x'_\epsilon)$ where $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$; in particular, U is open with respect to the usual topology on $\mathbf{I}$, and it is Lesbegue measurable.*

Proof. Let us show that the open interval $(x_\epsilon, x'_\epsilon)$ is contained in U. Let $x = \sum \tau\widetilde{Q}$ where $x \in (x_\epsilon, x'_\epsilon)$. By Theorem 6,

$$\sum_{m=1}^{\ell} \zeta^m + \bar\beta^{b+1} <_{\mathrm{d}} \tau <_{\mathrm{d}} \sum_{m=1}^{\ell} \zeta^m + \beta^b.$$

This implies that $\tau = \sum_{m=1}^{\ell} \zeta^m + \mu$ where $\bar\beta^{b+1} <_{\mathrm{d}} \mu$. Thus, the sum is the $\overline{L^*}$-block decomposition of $\tau \in \widetilde{\mathcal{E}}^* - \overline{\mathcal{E}^*}$, i.e., $x \in U$.

By Corollary 1, given $x = \sum \tau\widetilde{Q} \in U$, there is a unique full $\overline{L^*}$-block decomposition of $\tau = \sum_{m=1}^{\ell} \zeta^m + \mu$ where b is the largest $\overline{L^*}$-support index, $\bar\beta^{b+1} <_{\mathrm{d}} \mu$, and the union of the support intervals of ζ^m for $1 \le m \le \ell$ is $[1, b]$. Thus, by Theorem 6, if $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \bar\beta^{b+1}$, then $x_\epsilon < x < x'_\epsilon$.

Notice that, by Corollary 1, given $x \in U$, the coefficient function $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$ described above is uniquely determined, and this implies that the intervals $(x_\epsilon, x'_\epsilon)$ are disjoint. This concludes the proof of the fact that U is a disjoint union of the intervals $(x_\epsilon, x'_\epsilon)$. □

The length of the interval $(x_\epsilon, x_{\epsilon'})$ described in Proposition 3 is

$$I_b := \widetilde{\omega}^b - \sum \bar\beta^{b+1}\widetilde{Q} \tag{13}$$

where $b \ge 1$, and to find the measure of U, we need to figure out the number of intervals $(x_\mu, x_{\mu'})$ where $\mu \in \overline{\mathcal{E}^*} - \mathcal{E}^*$, for which $x_{\mu'} - x_\mu = I_b$. Recall the list $L = (e_1, \dots, e_N)$, which determines the periodic Zeckendorf collection $\mathcal{E}^*$.

Lemma 15. *Let $i \le b$ be two positive integers such that $b - i + 1 \equiv j \pmod N$ for some $1 \le j \le N$. The number of proper L^*-blocks with support interval $[i, b]$ is equal to e_j.*

Proof. Let ζ be a member of $\mathcal{E}^*$ supported on $[i, b]$. Notice that ζ is a proper L^*-block with support interval $[i, b]$ if and only if $0 \le \zeta_b < e_j$ and $\zeta_k = \bar\beta^i_k$ for all k such that $i \le k < b$. Since the number of the possibilities for ζ_b is e_j, we prove the lemma. □

The value C_b described in Proposition 4 below is the number of intervals $(x_\mu, x_{\mu'})$ whose length is I_b defined in Equation (13).

Proposition 4. *Let S_n for $n \geq 1$ be the subset of $\epsilon \in \mathcal{E}^*$ such that $\epsilon = \sum_{m=1}^{\ell} \zeta^m$ is an L^*-block decomposition and the support interval of ζ^ℓ is $[i, n]$ for some $i \leq n$, and let $S_0 := \{0\}$. Let C_n for $n \geq 0$ be the number of elements in S_n. Then,*

$$\begin{aligned} C_k &= e_1 C_{k-1} + \cdots + e_k C_0 \quad \text{for } k = 1, \ldots, N \\ C_n &= \sum_{n=1}^{N-1} e_k C_{n-k} + (1 + e_N) C_{n-N} \quad \text{for all } n \geq N+1. \end{aligned} \tag{14}$$

Proof. Let ζ be a proper L^*-block with support interval $[1, 1]$. Then, by Lemma 15, there are e_1 such blocks, and hence, $C_1 = e_1$. We shall calculate C_n inductively.

Given an integer $n \geq 1$, let T_i for $1 \leq i \leq n$ be the set of proper L^*-blocks with support interval $[i, n]$. Notice that the uniqueness of the proper L^*-block decomposition implies that S_n is the disjoint union of

$$S_k + T_{k+1} := \{\epsilon + \mu : \epsilon \in S_k,\ \mu \in T_{k+1}\}$$

for $k = 0, \ldots, n-1$, where the cardinality of $S_k + T_{k+1}$ is equal to $C_k \cdot \# T_{k+1} = C_k e_j$ for $1 \leq j \leq N$ such that $n - k \equiv j \pmod{N}$. Thus,

$$C_n = C_{n-1} e_1 + C_{n-2} e_2 + \cdots + C_1 e_{t-1} + C_0 e_t$$

where $1 \leq t \leq N$ and $t \equiv n \pmod{N}$, and the coefficients of C_k periodically repeat. By induction, if $n \geq N+1$, then

$$\begin{aligned} C_n &= e_1 C_{n-1} + \cdots + e_N C_{n-N} + (e_1 C_{n-N-1} + \cdots + e_t C_0) \\ &= e_1 C_{n-1} + \cdots + e_{N-1} C_{n-N+1} + (1 + e_N) C_{n-N}. \end{aligned}$$

□

Proposition 5. *Let $f(x) = 1 - \sum_{k=1}^{N-1} e_k x^k - (1 + e_N) x^N$. Then, the generating function $p(x) := \sum_{k=0}^{\infty} C_k x^k$ is identical to $(1 - x^N)/f(x)$ on an open interval containing* 0.

Proof. For a non-negative integer $t \leq N$, we have

$$\begin{aligned} x^t p(x) &= \sum_{k=0}^{\infty} C_k x^{k+t} = \sum_{k=t}^{\infty} C_{k-t} x^k = C_0 x^t + \cdots + C_{N-t} x^N + \sum_{k=N+1}^{\infty} C_{k-t} x^k \\ &= \sum_{k=t}^{N} C_{k-t} x^k + \sum_{k=N+1}^{\infty} C_{k-t} x^k. \end{aligned}$$

