#A27 INTEGERS 25 (2025)

ON MATRICES WHOSE ENTRIES ARE STIRLING NUMBERS OF THE SECOND KIND (I)

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Received: 6/3/24, Accepted: 3/11/25, Published: 3/26/25

Abstract

In this paper, we examine matrices whose entries are Stirling numbers of the second kind. Specifically, we explore various matrix decompositions of these matrices and compute their determinants. Utilizing these results, we subsequently derive multiple identities involving both Stirling numbers of the first and second kinds.

1. Introduction

Let k and n be two integers. We denote by $c(n,k)$ the number of permutations on n elements that have exactly k cycles in their cycle structures. The number $c(n,k)$ is known as a *signless Stirling number of the first kind*. We denote by $s(n,k)$ the

DOI: 10.5281/zenodo.15091069

signed Stirling number of the first kind (also known as *Stirling number of the first kind*), which is defined as $s(n,k) = (-1)^{n-k}c(n,k)$. We also denote by $S(n,k)$ the *Stirling number of the second kind*, which is defined as the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into exactly k nonempty subsets. It follows easily from the definition that for every integer $n \geq 1$,

$$s(n,k) = S(n,k) = 0, \text{ if } k > n,$$

and

$$c(n,0) = s(n,0) = S(n,0) = 0, \text{ and } S(n,1) = 1,$$

and for every integer $n \geq 0$,

$$c(n,n) = s(n,n) = S(n,n) = 1.$$

It is well-known that for every $k, n \geq 1$ the numbers $c(n,k)$ and $S(n,k)$ satisfy the following recurrence relations [7, pages 26, 74] (see also [1, Theorems 8.2.4 and 8.2.8]):

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k), \tag{1}$$

and

$$S(n,k) = S(n-1,k-1) + kS(n-1,k). \tag{2}$$

Alternatively, Stirling numbers $s(n,k)$ and $S(n,k)$ can also be introduced as follows (see [1, p. 282]). In fact, they can be defined as the coefficients in the following expansion of a variable x:

$$[x]_n = \sum_{k=0}^{n} s(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S(n,k)[x]_k, \tag{3}$$

where

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1),$$

is the falling factorial (with $[x]_0 = 1$).

Given a matrix A, we use the notation $A_{i,j}$ to denote the entry of A in the ith row and jth column. The $n \times n$ *Stirling matrices of the first kind* $s(n) = [s_{i,j}]_{1 \leq i,j \leq n}$ and *of the second kind* $S(n) = [S_{i,j}]_{1 \leq i,j \leq n}$, are defined by

$$s_{i,j} = \begin{cases} s(i,j), & \text{if } i \geq j; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$S_{i,j} = \begin{cases} S(i,j), & \text{if } i \geq j; \\ 0, & \text{otherwise,} \end{cases}$$

respectively. The Stirling matrices $s(5)$ and $S(5)$, for example, are

$$s(5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix} \quad \text{and} \quad S(5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}.$$

These matrices have been studied by several authors, for instance, see [2]-[6] and [8]. It follows especially from Equation (3) that the Stirling matrices of the first kind and the second kind are inverses of each other, that is,

$$S(n)^{-1} = s(n) \quad \text{and} \quad s(n)^{-1} = S(n),$$

and this is why we study only the matrices associated with the Stirling matrix of the second kind $S(n)$ in this paper.

Given an integer $m \geq 0$, we define an $n \times n$ matrix $\mathsf{S}^{[m]}(n) = [\mathsf{S}^{[m]}_{i,j}]_{1 \leq i,j \leq n}$ with entries:

$$\mathsf{S}^{[m]}_{i,j} = S(i+m, j) \quad \text{for} \quad 1 \leq i, j \leq n.$$

The matrices $\mathsf{S}^{[1]}(5)$ and $\mathsf{S}^{[3]}(5)$, for instance, are given by

$$\mathsf{S}^{[1]}(5) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \\ 1 & 31 & 90 & 65 & 15 \end{bmatrix} \quad \text{and} \quad \mathsf{S}^{[3]}(5) = \begin{bmatrix} 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \\ 1 & 31 & 90 & 65 & 15 \\ 1 & 63 & 301 & 350 & 140 \\ 1 & 127 & 966 & 1701 & 1050 \end{bmatrix}.$$

We observe that the entries of $\mathsf{S}^{[m]}(n)$ satisfy the following recurrence relation:

$$\mathsf{S}^{[m]}_{i,j} = \mathsf{S}^{[m]}_{i-1,j-1} + j\mathsf{S}^{[m]}_{i-1,j}, \quad (2 \leq i, j \leq n), \tag{4}$$

with initial conditions

$$\mathsf{S}^{[m]}_{i,1} = 1, \quad \mathsf{S}^{[m]}_{1,j} = S(m+1, j), \quad (i, j \geq 1). \tag{5}$$

Among other results, we will obtain a matrix decomposition of $\mathsf{S}^{[m]}(n)$ (see Theorem 2) and through this decomposition, we will obtain its determinant:

$$\det \mathsf{S}^{[m]}(n) = \prod_{i=1}^{n} i^m = n!^m.$$

Finally, we obtain several identities related to the Stirling numbers of the first and second kind.

