#A28 INTEGERS 25 (2025)

FIXED HOOKS IN ARBITRARY COLUMNS OF PARTITIONS

Philip Cuthbertson
Department of Mathematics, Michigan Technological University, Houghton, Michigan
pecuthbe@mtu.edu

Received: 6/26/24, Accepted: 3/11/25, Published: 3/26/25

Abstract

In a paper by the author, Hemmer, Hopkins, and Keith, the concept of a fixed point in a sequence was applied to the sequence of first column hook lengths of a partition. In this paper, we generalize this notion to fixed hook lengths in an arbitrary column of a partition. We establish combinatorial connections between these fixed hooks and colored partitions that have interesting gap and mex-like conditions. Additionally, we obtain several generating functions for hook lengths of a given fixedness by hook length or part size in unrestricted partitions, as well as some classical restrictions such as odd and distinct partitions.

1. Introduction

Recall that a weakly decreasing sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ is called a *partition* of n if $\lambda_1 + \lambda_2 + \ldots + \lambda_r = n$ and then each λ_i is called a part. The *Young diagram* of a partition is a representation of the partition in which each part is represented by a row of λ_i many squares justified to the top right. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n and λ' denote its conjugate. For a square (i, j) in the Young diagram of λ, define $h_{i,j}(\lambda) = \lambda_i + \lambda'_j - i - j + 1$ to be the *hook length of (i, j)*. In terms of the Young diagram, this is the sum of the number of squares in the ith row and column at least j, called the *arm*, and the number of squares in the jth column and row at least i, called the *leg*.

In [5] Blecher and Knopfmacher defined the notion of a fixed point in a partition so that the partition λ has a *fixed point* if there is some i such that $\lambda_i = i$. This was extended by Hopkins and Sellers [9] so that λ is said to have an *h-fixed point* if $\lambda_i = i + h$ for some i. Instead of looking at the sequence of parts, the author, Hemmer, Hopkins, and Keith in [8] considered instead $\{h_{1,1}(\lambda), h_{2,1}(\lambda), \ldots, h_{t,1}(\lambda)\}$, the sequence of first column hook lengths. This sequence uniquely defines the partition λ and is notable in representation theory. Here we concern ourselves with the

DOI: 10.5281/zenodo.15091075

7	5	4	2
6	4	3	1
4	2	1	
1			

Figure 1: Young diagram of $(4,4,3,1)$ labeled by hook lengths

sequence of mth column hook lengths $\{h_{1,m}(\lambda), h_{2,m}(\lambda), \ldots, h_{t,m}(\lambda)\}$. We can no longer recover the entire partition from this sequence since there is no information about the parts of size less than m.

We also define the q-Pochhammer symbol along with the Gaussian binomial coefficient and make use of their standard notation and combinatorial interpretations (see, for instance, Andrews [2]):

$$(a;b)_n := \prod_{k=0}^{n-1}(1-ab^k),$$
$$(a;b)_\infty := \prod_{k=0}^{\infty}(1-ab^k),$$
$$\binom{a}{b}_q := \frac{(q;q)_a}{(q;q)_b(q;q)_{a-b}}.$$

We now generalize some of the combinatorial theorems from [8], noting that in each case their original theorems can be obtained by setting $m=1$. Their Theorem 2.1 is generalized in the following theorem.

Theorem 1. *The number of partitions of n having a 0-fixed hook in the mth column is equal to the sum over L of the number of times across all partitions of n, with two colors of parts $1,2,\ldots,m-1$, that a part of size L appears exactly $L+m-1$ times in the first color, but $L+1, L+2, \ldots, L+2m-2$ are not parts in the first color.*

Example 1. Using $m=3$ and looking at partitions of 10 we find the two sets given in Table 1.

Generalizing Theorems 3.3, 3.4, and 4.3 from [8] we have the following theorems.

Theorem 2. *The number of partitions of n with an h-fixed hook in the mth column arising from a part of size m equals the number of times in all partitions of $n-mh$ where m appears as a part exactly once, there are no parts of sizes $m+1, m+2, \ldots, 2m-1$, and there are at least $-h$ parts of size at least $2m$.*

Generalizing Theorem 3.4 from [8] we get the following theorem.

Description 1	Description 2
$(6,4)$	$(7,1^3)$
$(5,4,1)$	$(6,1,1^3)$
$(4,4,2)$	$(2,2^4)$
$(4,4,1,1)$	$(2^4,1^2)$
$(4,3,3)$	$(2^4,1,1)$
$(3,3,3,1)$	$(2^4,1^2)$
$(3,2,2,2,1)$	$(2^3,1^1,1^3)$
$(3,2,2,1,1,1)$	$(2^2,1^3,1^3)$
$(3,2,1,1,1,1,1)$	$(2,1^5,1^3)$
$(3,1,1,1,1,1,1,1)$	$(1^7,1^3)$

Table 1: Example of Theorem 1.