Below we shall calculate $p(x)-\sum_{t=1}^{N-1} e_t x^t p(x)-(1+e_N)x^N p(x)$, and by Equation (14), the sums over $k\geq N+1$ shall be cancelled out:

$$\begin{aligned}
p(x)-&(e_1xp(x)+\cdots+e_{N-1}x^{N-1}p(x)+(1+e_N)x^Np(x))\\
&=\sum_{k=0}^{N} C_kx^k+\sum_{k=N+1}^{\infty} C_kx^k\\
&\quad-\sum_{t=1}^{N-1}\left(\sum_{k=t}^{N} e_tC_{k-t}x^k+\sum_{k=N+1}^{\infty} e_tC_{k-t}x^k\right)-(1+e_N)C_0x^N\\
&\quad-\sum_{k=N+1}^{\infty}(1+e_N)C_{k-N}x^k\\
&=\sum_{k=0}^{N} C_kx^k-\sum_{t=1}^{N-1}\sum_{k=t}^{N} e_tC_{k-t}x^k-(1+e_N)C_0x^N
\end{aligned}$$

$$\begin{aligned}
&=\sum_{k=0}^{N} C_kx^k-\sum_{t=1}^{N-1}\sum_{k=t}^{N-1} e_tC_{k-t}x^k-\sum_{t=1}^{N-1} e_tC_{N-t}x^N-(1+e_N)C_0x^N\\
&=\sum_{k=0}^{N} C_kx^k-\sum_{k=1}^{N-1} x^k\sum_{t=1}^{k} e_tC_{k-t}-\sum_{t=1}^{N-1} e_tC_{N-t}x^N-(1+e_N)C_0x^N\\
&=\sum_{k=0}^{N} C_kx^k-\sum_{k=1}^{N-1} C_kx^k-x^N\sum_{t=1}^{N-1} e_tC_{N-t}-(1+e_N)C_0x^N=1-x^N.
\end{aligned}$$

Thus, $p(x)f(x)=1-x^N$, and it proves the result. □

Definition 24. Let ρ denote the real number $\sum\bar{\beta}^1\widetilde{Q}$ (where $\bar{\beta}^1$ is the maximal $\overline{L^*}$-block at index 1).

Lemma 16. *Let $\ell\geq 1$ be an integer. Then, $\sum\bar{\beta}^\ell\widetilde{Q}=\widetilde{\omega}^{\ell-1}\rho$.*

Proof. Notice that $\bar{\beta}^\ell=(\bar{0},\overline{e_1,\ldots,e_N})$ implies

$$\sum\bar{\beta}^\ell\widetilde{Q}=\sum_{k=\ell}^{\infty}\bar{\beta}^\ell_k\widetilde{\omega}^k=\sum_{k=\ell}^{\infty}\bar{\beta}^1_{k-\ell+1}\widetilde{\omega}^k=\widetilde{\omega}^{\ell-1}\sum_{k=\ell}^{\infty}\bar{\beta}^1_{k-\ell+1}\widetilde{\omega}^{k-\ell+1}=\rho\widetilde{\omega}^{\ell-1}.$$

□

Theorem 11. *The open subset U has Lesbegue measure* 1.

Proof. Let $\epsilon=\sum_{m=1}^{\ell}\zeta^m+\bar{\beta}^{b+1}$ be an $\overline{L^*}$-block decomposition where $b\geq 0$ is the largest $\overline{L^*}$-support index of ϵ. Let

$$\epsilon^0=\sum_{m=1}^{\ell}\zeta^m,\ x_\epsilon=\sum\epsilon\widetilde{Q},\ \text{and}\ x'_\epsilon=\sum(\sum_{m=1}^{\ell}\zeta^m+\beta^b)\widetilde{Q},$$

which is interpreted as 1 if $b = 0$, i.e., $\epsilon = \bar{\beta}^1$. By Proposition 3, U is the disjoint union of $(x_\epsilon, x'_\epsilon)$ where ϵ^0 varies over $\mathcal{E}^*_\circ$, and it is Lesbegue measurable.

Recall the set S_b and the number C_b defined in Proposition 4. Notice that, by Lemma 16, the length of $(x_\epsilon, x'_\epsilon)$ is $\widetilde{\omega}^b - \widetilde{\omega}^b\rho = \widetilde{\omega}^b(1-\rho)$. By Proposition 5, the measure of U is

$$\sum_{b=0}^{\infty} C_b\widetilde{\omega}^b(1-\rho) = (1-\rho)\frac{1-\widetilde{\omega}^N}{f(\widetilde{\omega})}$$

where $f(x)$ is the characteristic polynomial defined in the proposition. Let us rearrange ρ as follows. Notice that $e_1\widetilde{\omega} + \cdots + e_N\widetilde{\omega}^N = 1 - f(\widetilde{\omega}) - \widetilde{\omega}^N$. Then,

$$\begin{aligned}\rho &= (e_1\widetilde{\omega} + \cdots + e_N\widetilde{\omega}^N) + \widetilde{\omega}^N(e_1\widetilde{\omega} + \cdots + e_N\widetilde{\omega}^N) + \cdots \\ &= (1 - f(\widetilde{\omega}) - \widetilde{\omega}^N)(1 + \widetilde{\omega}^N + \widetilde{\omega}^{2N} + \cdots) = 1 - \frac{f(\widetilde{\omega})}{1-\widetilde{\omega}^N},\end{aligned}$$

$$\text{which implies} \quad (1-\rho)\frac{1-\widetilde{\omega}^N}{f(\widetilde{\omega})} = 1.$$

Therefore, the measure of U is equal to 1. □

4.2. The Local and Global Maximum and Minimum Values

Theorem 12 below is the technical version of Theorem 2, and we prove it in this section. Recall that, by Theorem 9, δ^* is locally decreasing on U, and hence, $\delta^*(x)$ is not a local extremum value if $x \in U$.

Theorem 12. *The function δ^* does not assume a local extremum value at points in* $\mathrm{eval}_{\widetilde{Q}}((\widetilde{\mathcal{E}^*} - \overline{\mathcal{E}^*}) \cup (\mathcal{E}^* - \mathcal{E}^*_\circ))$. *It assumes a local minimum value at points in $\mathcal{E}^*_\circ$, and a local maximum value at points in* $\mathrm{eval}_{\widetilde{Q}}(\overline{\mathcal{E}^*} - \mathcal{E}^*)$.

The following terminology is explained in Lemma 17 below.

Definition 25. Let p denote the smallest positive integer such that $\gamma\widetilde{\omega}^{p-1} < \omega^p$, and let us call the integer *the exponent of the generic upper bound of δ^**.