The rest of this paper is organized as follows: Additional notation, definitions and auxiliary results are collected in Section 2. In Section 3, we consider the matrices $\mathsf{S}^{[m]}(n)$, and formulate one of the main results of the paper – Theorem 2. In Section 4, we derive some identities involving the Stirling numbers of the first and second kind.

2. Auxiliary Results

In this section, we collect some relevant facts and auxiliary results which will be used later. First, we introduce the $n \times n$ upper triangular matrix $V(n) = [V_{i,j}]_{1 \le i,j \le n}$ by

$$V_{i,j} = \begin{cases} j, & \text{if } j = i; \\ 1, & \text{if } j = i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Also, let $\tilde{V}(n) = [\tilde{V}_{i,j}]_{1 \le i,j \le n}$ be the $n \times n$ upper triangular matrix whose entries are given by

$$\tilde{V}_{i,j} = \begin{cases} (-1)^{j-i}\frac{(i-1)!}{j!}, & \text{if } j \ge i; \\ 0, & \text{otherwise.} \end{cases}$$

For instance, if $n = 5$, the matrices $V(5)$ and $\tilde{V}(6)$ are

$$V(5) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \tilde{V}(5) = \begin{bmatrix} 1 & -1/2 & 1/6 & -1/24 & 1/120 \\ 0 & 1/2 & -1/6 & 1/24 & -1/120 \\ 0 & 0 & 1/3 & -1/12 & 1/60 \\ 0 & 0 & 0 & 1/4 & -1/20 \\ 0 & 0 & 0 & 0 & 1/5 \end{bmatrix}.$$

The following result shows that $V(n)$ and $\tilde{V}(n)$ are inverses of each other.

Lemma 1. *For any positive integer n, we have $V(n)^{-1} = \tilde{V}(n)$.*

Proof. Let $V = V(n)$, $\tilde{V} = \tilde{V}(n)$, and $I = I_n$, the identity matrix of order n. It suffices to show that $V \cdot \tilde{V} = I$. To prove this, we compute the (i, j)-entry of $V \cdot \tilde{V}$. Since V and $\tilde{V}$ are both upper triangular matrices, their product is also an upper triangular matrix, therefore, it is enough to consider only two cases: $i = j$ and $j > i$. If $i = j$, then by direct calculation, we obtain

$$(V \cdot \tilde{V})_{i,i} = \sum_{l=1}^{n} V_{i,l}\tilde{V}_{l,i} = V_{i,i}\tilde{V}_{i,i} = i \cdot \frac{1}{i} = 1,$$

and if $j > i$, then

$$\begin{aligned}(V \cdot \tilde{V})_{i,j} &= \sum_{l=1}^{n} V_{i,l}\tilde{V}_{l,j} = V_{i,i}\tilde{V}_{i,j} + V_{i,i+1}\tilde{V}_{i+1,j} \\ &= i \cdot \frac{(-1)^{j-i}(i-1)!}{j!} + 1 \cdot \frac{(-1)^{j-i-1}i!}{j!} = 0.\end{aligned}$$

This completes the proof of the lemma. □

Now, we define the following matrices:

- the $n \times n$ matrix $E(n) = [E_{i,j}]_{1 \le i,j \le n}$, with $E_{i,j} = (j+1)^{i-1}$.
- the $n \times n$ upper triangular matrix $Q^{[N]}(n) = [Q^{[N]}_{i,j}]_{1 \le i,j \le n}$, with

$$Q^{[N]}_{i,j} = \begin{cases} (-1)^{j-i+1} \frac{i^{N-1}}{(i-1)!(j-i-1)!}, & \text{if } j \ge i+1; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