Lemma 17. *If $\epsilon \in \widetilde{\mathcal{E}}^*$, then $\delta^*[\epsilon] < \sum \overline{\mathrm{rev}}(\epsilon)Q / \sum \epsilon\widetilde{Q} < \frac{1}{\gamma}(\omega/\widetilde{\omega})^p$.*

Proof. Notice that $\sum \epsilon\widetilde{Q} < \left(\sum \epsilon\widetilde{Q}\right)^\gamma$ for all $\epsilon \in \widetilde{\mathcal{E}}^*$, so $\delta^*[\epsilon] < \sum \overline{\mathrm{rev}}(\epsilon)Q / \sum \epsilon\widetilde{Q}$. Since $\overline{\mathrm{rev}}(\epsilon) \le \bar{\beta}^1$, we have $\sum \overline{\mathrm{rev}}(\epsilon)Q \le 1$. Since the set of values of δ^* is equal to that of $\delta^*_1 : [\widetilde{\omega}, 1) \to \mathbb{R}$, we may assume that $\epsilon_1 \ge 1$, i.e., $\sum \epsilon\widetilde{Q} \ge \widetilde{\omega}$. Then,

$$\delta^*[\epsilon] < \sum \overline{\mathrm{rev}}(\epsilon)Q / \sum \epsilon\widetilde{Q} \le 1/\widetilde{\omega} < \frac{1}{\gamma}(\omega/\widetilde{\omega})^p.$$

□

Lemma 18. *Let $\epsilon \in \widetilde{\mathcal{E}}^*$, $x = \sum \epsilon \widetilde{Q}$, and $y = \sum \overline{\text{rev}}(\epsilon) Q$. If Δx and Δy are positive real numbers such that $\omega^p / \widetilde{\omega}^p \leq \Delta y / \Delta x$, then $y/x^\gamma < (y + \Delta y)/(x + \Delta x)^\gamma$.*

Proof. Since $\frac{\omega^p}{\widetilde{\omega}^p} \leq \frac{\Delta y}{\Delta x}$, Lemma 17 implies $\frac{y}{x} < \frac{1}{\gamma} \frac{\omega^p}{\widetilde{\omega}^p} \leq \frac{1}{\gamma} \frac{\Delta y}{\Delta x}$. Notice that

$$\frac{y}{x} < \frac{1}{\gamma} \frac{\Delta y}{\Delta x}, \text{ which implies } \quad 1 + \gamma \frac{\Delta x}{x} < 1 + \frac{\Delta y}{y}.$$

Since $(1+t)^\gamma < 1 + \gamma t$ for real numbers $t > 0$, we have $\left(1 + \frac{\Delta x}{x}\right)^\gamma < 1 + \gamma \frac{\Delta x}{x}$, and hence,

$$\left(1 + \frac{\Delta x}{x}\right)^\gamma < 1 + \gamma \frac{\Delta x}{x} < 1 + \frac{\Delta y}{y}.$$

It follows

$$\left(1 + \frac{\Delta x}{x}\right)^\gamma < 1 + \frac{\Delta y}{y}, \text{which implies } \quad \frac{y}{x^\gamma} < \frac{y + \Delta y}{(x + \Delta x)^\gamma}.$$

□

The following corollary is useful for further reducing the finite cases listed in Theorem 13 and Theorem 14 below.

Corollary 3. *Suppose that $\epsilon \in \widetilde{\mathcal{E}}^*$. If $n \geq p$, then*

$$\delta^*[\epsilon] = \frac{\sum \overline{\text{rev}}(\epsilon) Q}{\left(\sum \epsilon \widetilde{Q}\right)^\gamma} < \frac{\sum \overline{\text{rev}}(\epsilon) Q + Q_n}{\left(\sum \epsilon \widetilde{Q} + \widetilde{Q}_n\right)^\gamma}.$$

Proof. If $p \leq n$, then $\omega^p / \widetilde{\omega}^p \leq Q_n / \widetilde{Q}_n = \omega^n / \widetilde{\omega}^n$. By Lemma 18, we prove the result. □

Proposition 6. *The function δ^* assumes a local minimum value at the points in* $\text{eval}_{\widetilde{Q}}(\mathcal{E}^*_\circ)$.

Proof. Let $\epsilon = \sum_{m=1}^{\ell} \zeta^m$ be an L^*-block decomposition where ζ^ℓ is nonzero and $[i, b]$ is the support interval of ζ^ℓ, and let $x = \sum \epsilon \widetilde{Q}$. For sufficiently small positive real numbers Δx, we have $x + \Delta x = \sum \epsilon^+ \widetilde{Q}$ where $\epsilon^+ \in \widetilde{\mathcal{E}}^*$, and by Lemma 5, $\epsilon^+ = \epsilon + \tau$ where $\tau \in \widetilde{\mathcal{E}}^*$ such that $\text{ord}^*(\tau) = n > b + 1$ is sufficiently large, i.e., it is an $\widetilde{L}^*$-decomposition. By Lemma 10 and 13, $\overline{\text{rev}}(\epsilon^+) = \epsilon + \overline{\text{rev}}(\tau)$, and we have

$$\Delta y := \sum \overline{\text{rev}}(\epsilon^+) Q - \sum \epsilon Q = \sum \overline{\text{rev}}(\tau) Q \geq \omega^n \text{ and } \Delta x = \sum \tau \widetilde{Q} < \widetilde{\omega}^{n-1}$$

by Theorem 6. Thus, $\Delta y / \Delta x > \widetilde{\omega}(\omega/\widetilde{\omega})^n \to \infty$ as $n \to \infty$. By Lemma 18, $\delta^*[\epsilon] < \delta^*[\epsilon^+]$.

Since ζ^ℓ is non-zero, there is a largest index r such that $\zeta^\ell_r > 0$, and $\xi := \zeta^\ell - \beta^r$ is a proper L^*-block with support interval $[i, r]$. Then,

$$\epsilon^- := \sum_{m=1}^{\ell-1} \zeta^m + \xi + \bar{\beta}^{r+1} \in \overline{\mathcal{E}^*}$$

is an $\overline{L^*}$-block decomposition, and let $x^- := \sum \epsilon^- \widetilde{Q}$. By Theorem 9 and Proposition 3, δ^* is decreasing on (x^-, x), and the continuity of δ^* implies that $\delta^*(x_1) > \delta^*(x)$ for all $x_1 \in (x^-, x)$. ☐

Proposition 7. *The function δ^* does not assume a local extremum value at the points in $\mathrm{eval}_{\widetilde{Q}}(\mathcal{E}^* - \mathcal{E}^*_\circ)$.*

Proof. Let $\epsilon = \sum_{m=1}^\infty \zeta^m$ be the non-zero L^*-block decomposition. For each m, there is the largest index r in the support interval of ζ^m and $\zeta^m_r > 0$. Then, $\epsilon - \beta^r \in \mathcal{E}^*$, and by Corollary 3, $\delta^*[\epsilon - \beta^r] < \delta^*[\epsilon]$ for infinitely many r. So, $\delta^*[\epsilon]$ is not a local minimum.

Let us show that it is not a local maximum. Given $t \geq 1$, let $[i, \ell - 1]$ be the support interval of ζ^t. Let $\epsilon^+ := \sum_{m=1}^t \zeta^m + \beta^{\ell-1} + \beta^r$ where $r = \ell + N$. Then, the proper L^*-block $\xi := \zeta^t + \beta^{\ell-1}$ has support interval $[i, c]$ such that $c \leq \ell + N - 1$ since $\mathcal{E}^*$ is periodic and $\bar{\beta}^i_{i+kN} = e_1 \geq 1$ for $k \geq 0$. Since $r > c$, $\epsilon^+ = \sum_{m=1}^{t-1} \zeta^m + \xi + \beta^r$ is an L^*-block decomposition. Thus,

$$\begin{aligned}
\Delta y &:= \sum \epsilon^+ Q - \sum \epsilon Q = Q_{\ell-1} + Q_r - \sum (\textstyle\sum_{m=t+1}^\infty \zeta^m) Q > Q_r = \omega^r, \\
\Delta x &:= \sum \epsilon^+ \widetilde{Q} - \sum \epsilon \widetilde{Q} = \widetilde{Q}_{\ell-1} + \widetilde{Q}_r - \sum (\textstyle\sum_{m=t+1}^\infty \zeta^m) \widetilde{Q} \\
&\leq \widetilde{Q}_{\ell-1} + \widetilde{Q}_r < 2\widetilde{\omega}^{\ell-1},
\end{aligned}$$

$$\text{which implies} \quad \frac{\Delta y}{\Delta x} > \frac{\omega^r}{2\widetilde{\omega}^{\ell-1}} = \frac{\omega^{\ell+N-1}}{2\widetilde{\omega}^{\ell-1}}$$

where $\omega^\ell / \widetilde{\omega}^\ell \to \infty$ as $\ell \to \infty$. Thus, for sufficiently large t, $\Delta y / \Delta x > \omega^p / \widetilde{\omega}^p$. Notice that

$$y + \Delta y = \sum \epsilon^+ Q = \sum \overline{\mathrm{rev}}(\epsilon^+) Q \text{ and } x + \Delta x = \sum \epsilon^+ \widetilde{Q}$$

where $\epsilon^+ \in \mathcal{E}^*$. By Lemma 18, $\delta^*[\epsilon] < \delta^*[\epsilon^+]$. Since t can be arbitrarily large, this proves that $\delta^*[\epsilon]$ is not a local maximum. ☐

Proposition 8. *The function δ^* assumes a local maximum value at the points x in $\mathrm{eval}_{\widetilde{Q}}(\overline{\mathcal{E}^*} - \mathcal{E}^*)$. Moreover, $x \in \mathrm{eval}_{\widetilde{Q}}(\overline{\mathcal{E}^*} - \mathcal{E}^*)$ is locally the only value that attains the local maximum.*

Proof. Let $\epsilon = \sum_{m=1}^{\ell} \zeta^{\ell} + \bar\beta^{b+1}$ be an $\overline{L^*}$-block decomposition of $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$, and let $x = \sum \epsilon \widetilde{Q}$. By Theorem 9 and Proposition 3, we have $\delta^*(x) > \delta^*(x')$ for all values $x' > x$ that are sufficiently close to x.

Let us show that $\delta^*(x) > \delta^*(x')$ for all values $x' < x$ that are sufficiently close to x. Given an integer $n \geq b+1$, let $\epsilon^n := \sum_{m=1}^{\ell} \zeta^{\ell} + \mathrm{res}_n(\bar\beta^{b+1}) \in \mathcal{E}^*_\circ$, let $x_n := \sum \epsilon^n \widetilde{Q}$, which approaches x as $n \to \infty$, and let $\widehat{x}_n \in I_n := [x_n, x]$ such that $\delta^*(\widehat{x}_n) = \max\{\delta^*(z) : z \in I_n\}$. Let $\widehat{x}_n = \sum \epsilon^- \widetilde{Q}$ for $\epsilon^- \in \widetilde{\mathcal{E}}^*$. By Proposition 6, $\delta^*(x_n)$ is a local minimum, and hence, $\widehat{x}_n \neq x_n$. Let us claim that $\epsilon^- \in \overline{\mathcal{E}^*} - \mathcal{E}^*$. By Theorem 11, Propositions 6 and 7, if $\epsilon^- \in \mathcal{E}^* \cup (\widetilde{\mathcal{E}}^* - \overline{\mathcal{E}^*})$ and $\widehat{x}_n > x_n$, then δ^* assumes a value higher than $\delta^*(\widehat{x}_n)$ at values arbitrarily near $\widehat{x}_n$, which contradicts the choice of $\widehat{x}_n$.

Below we shall show that if $x' = \sum \epsilon' \widetilde{Q} < x$, $\epsilon' \in \overline{\mathcal{E}^*} - \mathcal{E}^*$, and x' is sufficiently close to x, then $\delta^*(x') < \delta^*(x)$. This shall imply that $\delta^*(x)$ is a local maximum, and x is locally the only value that attains the local maximum.

Let $x' = \sum \epsilon' \widetilde{Q} < x$ where $\epsilon' \in \overline{\mathcal{E}^*} - \mathcal{E}^*$ such that x' is sufficiently close to x. By Lemma 5 applied to $\widetilde{\mathcal{E}}^*$, there are proper L^*-blocks ζ^m for $\ell+1 \leq m \leq t$ such that

$$\epsilon' = \sum_{m=1}^{\ell} \zeta^{\ell} + \sum_{m=\ell+1}^{t} \zeta^m + \bar\beta^{e+1}$$

is an $\overline{L^*}$-block decomposition, and $[b+1, c]$ is the support interval of $\zeta^{\ell+1}$ where $c \to \infty$ as $x' \to x$. Notice that the periodic property of $\mathcal{E}^*$ and Lemma 5 imply that there is an index r such that $c+1 \leq r \leq c+N$ and $\mathrm{res}^r(\bar\beta^{b+1}) = (\bar 0, \overline{e_1, \ldots, e_N})$. Recall $\rho := \sum \bar\beta^1 \widetilde{Q}$ and Lemma 16. Then,

$$\begin{aligned}
\Delta x := x - x' &= \sum \epsilon \widetilde{Q} - \sum \epsilon' \widetilde{Q} \leq \rho \widetilde{\omega}^b - \sum \zeta^{\ell+1} \widetilde{Q} \\
&= \rho \widetilde{\omega}^b - \Big(\sum \bar\beta^{b+1} \widetilde{Q} - \sum_{k=c}^{r-1} (\bar\beta^{b+1}_k - \zeta^{\ell+1}_k) \widetilde{\omega}^k - \sum \bar\beta^r \widetilde{Q}\Big) \\
&= \rho \widetilde{\omega}^b - \Big(\rho \widetilde{\omega}^b - \sum_{k=c}^{r-1} (\bar\beta^{b+1}_k - \zeta^{\ell+1}_k) \widetilde{\omega}^k - \rho \widetilde{\omega}^{r-1}\Big), \quad \mathrm{res}^{c+1}(\zeta^{\ell+1}) = 0.
\end{aligned}$$

Let M be the maximum value of e_k for $k = 1, \ldots, N$;

$$\begin{aligned}
\Delta x &\leq \sum_{k=c}^{r-1} (\bar\beta^{b+1} - \zeta^{\ell+1}) \widetilde{\omega}^k + \rho \widetilde{\omega}^{r-1} \leq \widetilde{\omega}^{r-1} \left(\sum_{k=c}^{r-1} M \widetilde{\omega}^{k-r+1} + \rho \right), \ r - N \leq c \\
&\leq \widetilde{\omega}^{r-1} \left(\sum_{k=r-N}^{r-1} M \widetilde{\omega}^{k-r+1} + \rho \right) \leq \widetilde{\omega}^{r-1} \left(\sum_{k=0}^{N-1} M \widetilde{\phi}^k + \rho \right) \leq \lambda \widetilde{\omega}^c.
\end{aligned}$$

where $\lambda = \sum_{k=0}^{N-1} M \widetilde{\phi}^k + \rho$. Let us estimate $\Delta y := \sum \epsilon Q - \sum \epsilon' Q$. By Equation