For instance, if $n = 5$, the matrices $E(5)$ and $Q^{[3]}(5)$ are

$$E(5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 4 & 9 & 16 & 25 & 36 \\ 8 & 27 & 64 & 125 & 216 \\ 16 & 81 & 256 & 625 & 1296 \end{bmatrix} \quad \text{and} \quad Q^{[3]}(5) = \begin{bmatrix} 0 & 1 & -1 & 1/2 & -1/6 \\ 0 & 0 & 4 & -4 & 2 \\ 0 & 0 & 0 & 9/2 & -9/2 \\ 0 & 0 & 0 & 0 & 8/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 2. *Let N, n be positive integers. Then $Q^{[N+1]}(n)$ has the following matrix decomposition:*

$$Q^{[N+1]}(n) = Q^{[N]}(n) \cdot \big(V(n) - I_n\big).$$

Proof. Again, we compute the (i,j)-entry of $Q^{[N]}(n) \cdot \big(V(n) - I_n\big)$. Note, first of all, that

$$\big(V(n) - I_n\big)_{i,j} = \begin{cases} j-1, & \text{if } j = i; \\ 1, & \text{if } j = i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Now, for every $1 \le i, j \le n$, we have

$$\begin{aligned} \big(Q^{[N]}(n)\big(V(n) - I_n\big)\big)_{i,j} &= \sum_{l=1}^{n} Q^{[N]}_{i,l} \big(V(n) - I_n\big)_{l,j} \\ &= Q^{[N]}_{i,j-1} \big(V(n) - I_n\big)_{j-1,j} + Q^{[N]}_{i,j} \big(V(n) - I_n\big)_{j,j} \\ &= Q^{[N]}_{i,j-1} + (j-1) Q^{[N]}_{i,j}. \end{aligned}$$

We calculate $Q^{[N]}_{i,j-1} + (j-1)Q^{[N]}_{i,j}$ in three cases: $j \le i$, $j = i+1$, and $j \ge i+2$.

`Case 1:` $j \le i$: In this case, $Q^{[N]}_{i,j-1} = Q^{[N]}_{i,j} = 0$, and hence

$$\big(Q^{[N]}(n) \cdot \big(V(n) - I_n\big)\big)_{i,j} = Q^{[N]}_{i,j-1} + \big(j-1\big) Q^{[N]}_{i,j} = 0.$$

`Case 2:` $j = i+1$: In this case, we have $Q^{[N]}_{i,j-1} = Q^{[N]}_{i,i} = 0$ and

$$Q^{[N]}_{i,j} = Q^{[N]}_{i,i+1} = (-1)^{(i+1)-i+1} \frac{i^{N-1}}{(i-1)!((i+1)-i-1)!} = \frac{i^{N-1}}{(i-1)!},$$

hence we obtain

$$\left(Q^{[N]}(n)\cdot\left(V(n)-I_n\right)\right)_{i,j}=Q^{[N]}_{i,j-1}+(j-1)Q^{[N]}_{i,j}=Q^{[N]}_{i,i}+iQ^{[N]}_{i,i+1}=\frac{i^N}{(i-1)!}.$$

`Case 3:` $j\geq i+2$: In this case, we obtain

$$\begin{aligned}\left(Q^{[N]}(n)\cdot\left(V(n)-I_n\right)\right)_{i,j}&=Q^{[N]}_{i,j-1}+(j-1)Q^{[N]}_{i,j}\\&=(-1)^{j-i}\frac{i^{N-1}}{(i-1)!(j-i-2)!}+(j-1)(-1)^{j-i+1}\frac{i^{N-1}}{(i-1)!(j-i-1)!}\\&=(-1)^{j-i+1}\frac{i^N}{(i-1)!(j-i-1)!}.\end{aligned}$$

Therefore, combining the above cases, we have

$$\begin{aligned}\left(Q^{[N]}(n)\cdot\left(V(n)-I_n\right)\right)_{i,j}&=\begin{cases}(-1)^{j-i+1}\frac{i^N}{(i-1)!(j-i-1)!}, & \text{if } j\geq i+2;\\ \frac{i^N}{(i-1)!}, & \text{if } j=i+1;\\ 0, & \text{otherwise,}\end{cases}\\&=\begin{cases}(-1)^{j-i+1}\frac{i^N}{(i-1)!(j-i-1)!}, & \text{if } j\geq i+1;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

This shows that

$$\left(Q^{[N]}(n)\left(V(n)-I_n\right)\right)_{i,j}=Q^{[N+1]}_{i,j},$$

for every $1\leq i,j\leq n$, and the proof is complete. □

Corollary 1. *Let N,n be positive integers. Then, we have the following matrix identity:*

$$Q^{[N]}(n)=Q^{[1]}(n)\cdot\left(V(n)-I_n\right)^{N-1}.$$

Proof. The result follows by repeatedly applying Lemma 2. □

Lemma 3. *For any positive integer n, we have the following matrix identity:*

$$S(n)\cdot\left(V(n)-I_n\right)=E(n)\cdot Q^{[1]}(n). \tag{7}$$

Proof. Let $\mathrm{R}_1(A)$ (resp. $\mathrm{C}_1(A)$) denote the first row (resp. the first column) of a matrix A. Using this notation, to prove Equation (7), it suffices to establish

$$\mathrm{C}_1\left(S(n)\cdot\left(V(n)-I_n\right)\right)=\mathrm{C}_1\left(E(n)\cdot Q^{[1]}(n)\right), \tag{8}$$