(6), $\sum(\sum_{m=\ell+2}^{t}\zeta^m)Q + \sum\bar{\beta}^{e+1}Q = \sum(\sum_{m=\ell+2}^{t}\zeta^m)Q + Q_e < Q_c$. So,

$$\begin{aligned}\Delta y &= \sum\bar{\beta}^{b+1}Q - \sum\left(\sum_{m=\ell+1}^{t}\zeta^m + \bar{\beta}^{e+1}\right)Q \geq \omega^b - \sum(\zeta^{\ell+1}+\beta^c)Q \\ &= \omega^b - (\sum\bar{\beta}^{b+1}Q - \sum_{k=c}^{r-1}(\bar{\beta}_k^{b+1} - \zeta_k^{\ell+1})\widetilde{\omega}^k - \sum\bar{\beta}^rQ + Q_c) \\ &= \omega^b - (\omega^b - \sum_{k=c}^{r-1}(\bar{\beta}_k^{b+1} - \zeta_k^{\ell+1})\omega^k - \omega^{r-1} + \omega^c) \\ &= \sum_{k=c}^{r-1}(\bar{\beta}_k^{b+1} - \zeta_k^{\ell+1})\omega^k + \omega^{r-1} - \omega^c \\ &\geq \omega^c + \omega^{r-1} - \omega^c = \omega^{r-1} \geq \omega^{c+N-1}.\end{aligned}$$

Thus, $\frac{\Delta y}{\Delta x} \geq \left(\frac{\omega}{\widetilde{\omega}}\right)^c \frac{\omega^{N-1}}{\lambda}$, and since $c \to \infty$ as $\Delta x \to 0$, it follows $\Delta y/\Delta x \to \infty$. By Lemma 18, $\delta^*(x - \Delta x) < \delta^*(x)$. This proves that $\delta^*(x') < \delta^*(x)$, and it concludes the proof of the proposition. □

This concludes the proof of Theorem 12. Using the following theorems, we reduce the task of finding the global extremum values to a finite search, which is the assertion of Theorem 1, Part (2). Let

$$p^* := p + 1 + \frac{N\ln\phi + \ln(1 - \rho + \widetilde{\omega}^N)}{\ln\widetilde{\phi} - \ln\phi}. \tag{15}$$

Theorem 13. *The maximum value of δ^* is obtained only at $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$. Suppose that $\epsilon = \sum_{m=1}^{t}\zeta^m + \bar{\beta}^\ell$ is an $\overline{L^*}$-block decomposition with the largest $\overline{L^*}$-support index $\ell - 1 \geq 0$, and $\delta^*[\epsilon]$ is the maximum value. Then, $\ell < \max\{2, p^*\}$.*

Proof. By Theorem 12, the global maximum is obtained only on $\overline{\mathcal{E}^*} - \mathcal{E}^*$, and let ϵ be a member of $\overline{\mathcal{E}^*} - \mathcal{E}^*$ with the $\overline{L^*}$-block decomposition as described in the statement where $\ell \geq 1$. Suppose that $\ell \geq \max\{2, p^*\} \geq 2$. Let

$$\epsilon^+ := \sum_{m=1}^{t}\zeta^m + \beta^{\ell-1} + \beta^r$$

where $r = \ell + N - 1$. Then, $\epsilon^+ \in \mathcal{E}^*$, and $\mathrm{res}^\ell(\epsilon) = \bar{\beta}^\ell$. Thus,

$$\begin{aligned}\Delta y &:= \sum\epsilon^+Q - \sum\epsilon Q = Q_r \quad \text{by Equation (6), and} \\ \Delta x &:= \sum\epsilon^+\widetilde{Q} - \sum\epsilon\widetilde{Q} = \widetilde{Q}_{\ell-1} - \sum\bar{\beta}^\ell\widetilde{Q} + \widetilde{Q}_r.\end{aligned}$$

Thus, by Lemma 16,

$$\Delta x = \widetilde{\omega}^{\ell-1} - \rho\widetilde{\omega}^{\ell-1} + \widetilde{\omega}^{\ell+N-1} = \widetilde{\omega}^{\ell-1}(1-\rho+\widetilde{\omega}^N),$$

and $\ell \geq p^*$ implies

$$\frac{\Delta y}{\Delta x} = \frac{\omega^{\ell+N-1}}{\widetilde{\omega}^{\ell-1}(1-\rho+\widetilde{\omega}^N)} = \frac{\widetilde{\phi}^{\ell-1}}{\phi^{\ell+N-1}(1-\rho+\widetilde{\omega}^N)} \geq \frac{\widetilde{\phi}^p}{\phi^p} = \frac{\omega^p}{\widetilde{\omega}^p}.$$

By Lemma 18, $\delta^*[\epsilon] < \delta^*[\epsilon^+]$, and it implies that $\delta^*[\epsilon]$ is not a maximum value. Therefore, $\ell < \max\{2, p^*\}$. □

Theorem 14. *Let $p^\dagger := \max\{2, p\}$. Then, there is $\epsilon \in \mathcal{E}^*$ such that $\operatorname{res}^{p^\dagger}(\epsilon) = 0$ and $\delta^*[\epsilon]$ is the minimum value. In particular, if $p \leq 2$, then the minimum value is $\delta^*[\beta^1] = 1$.*

Proof. By Theorem 12, the minimum value is obtained by $\epsilon \in \mathcal{E}^*_\circ$, i.e., $\epsilon = \sum_{m=1}^{\ell} \zeta^m$, which is the non-zero L^*-block decomposition. Since $\delta_1 : [\widetilde{\omega}, 1) \to \mathbb{R}$ takes the same set of values as δ, assume without loss of generality that $\epsilon_1 \geq 1$. Let $[i, s]$ be the support interval of ζ^ℓ, and let r be the largest integer such that $\zeta^\ell_r > 0$. Then, $\zeta^\ell - \beta^r$ is a proper L^*-block with support interval $[i, r]$, and hence $\epsilon^- := \epsilon - \beta^r \in \mathcal{E}^*_\circ$. Suppose that $\delta^*[\epsilon]$ is the minimum value of δ^*. If $r \geq p^\dagger \geq 2$, then $\epsilon^-_1 \geq 1$, so ϵ^- is non-zero. By Lemma 3, $\delta^*[\epsilon^-] < \delta^*[\epsilon^- + \beta^r] = \delta^*[\epsilon]$ contradicting that $\delta^*[\epsilon]$ is a minimum. Thus, $r < p^\dagger$, which proves that $\operatorname{res}^{p^\dagger}(\epsilon) = 0$. □

5. Examples

We consider several examples in this section, and use the results in Section 4 to find the global maximum and minimum values of δ^*. Recall the constant α defined in Theorem 7, and for the closure of calculations, we introduce the following theorem. The idea of the proof of the following is also available in [18, Theorem 2.4].