and

$$\mathrm{R}_1\left(S(n)\cdot\left(V(n)-I_n\right)\right)=\mathrm{R}_1\left(E(n)\cdot Q^{[1]}(n)\right), \tag{9}$$

and also the remaining entries of $S(n)\cdot\big(V(n)-I_n\big)$ and $E(n)\cdot Q^{[1]}(n)$ satisfy the same recurrence relation, defined by

$$A_{i,j}=A_{i-1,j-1}+jA_{i-1,j},\quad i,j=2,3,\dots,n. \tag{10}$$

Let us do the required calculations. Let $i\geq 1$. Since

$$\mathrm{C}_1\big(V(n)-I_n\big)=\mathrm{C}_1(Q^{[1]}(n))=0,$$

it is clear that

$$\Big(S(n)\cdot\big(V(n)-I_n\big)\Big)_{i,1}=\big(E(n)\cdot Q^{[1]}(n)\big)_{i,1}=0,\quad i\geq 1,$$

and Equation (8) follows. Let us for the moment assume that $j\geq 2$. The $(1,j)$-entry of $S(n)\cdot\big(V(n)-I_n\big)$ is

$$\begin{aligned}\Big(S(n)\cdot\big(V(n)-I_n\big)\Big)_{1,j}&=\sum_{l=1}^{n}S_{1,l}\big(V(n)-I_n\big)_{l,j}\\&=S_{1,j-1}\big(V(n)-I_n\big)_{j-1,j}+S_{1,j}\big(V(n)-I_n\big)_{j,j}\\&=S_{1,j-1}+\big(j-1\big)S_{1,j}\\&=\begin{cases}1, & \text{if } j=2;\\ 0, & \text{if } j\geq 3.\end{cases}\end{aligned} \tag{11}$$

On the other hand, since $E_{1,l}=1$ for all l and $Q^{[1]}_{l,j}=0$ for all $l\geq j$, we obtain

$$\big(E(n)\cdot Q^{[1]}(n)\big)_{1,j}=\sum_{l=1}^{n}E_{1,l}Q^{[1]}_{l,j}=\sum_{l=1}^{j-1}Q^{[1]}_{l,j}. \tag{12}$$

Now, we rewrite $Q^{[1]}_{l,j}$, slightly,

$$Q^{[1]}_{l,j}=(-1)^{j-l+1}\frac{1}{(l-1)!(j-l-1)!}=(-1)^{j-l+1}\frac{1}{(j-2)!}\binom{j-2}{l-1}.$$

If this is substituted in Equation (12), then we obtain

$$\begin{aligned}\big(E(n)\cdot Q^{[1]}(n)\big)_{1,j}&=\sum_{l=1}^{j-1}(-1)^{j-l+1}\frac{1}{(j-2)!}\binom{j-2}{l-1}\\&=\frac{1}{(j-2)!}\sum_{l=0}^{j-2}(-1)^{j-l}\binom{j-2}{l}=\begin{cases}1, & \text{if } j=2;\\ 0, & \text{if } j\geq 3.\end{cases}\end{aligned} \tag{13}$$

Comparing Equations (11) and (13) gives us Equation (9).

Finally, we show that the entries $\big(E(n)\cdot Q^{[1]}(n)\big)_{i,j}$ and $\big(S(n)\cdot\big(V(n)-I_n\big)\big)_{i,j}$ satisfy Equation (10). Hence, from now on we assume $2\le i,j\le n$. First, since $Q^{[1]}_{l,j}=0$ for all $l\ge j$, we see that

$$\big(E(n)\cdot Q^{[1]}(n)\big)_{i,j}=\sum_{l=1}^{j-1}E_{i,l}Q^{[1]}_{l,j}=\sum_{l=1}^{j-1}(l+1)^{i-1}(-1)^{j-l+1}\frac{1}{(l-1)!(j-l-1)!}.$$

Similarly, we have

$$\big(E(n)\cdot Q^{[1]}(n)\big)_{i-1,j-1}=\sum_{l=1}^{j-2}(l+1)^{i-2}(-1)^{j-l}\frac{1}{(l-1)!(j-l-2)!},\tag{14}$$

and

$$\big(E(n)\cdot Q^{[1]}(n)\big)_{i-1,j}=\sum_{l=1}^{j-1}(l+1)^{i-2}(-1)^{j-l+1}\frac{1}{(l-1)!(j-l-1)!}.\tag{15}$$