Theorem 15. *Let $\mathcal{E}$ be the periodic Zeckendorf collections for positive integers determined by a list $L = (e_1, \dots, e_N)$. Let f be the characteristic polynomial of L for positive integers, defined in Definition 14, and let ϕ be its dominating (real) zero. Let H be the fundamental sequence of $\mathcal{E}$. Then,*

$$\lim_{n\to\infty} \frac{H_n}{\phi^{n-1}} = \frac{1}{f'(\phi)} \sum_{k=1}^{N} \frac{H_k}{(k-1)!} \left[\frac{d^{k-1}}{dx^{k-1}} \frac{f(x)}{x-\phi} \right]_{x=0}.$$

If $H_k = B^{k-1}$ for $1 \leq k \leq N$ and $B \neq \phi$, then

$$\lim_{n\to\infty} \frac{H_n}{\phi^{n-1}} = \frac{f(B)}{(B-\phi)f'(\phi)}.$$

Proof. Suppose that f has no repeated zeros. Let $\lambda_1, \dots, \lambda_{N-1}$ be the remaining distinct zeros of f. Given a complex number x, let $\mathbf{r}_k(x) = (\lambda_1^{k-1}, \dots, \lambda_{N-1}^{k-1}, x)$ for $1 \le k \le N$ be row vectors, and let $M(x_1, \dots, x_N)$ be the matrix whose kth row is $\mathbf{r}_k(x_k)$ where x_k are complex numbers. By Binet's formula,

$$H_n = \sum_{k=1}^{N-1} \alpha_k \lambda_k^{n-1} + \alpha \phi^{n-1}$$

for constants α_k and α. By specializing this formula at $1 \le n \le N$, we have a system of linear equations for α_k and α, and by Cramer's rule, we have

$$\alpha = \det M(H_1, \dots, H_N) / \det M(1, \phi, \dots, \phi^{N-1}).$$

By the determinant formula of the Vandermonde matrix,

$$\det M(1, \phi, \dots, \phi^{N-1}) = E \prod_{k=1}^{N-1} (\phi - \lambda_k)$$

where E is a quantity determined only by $\lambda_1, \dots, \lambda_{N-1}$, and in particular, E is independent of ϕ. Since $f(x) = (x - \phi)g(x)$ for $g(x) \in \mathbb{R}[x]$,

$$f'(\phi) = g(\phi) = \prod_{k=1}^{N-1} (\phi - \lambda_k).$$

Hence, $\det M(1, \phi, \dots, \phi^{N-1}) = E f'(\phi)$.

Let x be a polynomial variable. Then,

$$\det M(1, x, \dots, x^{N-1}) = E f(x)/(x - \phi)$$

since it is a Vandermonde matrix. For $1 \le k \le N$, let C_k be the cofactor of $M(1, x, \dots, x^{N-1})$ at the position (k, N), so that

$$\det M(1, x, \dots, x^{N-1}) = \sum_{k=0}^{N-1} C_{k+1} x^k.$$

Then, for $1 \le k \le N$, we have $E f(x)/(x - \phi) = \sum_{k=0}^{N-1} C_{k+1} x^k$, which implies

$$C_k = \frac{E}{(k-1)!} \left[\frac{d^{k-1}}{dx^{k-1}} \frac{f(x)}{x - \phi} \right]_{x=0}.$$

Then,

$$\det M(H_1, \dots, H_N) = \sum_{k=1}^{N} C_k H_k = \sum_{k=1}^{N} \frac{E H_k}{(k-1)!} \left[\frac{d^{k-1}}{dx^{k-1}} \frac{f(x)}{x - \phi} \right]_{x=0},$$

which implies

$$\alpha = \frac{\det M(H_1, \ldots, H_N)}{\det M(1, \phi, \ldots, \phi^{N-1})} = \frac{1}{f'(\phi)} \sum_{k=1}^{N} \frac{H_k}{(k-1)!} \left[\frac{d^{k-1}}{dx^{k-1}} \frac{f(x)}{x-\phi} \right]_{x=0}.$$

If $H_k = B^{k-1}$ for $1 \le k \le N$ and $B \ne \phi$, then

$$\det M(H_1, \ldots, H_N) = E \prod_{k=1}^{N-1} (B - \lambda_k) = E \frac{f(B)}{B-\phi},$$
$$\text{which implies} \quad \alpha = \frac{\det M(H_1, \ldots, H_N)}{\det M(1, \phi, \ldots, \phi^{N-1})} = \frac{f(B)}{(B-\phi) f'(\phi)}.$$

Suppose that f has repeated zeros. The dominant zero ϕ remains simple, and let d_k be the multiplicity of λ_k for $1 \le k \le N'$ where N' is the number of distinct zeros of f other than ϕ. Let the rows of $M(x_1, \ldots, x_N)$ consist of the $d_k - 1$ consecutive derivatives of λ_j^{k-1} where λ_j is treated as a variable. For example, if λ_1 is the only repeated zero, and it has multiplicity 3, then

$$\begin{aligned} \mathbf{r}_k(x_k) &= (\lambda_1^{k-1}, (k-1)\lambda_1^{k-2}, (k-1)(k-2)\lambda_1^{k-3}, \lambda_2^{k-1}, \ldots, \lambda_{N-3}^{k-1}, x_k) \\ &= \left(\lambda_1^{k-1}, \tfrac{k-1}{\lambda_1} \lambda_1^{k-1}, \tfrac{(k-1)(k-2)}{\lambda_1^2} \lambda_1^{k-1}, \lambda_3^{k-1}, \ldots, \lambda_{N-1}^{k-1}, x_k \right). \end{aligned}$$

See [2, Section 1] for more examples. Let us use these components as a basis for Binet's formula. By [2, Theorem 1] or [14],

$$\det M(1, x, \ldots, x^{N-1}) = E f(x)/(x - \phi)$$

where E depends only on λ_k and their multiplicities, and in particular, E is independent of x. Also, $\det M(1, \phi, \ldots, \phi^{N-1}) = E f'(\phi)$. Using this result, the argument for f without repeated roots applies seamlessly to this situation, and we leave it to the reader. □

Recall the exponent p of the generic upper bound of δ^* from Definition 25, and the values p^* and $p^\dagger$ described in Theorems 13 and 14.

5.1. The Nth Order Zeckendorf Base-N Expansions

Let $\mathcal{E}^*$ be the periodic collection for $\mathbf{I}$ determined by $L = (1, 0)$, and let $\widetilde{\mathcal{E}}$ be the one for $\mathbf{I}$ determined by the list $\widetilde{L} = (1, 1)$. This is an example introduced in Section 1 called the Zeckendorf binary expansions. Then, the exponent of generic upper bound of δ^* is $p = 2$, and $p^* \approx 4.998$. The value of $p^\dagger$ is 2, so $\delta^*[\beta^1] = 1$ is the minimum value. For the maximum values, consider $\epsilon = \sum_{m=1}^{t} \zeta^m + \bar{\beta}^\ell$, which is an $\overline{L^*}$-block decomposition with the largest $\overline{L^*}$-support index $\ell - 1$. By Theorem 13 with $p = 2$, we have $\ell \leq 4$, and hence, the only possibilities of ϵ are $(1, 0, 0, \overline{1, 0})$ and $\bar{\beta}^1$. It turns out that $\delta^*[\bar{\beta}^1]$ is the largest, and its value is $1/\rho^\gamma = (3/2)^\gamma$. Using Theorem 15 and Proposition 2, we obtain the result (1).

We may generalize it to the setting of the following two lists of length N.

Definition 26. Let $N \geq 2$ be an integer. The periodic Zeckendorf collections $\mathcal{E}$ for positive integers and $\mathcal{E}^*$ determined by $L = (1, 1, \ldots, 1, 0)$ where 1 is repeated $N-1$ times are called *the Nth order Zeckendorf collections for positive integers and for* $\mathbf{I}$, respectively. The periodic Zeckendorf collections $\widetilde{\mathcal{E}}$ for positive integers and $\widetilde{\mathcal{E}}^*$ for $\mathbf{I}$ determined by $\widetilde{L} = (N-1, N-1, \ldots, N-1)$ are called *the base-N collections for positive integers and for* $\mathbf{I}$, respectively.

Let us demonstrate the 3rd order Zeckendorf collections for positive integers, i.e., $N = 3$. Since $L = (1, 1, 0)$, the collection $\mathcal{E}$ consists of coefficient functions that have no three consecutive 1s. For example, $100 = 3^0 + 2 \cdot 3^2 + 3^4 = \widetilde{H}_1 + 2\widetilde{H}_3 + \widetilde{H}_5$ is not a 3rd order Zeckendorf expression while $1000 = \widetilde{H}_1 + \widetilde{H}_4 + \widetilde{H}_6 + \widetilde{H}_7$ is a 3rd order Zeckendorf expression.

Theorem 13 is what we could do for general cases, but for the setup considered in Definition 26, we obtain a better version of the theorem using Lemma 18 alone.

Lemma 19. *Let $\mathcal{E}^*$ be the periodic Zeckendorf collection defined in Definition 26 where $N \geq 3$, and let $\epsilon \in \mathcal{E}^*$. Then, the exponent of generic upper bound of δ^* is $p = 1$. If $\epsilon_n = 1$ and ζ is a proper L^*-block of ϵ with support interval $[i, n-1]$ for $n \geq 3$, then $\epsilon' := \epsilon + \beta^{n-1} - \beta^n \in \mathcal{E}^*$ and $\delta^*[\epsilon] < \delta^*[\epsilon']$.*

Proof. Let us prove that $p = 1$. Recall Notation 1. Notice that $\widetilde{\omega} = 1/N$, and the characteristic polynomial of L for positive integers is

$$f_N(x) = x^N - (x^{N-1} + \cdots + x + 1).$$

Since $f_N(2) = 2^N - (2^N - 1) = 1$, we have $\phi < 2$, and hence, $\omega > 1/2$. If ϕ' is the dominant real zero of f_{N-1}, then the collection for positive integers determined by $L_0 = (1, \ldots, 1, 0)$ with $N-2$ copies of 1 is a subcollection of the one determined by L, and hence, by Theorem 7, we have $\phi' < \phi$. This implies that $\omega \leq \omega_0$ where ω_0 is the reciprocal of the dominant zero of $x^3 - (x^2 + x + 1)$, which is less than 0.55. If $N \geq 4$ and $p = 1$, then

$$\phi^p \ln \phi < 2 \ln 2 = \ln 4 \leq N^{p-1} \ln N,$$

which implies $\gamma \widetilde{\omega}^{p-1} < \omega^p$ where γ is the ratio defined in Theorem 7. The inequality also holds for $N = 3$ as well.

Suppose that $\epsilon_n = 1$, and that ζ is a proper L^*-block of ϵ with support interval $[i, n-1]$ for $n \geq 3$. Let $\epsilon' := \epsilon + \beta^{n-1} - \beta^n \in \mathcal{E}^*$, and recall the fundamental sequences Q and $\widetilde{Q}$ from Notation 1. Let

$$x = \sum \epsilon\widetilde{Q},\ y = \sum \epsilon Q,\ x' = \sum \epsilon'\widetilde{Q},\ \text{and } y' = \sum \epsilon' Q.$$

Let us show that $\epsilon' \in \mathcal{E}^*$. If $(\zeta_{n-2}, \zeta_{n-1}) = (0,0)$, then $(\epsilon'_{n-2}, \epsilon'_{n-1}, \epsilon'_n) = (0,1,0)$, and hence, $\epsilon' \in \mathcal{E}^*$. If $(\zeta_{n-2}, \zeta_{n-1}) = (1,0)$, then by the definition of the support interval of a proper L^*-block, the number of 1s preceding the entry $\zeta_{n-1} = 0$ must be less than $N-1$ since $\epsilon_n = 1$. Thus, $(\epsilon'_{n-2}, \epsilon'_{n-1}, \epsilon'_n) = (1,1,0)$, and the number of 1s preceding the entry $\epsilon'_n = 0$ is less than or equal to $N-1$. Hence, $\epsilon' \in \mathcal{E}^*$.

Thus, we have $\delta^*(x') = y'/(x')^\gamma$. Let

$$\begin{aligned}\Delta x &= x' - x = \widetilde{\omega}^{n-1} - \widetilde{\omega}^n,\\ \Delta y &= y' - y = \omega^{n-1} - \omega^n = \omega^{n-1}(1-\omega).\end{aligned}$$

Notice that $n \geq 3$, and $(1-x)x$ is an increasing function on $[0,1/2]$ and decreasing on $[1/2,1]$. Then, $\frac{1}{N} = \widetilde{\omega} \leq \frac{1}{3} < \frac{1}{2} < \omega \leq \omega_0 < .55$, $\frac{1}{2} - \widetilde{\omega} \geq \frac{1}{6} > 0.1$, and $\omega - \frac{1}{2} < .05$. Since the graph of $(1-x)x$ is symmetric about the vertical line $x = \frac{1}{2}$, it follows that $(1-\widetilde{\omega})\widetilde{\omega} < (1-\omega)\omega$, and hence,

$$\frac{\Delta y}{\Delta x} = \frac{1-\omega}{1-\widetilde{\omega}}\left(\frac{\omega^{n-1}}{\widetilde{\omega}^{n-1}}\right) \geq \frac{1-\omega}{1-\widetilde{\omega}}\left(\frac{\omega^2}{\widetilde{\omega}^2}\right) > \frac{\omega}{\widetilde{\omega}} = \frac{\omega^p}{\widetilde{\omega}^p}. \tag{16}$$

Thus, by Lemma 18, $\delta^*(x) < \delta^*(x + \Delta x)$ where $N \geq 3$. □

By Theorem 8, if $\delta^*[\epsilon]$ is the maximum and $\epsilon_1 \geq 1$, then $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$. Let $\epsilon = \sum_{m=1}^{\ell} \zeta^m + \bar{\beta}^{b+1}$ be an $\overline{L^*}$-block decomposition where $b \geq 2$. Since ζ^ℓ is a proper L^*-block, we may shift the entry $\epsilon_{b+1} = 1$, and obtain a higher value of δ^* by Lemma 19. Thus, $b < 2$, and $\epsilon_1 \geq 1$ implies that $b = 0$, i.e., $\epsilon = \bar{\beta}^1$. By Theorem 14, the minimum is $\delta^*[\beta^1]$. By Theorem 15, Proposition 2, and the above calculations, we proved the following.