If we substitute Equations (14) and (15) into Equation (10), we see that

$$\begin{aligned}
&\big(E(n)\cdot Q^{[1]}(n)\big)_{i-1,j-1}+j\big(E(n)\cdot Q^{[1]}(n)\big)_{i-1,j}\\
&\quad=\sum_{l=1}^{j-2}(l+1)^{i-2}(-1)^{j-l}\frac{1}{(l-1)!(j-l-2)!}\\
&\qquad\quad+j\sum_{l=1}^{j-1}(l+1)^{i-2}(-1)^{j-l+1}\frac{1}{(l-1)!(j-l-1)!}\\
&\quad=\sum_{l=1}^{j-2}(-1)^{j-l+1}\frac{(l+1)^{i-1}}{(l-1)!(j-l-1)!}+\frac{j^{i-1}}{(j-2)!}\\
&\quad=\sum_{l=1}^{j-1}(-1)^{j-l+1}\frac{(l+1)^{i-1}}{(l-1)!(j-l-1)!}=\big(E(n)\cdot Q^{[1]}(n)\big)_{i,j}.
\end{aligned}$$

Next, we observe that

$$\begin{aligned}
\big(S(n)\cdot\big(V(n)-I_n\big)\big)_{i,j}&=\sum_{l=1}^{n}S_{i,l}\big(V(n)-I_n\big)_{l,j}\\
&=S_{i,j-1}\big(V(n)-I_n\big)_{j-1,j}+S_{i,j}\big(V(n)-I_n\big)_{j,j}\\
&=S(i,j-1)+(j-1)S(i,j).
\end{aligned}$$

In a similar way, we obtain that

$$\big(S(n)\cdot\big(V(n)-I_n\big)\big)_{i-1,j-1}=S(i-1,j-2)+(j-2)S(i-1,j-1),\tag{16}$$

and

$$\big(S(n)\cdot\big(V(n)-I_n\big)\big)_{i-1,j}=S(i-1,j-1)+(j-1)S(i-1,j).\tag{17}$$

Again, if we substitute Equations (16) and (17) into Equation (10), we see that

$$\big(S(n)\cdot\big(V(n)-I_n\big)\big)_{i-1,j-1}+j\big(S(n)\cdot\big(V(n)-I_n\big)\big)_{i-1,j}$$
$$=S(i-1,j-2)+(j-2)S(i-1,j-1)+jS(i-1,j-1)+j(j-1)S(i-1,j)$$
$$=S(i-1,j-2)+(j-1)S(i-1,j-1)+(j-1)\left[S(i-1,j-1)+jS(i-1,j)\right]$$
$$=S(i,j-1)+(j-1)S(i,j)\quad\text{(by Equation (2))}$$
$$=\big(S(n)\cdot\big(V(n)-I_n\big)\big)_{i,j}.$$

The proof is now complete. □

Finally, a relationship between the matrices $S(n)$, $V(n)$, $E(n)$, and $Q^{[N]}(n)$ is established.

Theorem 1. *Let N, n be positive integers. Then, we have the following matrix identity:*

$$S(n)\cdot\big(V(n)-I_n\big)^N=E(n)\cdot Q^{[N]}(n).$$

Proof. It follows that

$$\begin{aligned}S(n)\cdot\big(V(n)-I_n\big)^N&=\big[S(n)\cdot\big(V(n)-I_n\big)\big]\big(V(n)-I_n\big)^{N-1}\\&=E(n)\cdot Q^{[1]}(n)\big(V(n)-I_n\big)^{N-1}\quad\text{(by Lemma 3)}\\&=E(n)\cdot Q^{[N]}(n),\ \text{(by Corollary 1)}\end{aligned}$$

as desired. □

3. The Matrices $\mathsf{S}^{[m]}(n)$

Preliminary observations show that the leading principal minors of the infinite matrix $\mathsf{S}^{[m]}(\infty)=[\mathsf{S}^{[m]}_{i,j}]_{i,j\geq 1}$ form an interesting integer sequence. In fact, we will prove that (see Theorem 2 below):

$$\det\mathsf{S}^{[m]}(n)=\prod_{i=1}^{n}i^m=n!^m.$$

To achieve the above result, we need to find an appropriate matrix decomposition of $\mathsf{S}^{[m]}(n)$. Let us look at some special cases first. A more interesting case may be when $m = 1$. In this case, for $n = 5$, we have:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \\ 1 & 31 & 90 & 65 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

We note that the above upper triangular matrix is $V(5)$, that is, $\mathsf{S}^{[1]}(5) = S(5) \cdot V(5)$. Similarly, for $m = 3$ and $n = 5$, we have:

$$\begin{bmatrix} 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \\ 1 & 31 & 90 & 65 & 15 \\ 1 & 63 & 301 & 350 & 140 \\ 1 & 127 & 966 & 1701 & 1050 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 7 & 6 & 1 & 0 \\ 0 & 8 & 19 & 9 & 1 \\ 0 & 0 & 27 & 37 & 12 \\ 0 & 0 & 0 & 64 & 61 \\ 0 & 0 & 0 & 0 & 125 \end{bmatrix}.$$

The above upper triangular matrix is in fact $V^3(5)$, that is, $\mathsf{S}^{[3]}(5) = S(5) \cdot V^3(5)$. The existence of such matrix decompositions for the matrices $\mathsf{S}^{[m]}(n)$ is therefore guaranteed, as the following theorem shows.