Theorem 16. *Let $N \geq 2$ be an integer, let ϕ be the dominant positive real zero of $f(x) = x^N - \sum_{k=0}^{N-1} x^k$, and let $\gamma := \log_N \phi$. Let $z(x)$ be the number of non-negative integers $n < x$ whose base-N expansions are Nth order Zeckendorf expressions. Then,*

$$\limsup_x \frac{z(x)}{x^\gamma} = \frac{\alpha}{P^\gamma \phi^N}, \quad \liminf_x \frac{z(x)}{x^\gamma} = \alpha \tag{17}$$

where

$$\alpha := \frac{f(N)}{(N-\phi)f'(\phi)}, \quad P := \frac{N^{N-1} - 1}{(N-1)(N^N - 1)N^{N-1}}.$$

5.2. Non-base-N Expansions

Let $\mathcal{E}$ be the collection for positive integers determined by $L = (2, 0, 1)$, and let $\widetilde{\mathcal{E}}$ be the one determined by $\widetilde{L} = (10, 4)$. By Equation (3), the fundamental sequence H of $\mathcal{E}$ is given by $(H_1, H_2, H_3) = (1, 3, 7)$ and $H_n = 2H_{n-1} + 2H_{n-3}$ for $n \geq 4$, and the fundamental sequence $\widetilde{H}$ of $\widetilde{\mathcal{E}}$ is given by $(\widetilde{H}_1, \widetilde{H}_2) = (1, 11)$ and $\widetilde{H}_n = 10\widetilde{H}_{n-1} + 5\widetilde{H}_{n-2}$ for $n \geq 3$. The characteristic polynomials of L and $\widetilde{L}$ for positive integers are $f(x) = x^3 - 2x^2 - 2$ and $\widetilde{f}(x) = x^2 - 10x - 5$, respectively, and $\phi \approx 2.36$ and $\widetilde{\phi} \approx 10.48$ are their dominant (real) zeros, respectively. Then, the exponent of generic upper bound of δ^* is $p = 1$, and $p^* \approx 3.59$.

By Theorem 13 and Corollary 3, the following are the possibilities of $\epsilon \in \overline{\mathcal{E}^*} - \mathcal{E}^*$ for the maximum values of δ_1. The symbol $\star$ indicates the repeating blocks $(\overline{2, 0, 1})$:

$$\bar{\beta}^1,\ (1, \star),\ (1, 1, \star).$$

For example, $\epsilon = (1, 0, \star) \in \overline{\mathcal{E}^*} - \mathcal{E}^*$ is not listed above since Corollary 3 implies $\delta^*[\epsilon] < \delta^*[\epsilon']$ where $\epsilon' = (1, 1, \star) \in \overline{\mathcal{E}^*} - \mathcal{E}^*$. According to the numerical calculations of δ^* at the above three tuples, δ^* attains the highest value at $\epsilon^M := (1, \overline{2, 0, 1})$, and

$$\delta^*[\epsilon^M] = \frac{\omega + \sum \bar{\beta}^2 Q}{(\widetilde{\omega} + \sum \bar{\beta}^2 \widetilde{Q})^\gamma} = \frac{2\omega}{(\widetilde{\omega} + \widetilde{\omega}\rho)^\gamma} = \frac{2}{(1 + \rho)^\gamma} \approx 1.8757.$$

Since $p = 1$, the minimum of δ^* is $\delta^*[\beta^1] = 1$.

Using Theorem 7 and 15, we find the exact values of the maximum and minimum values to be as follows:

$$\alpha := \lim_{n\to\infty} \frac{H_n}{\phi^{n-1}} = \frac{4 + 3\phi + 17\phi^2}{86}, \quad \widetilde{\alpha} := \lim_{n\to\infty} \frac{\widetilde{H}_n}{\widetilde{\phi}^{n-1}} = \frac{\widetilde{\phi}}{10}$$

$$\rho = (2\widetilde{\omega} + \widetilde{\omega}^3) + \widetilde{\omega}^3(2\widetilde{\omega} + \widetilde{\omega}^3) + \cdots = \frac{2\widetilde{\phi}^2 + 1}{\widetilde{\phi}^3 - 1} = \frac{-227 + 25\widetilde{\phi}}{182}.$$

Thus, by Proposition 2, we proved

$$\limsup_x \frac{z(x)}{x^\gamma} = \frac{2\alpha}{(\widetilde{\alpha}(1 + \rho))^\gamma} = \frac{364^\gamma}{43} \frac{4 + 3\phi + 17\phi^2}{(25 + 41\widetilde{\phi})^\gamma} \approx 2.2666,$$

$$\liminf_x \frac{z(x)}{x^\gamma} = \frac{\alpha}{\widetilde{\alpha}^\gamma} = \frac{10^\gamma}{86}(3 + 13\phi + 2\phi^2) \approx 1.2084.$$

6. Future Work

From Definition 26, recall the base-N collections for positive integers. If $\widetilde{\mathcal{E}}$ is the base-N collection for positive integers, and $\mathcal{E}$ is a periodic Zeckendorf subcollection of $\widetilde{\mathcal{E}}$ for positive integers, then the counting function $z(x)$ defined in Definition 21 turns out to be the summatory function of an N-regular sequence χ, which is defined in [1]. In [12], it is proved that given a regular sequence χ, the following asymptotic relation holds:

$$\sum_{k=1}^{n} \chi(k) \sim n^{\gamma} \Phi(\{\log_{\widetilde{\phi}}(n)\}) \tag{18}$$

where $\gamma < 1$ is a real number, and Φ is a continuous function on $(-\infty, \infty)$.

If $\widetilde{\mathcal{E}}$ is a periodic Zeckendorf collection but not a base-N collection, then the result of [12] does not apply as the definition of regular sequences is given in terms of base-N expansions. In our future work, we aim to establish the definition of *generalized regular sequences* associated with a periodic Zeckendorf collection for positive integers such that given a generalized regular sequence χ, the relation (18) holds where $\widetilde{\phi}$ is the dominant zero defined in Definition 14.

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