Theorem 2. *Let m, n be positive integers. Then, the following matrix decomposition holds:*

$$\mathsf{S}^{[m]}(n) = S(n) \cdot V(n)^m. \tag{18}$$

Also, we have

$$\det \mathsf{S}^{[m]}(n) = \det_{1 \le i,j \le n} [S(i+m, j)] = \prod_{i=1}^{n} i^m = n!^m.$$

Proof. First, we show that

$$\mathsf{S}^{[k]}(n) = \mathsf{S}^{[k-1]}(n) \cdot V(n), \tag{19}$$

for any integer $k \ge 1$. In order to do this, we observe first that

$$\begin{aligned} \left(\mathsf{S}^{[k-1]}(n) \cdot V(n)\right)_{i,1} &= \sum_{l=1}^{n} \mathsf{S}^{[k-1]}(n)_{i,l} V(n)_{l,1} \\ &= \mathsf{S}^{[k-1]}(n)_{i,1} V(n)_{1,1} \quad \text{(since } V(n) \text{ is upper triangular)} \\ &= 1 \quad \text{(since } \mathsf{S}^{[k-1]}(n)_{i,1} = V(n)_{1,1} = 1) \\ &= \mathsf{S}^{[k]}(n)_{i,1}, \end{aligned}$$

and then, for every $2 \le j \le n$, we obtain

$$\begin{aligned}
\big(\mathsf{S}^{[k-1]}(n)\cdot V(n)\big)_{i,j} &= \sum_{l=1}^{n} \mathsf{S}^{[k-1]}(n)_{i,l} V(n)_{l,j} \\
&= \mathsf{S}^{[k-1]}(n)_{i,j-1} V(n)_{j-1,j} + \mathsf{S}^{[k-1]}(n)_{i,j} V(n)_{j,j} \\
&= S(i+k-1, j-1) + jS(i+k-1, j) \\
&\quad \text{(by the definition of } V(n)) \\
&= S(i+k, j) \quad \text{(by Equation (2))} \\
&= \mathsf{S}^{[k]}(n)_{i,j}.
\end{aligned}$$

This completes the proof of Equation (19).

Now, repeatedly applying Equation (19), we get

$$\mathsf{S}^{[m]}(n) = \mathsf{S}^{[m-1]}(n)\cdot V(n) = \mathsf{S}^{[m-2]}(n)\cdot V(n)^2 = \cdots = \mathsf{S}^{[0]}(n)\cdot V(n)^m.$$

The decomposition (18) then follows by noting that $\mathsf{S}^{[0]}(n) = S(n)$.

Taking the determinant of both sides of Equation (18), we get

$$\det \mathsf{S}^{[m]}(n) = \det\big(S(n)\cdot V(n)^m\big) = \det S(n)\cdot \big(\det V(n)\big)^m = n!^m,$$

because $\det S(n) = 1$ and $\det V(n) = n!$. The proof is now complete. □

The following corollary is an immediate consequence of Theorem 2.

Corollary 2. *Let $m \ge 0$ be an integer. Then, for any positive integer n, we have*

$$\mathsf{S}^{[m+1]}(n) = \mathsf{S}^{[m]}(n)\cdot V(n),$$

and consequently,

$$\det \mathsf{S}^{[m+1]}(n) = n! \det \mathsf{S}^{[m]}(n).$$

Corollary 3. *Let $m_1, m_2 \ge 0$ be integers. Then, for any positive integer n, we have*

$$\mathsf{S}^{[m_1+m_2]}(n) = \mathsf{S}^{[m_1]}(n)\cdot s(n)\cdot \mathsf{S}^{[m_2]}(n).$$

Proof. It follows from Theorem 2 that for any integer $m \ge 0$, we have

$$V(n)^m = S(n)^{-1}\cdot \mathsf{S}^{[m]}(n).$$

Using this and the fact that $V(n)^{m_1+m_2} = V(n)^{m_1}\cdot V(n)^{m_2}$, we obtain

$$S(n)^{-1}\cdot \mathsf{S}^{[m_1+m_2]}(n) = S(n)^{-1}\cdot \mathsf{S}^{[m_1]}(n)\cdot S(n)^{-1}\cdot \mathsf{S}^{[m_2]}(n).$$

Since $S(n)^{-1} = s(n)$, the result follows by left-multiplying both sides by $S(n)$. □

Theorem 3. *Let m, n be positive integers. Then, the inverse of the matrix $\mathsf{S}^{[m]}(n)$ has the following matrix decomposition:*

$$\mathsf{S}^{[m]}(n)^{-1} = \tilde{V}(n)^m \cdot s(n).$$

Proof. Applying Theorem 2 and noting that $S(n)^{-1} = s(n)$, we obtain

$$\mathsf{S}^{[m]}(n)^{-1} = \big(S(n) \cdot V^m(n)\big)^{-1} = V^m(n)^{-1} \cdot S(n)^{-1} = \big(V(n)^{-1}\big)^m \cdot s(n).$$

The result then follows by Lemma 1. □

4. Identities Involving the Stirling Numbers

In this section, we give several identities involving the Stirling numbers of the first and second kind. We begin with the following observation.

Theorem 4. *Let i, j be integers with $i \geq j \geq 1$. Then, for any positive integer m, we have*

$$\sum_{l=1}^{i} s(i,l)S(l+m,j) = \begin{cases} i^m, & \text{if } i = j; \\ 0, & \text{if } i > j. \end{cases}$$

Proof. Let $i \geq j \geq 1$ be fixed. Choose the integer n so that $1 \leq i, j \leq n$, and consider the matrices $S(n)$, $V(n)^m$, and $\mathsf{S}^{[m]}(n)$ for any positive integer m. Then, by Theorem 2 and the fact that $S(n)^{-1} = s(n)$, we have

$$V(n)^m = S(n)^{-1} \cdot \mathsf{S}^{[m]}(n) = s(n) \cdot \mathsf{S}^{[m]}(n).$$

This implies that the (i, j)-entry on both sides must be equal, that is,

$$\big(V(n)^m\big)_{i,j} = \big(s(n) \cdot \mathsf{S}^{[m]}(n)\big)_{i,j}. \tag{20}$$

If $j = i$, then we have

$$i^m = \big(V(n)^m\big)_{i,i} = \big(s(n) \cdot \mathsf{S}^{[m]}(n)\big)_{i,i} = \sum_{l=1}^{n} s_{i,l}\mathsf{S}^{[m]}_{l,i} = \sum_{l=1}^{i} s(i,l)S(l+m,i).$$

Also, if $j < i$, since $V(n)^m$ is an upper triangular matrix and $s(n)$ is a lower triangular matrix, Equation (20) implies that

$$0 = \big(V(n)^m\big)_{i,j} = \big(s(n) \cdot \mathsf{S}^{[m]}(n)\big)_{i,j} = \sum_{l=1}^{n} s_{i,l}\mathsf{S}^{[m]}_{l,j} = \sum_{l=1}^{i} s(i,l)S(l+m,j).$$

The proof is complete. □

Theorem 5. *Let i, j, N be positive integers. Then, we have*

$$\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}S(i+l,j)=\sum_{k=1}^{j-1}(-1)^{j-k+1}\frac{k^{N-1}(k+1)^{i-1}}{(k-1)!(j-k-1)!}.$$

Proof. Let i, j be fixed. Choose an integer $n \geq 1$ large enough so that $1 \leq i, j \leq n$, and consider the matrices $E(n)$, $Q^{[N]}(n)$, $S(n)$, and $V(n)$. We have

$$\begin{aligned}
E(n)\cdot Q^{[N]}(n) &= S(n)\cdot\big(V(n)-I_n\big)^N \quad \text{(by Theorem 1)}\\
&= S(n)\cdot\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}V(n)^l \quad \text{(by Binomial Theorem)}\\
&= \sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}\left[S(n)\cdot V(n)^l\right]\\
&= \sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}\mathsf{S}^{[l]}(n). \quad \text{(by Theorem 2)}
\end{aligned}$$

Equating the (i, j)-entries on both sides of the above equation shows that

$$\big(E(n)\cdot Q^{[N]}(n)\big)_{i,j}=\Big(\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}\mathsf{S}^{[l]}(n)\Big)_{i,j}=\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}\mathsf{S}^{[l]}(n)_{i,j}.$$

By the definition of $\mathsf{S}^{[m]}(n)$, for each l, $\mathsf{S}^{[l]}(n)_{i,j} = S(i + l, j)$. Therefore, we can rewrite the above equation as

$$\big(E(n)\cdot Q^{[N]}(n)\big)_{i,j}=\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}S(i+l,j). \tag{21}$$

On the other hand, the left-hand side can be simplified as

$$\begin{aligned}
\big(E(n)\cdot Q^{[N]}(n)\big)_{i,j} &= \sum_{k=1}^{n}E(n)_{i,k}Q^{[N]}(n)_{k,j}=\sum_{k=1}^{n}E_{i,k}Q^{[N]}_{k,j}\\
&= \sum_{k=1}^{n}(k+1)^{i-1}Q^{[N]}_{k,j}=\sum_{k=1}^{j-1}(k+1)^{i-1}(-1)^{j-k+1}\frac{k^{N-1}}{(k-1)!(j-k-1)!},
\end{aligned} \tag{22}$$

where the last equality is obtained by Equation (6). Combining Equations (21) and (22), we obtain the conclusion of the theorem. □

As a corollary of Theorem 5, we have:

Corollary 4. *Let $i, j \geq 1$ be integers. Then, we have*

$$S(i+1,j)-S(i,j)=\sum_{k=1}^{j-1}(-1)^{j-k+1}\frac{(k+1)^{i-1}}{(k-1)!(j-k-1)!}.$$

Proof. This is immediate by applying Theorem 5 with $N = 1$. □

Theorem 6. *Let i, j, n be integers with $n \geq \max\{i, j\}$. Then, we have*

$$\sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \frac{(-1)^{k_2-k_1}(k_1-1)!}{k_2!} S(i+1, k_1) s(k_2, j) = \delta_{i,j},$$

where $\delta_{i,j}$ denotes the delta Kronecker function.

Proof. Setting $m = 1$ in Theorem 3, it follows that

$$\mathsf{S}^{[1]}(n) \cdot \tilde{V}(n) \cdot s(n) = I_n,$$

and hence

$$\left(\mathsf{S}^{[1]}(n) \cdot \tilde{V}(n) \cdot s(n)\right)_{i,j} = (I_n)_{i,j} = \delta_{i,j},$$

for each $i, j \geq 1$. Simplifying the left-hand side of this equation, we obtain

$$\begin{aligned}\left(\mathsf{S}^{[1]}(n) \cdot \tilde{V}(n) \cdot s(n)\right)_{i,j} &= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \mathsf{S}^{[1]}(n)_{i,k_1} \tilde{V}(n)_{k_1,k_2} s(n)_{k_2,j} \\ &= \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} S(i+1, k_1)(-1)^{k_2-k_1} \frac{(k_1-1)!}{k_2!} s(k_2, j),\end{aligned}$$

and the proof follows. □

Theorem 7. *Let $i, j \geq 1$ be integers. Then, we have*

$$\sum_{k=1}^{\min\{i+1,j\}} \frac{(-1)^{j-k}(k-1)!}{j!} S(i+1, k) = \begin{cases} S(i, j), & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $n \geq \max\{i, j\}$ be an integer. Applying Theorem 2 with $m = 1$, we have

$$\mathsf{S}^{[1]}(n) = S(n) \cdot V(n).$$

By Lemma 1, we can then obtain

$$\mathsf{S}^{[1]}(n) \cdot \tilde{V}(n) = S(n),$$

which implies that

$$\left(\mathsf{S}^{[1]}(n) \cdot \tilde{V}(n)\right)_{i,j} = S(n)_{i,j} = \begin{cases} S(i, j), & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Simplifying the left-hand side, we get

$$\left(\mathsf{S}^{[1]}(n) \cdot \tilde{V}(n)\right)_{i,j} = \sum_{k=1}^{n} \mathsf{S}^{[1]}(n)_{i,k} \tilde{V}(n)_{k,j} = \sum_{k=1}^{\min\{i+1,j\}} S(i+1, k) \frac{(-1)^{j-k}(k-1)!}{j!}.$$

This completes the proof. □

Theorem 8. *Let i, j, n be positive integers with $n \geq \max\{i, j\}$. Then, we have*

$$\sum_{k_1=1}^{n} \sum_{k_2=1}^{k_1} S(i+m_1, k_1)s(k_1, k_2)S(k_2+m_2, j) = S(i+m_1+m_2, j).$$

Proof. Using Corollary 3, we obtain

$$\begin{aligned} S(i+m_1+m_2, j) &= \left(\mathsf{S}^{[m_1+m_2]}(n)\right)_{i,j} \\ &= \left(\mathsf{S}^{[m_1]}(n) \cdot s(n) \cdot \mathsf{S}^{[m_2]}(n)\right)_{i,j} \\ &= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \mathsf{S}^{[m_1]}(n)_{i,k_1} s(n)_{k_1,k_2} \mathsf{S}^{[m_2]}(n)_{k_2,j} \\ &= \sum_{k_1=1}^{n} \sum_{k_2=1}^{k_1} S(i+m_1, k_1)s(k_1, k_2)S(k_2+m_2, j), \end{aligned}$$

as desired. □

Acknowledgement. Part of this work was completed while the first author held a visiting position at the Department of Applied Mathematics, School of Aeronautical and Space Engineering, University of Vigo, Ourense, Spain. She gratefully acknowledges the Department of Applied Mathematics at the University of Vigo for their hospitality.

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