Theorem 3. *The number of times in all partitions of n that an h-fixed hook arises from a part of size k in the mth column equals the number of partitions of $n+\binom{k-m+1}{2}-k(k-h-m+1))$ where there are two colors of parts $1,2,\ldots,m-1$, parts of size $k-m+1,\ldots,k+m-1$ do not appear in the first color, and at least one part of every size $1,2,\ldots,k-m$ appear in the first color.*

Generalizing Theorem 4.3 from [8] we have the following theorem.

Theorem 4. *The generating function for the number of mth column hooks of size k in all partitions of n is*

$$\frac{q^{km}}{(q^k;q)_\infty}\sum_{l=1}^{k}\frac{q^{-(l-1)(m-1)}(q^m;q)_{l-1}}{(q;q)_{l-1}(q;q)_{k-l}}=\frac{q^{m+k-1}}{(q^k;q)_\infty}\sum_{l=1}^{k}\frac{q^{(m-1)(k-l)}(q^m;q)_{l-1}}{(q;q)_{l-1}(q;q)_{k-l}}.$$

Interestingly, for a fixed m, the function in the above theorem stabilizes to q^k times the generating function for the number of parts of size $m-1$ appearing in all partitions of n. See, for example, the sequences A000070, A024786, and A024787 in the OEIS [10].

Theorem 5. *For a fixed $m\geq 2$,*

$$\lim_{k\to\infty}\frac{q^{m-1}}{(q^k;q)_\infty}\sum_{l=1}^{k}\frac{q^{(m-1)(k-l)}(q^m;q)_{l-1}}{(q;q)_{l-1}(q;q)_{k-l}}=\frac{q^{m-1}}{(1-q^{m-1})(q;q)_\infty}.$$

In order to prove the previous theorems, we first establish several generating functions that are also of independent interest.

2. Part Sizes

Figure 1 will be a useful reference for constructing the generating functions in the next two sections.

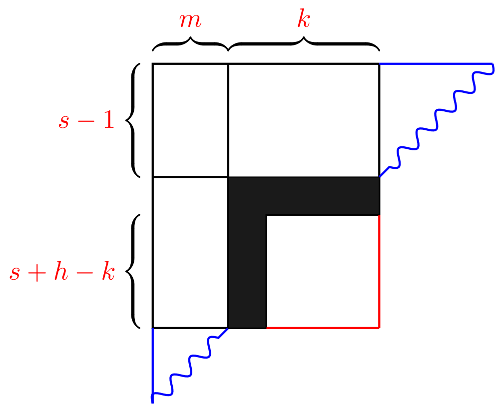

Figure 2: λ with an h-fixed hook $h_{s,m} = s + h$ at part $\lambda_s = k$.

The fixed hook is marked in black and we use the terminology "below the fixed hook" to refer to the red box in the diagram. Theorem 3.1 of [8] establishes a generating function that counts fixed points in the sequence of first column hook lengths in any partition of n. We include this theorem for completeness.

Theorem 6. *The generating function for the number of partitions of n, with an h-fixed hook arising from a part of size k, is given by*

$$\sum_{s=k-h}^{\infty} \frac{q^{(k+1)(s-1)+h+1}}{(q;q)_{s-1}} \binom{s+h-1}{k-1}_q = \sum_{s=0}^{\infty} \frac{q^{s(k+1)+k(k-h)}}{(q;q)_{s+k-h-1}} \binom{s+k-1}{k-1}_q.$$

We generalize the above to an arbitrary column in the following theorem.

Theorem 7. *The generating function for the number of partitions of n, with an h-fixed hook in the mth column arising from a part of size $k \geq m$, is given by*

$$\sum_{s=k-h-m+1}^{\infty} \frac{q^{s(k+m)+m(h-k+m-1)}}{(q;q)_{s-1}(q;q)_{m-1}} \binom{s+h-1}{k-m}_q$$
$$= \sum_{s=0}^{\infty} \frac{q^{s(k+m)+k(k-h-m+1)}}{(q;q)_{s+k-h-m}(q;q)_{m-1}} \binom{s+k-m}{k-m}_q.$$

Proof. Using Theorem 6 we extend the Young diagram by appending $m-1$ columns to the left. Each column must have length at least $2s+h-k$. This is generated by

$q^{(m-1)(2s+h-k)}/(q;q)_{m-1}$, giving

$$\sum_{s=k-h}^{\infty} \frac{q^{(k+1)(s-1)+h+1+(m-1)(2s+h-k)}}{(q;q)_{s-1}(q;q)_{m-1}} \binom{s+h-1}{k-1}_q.$$

Parts of size k are now of size $k+m-1$. After reindexing by substituting $k+m-1$ for k, one gets the theorem. The second line follows by substituting $s+k-h-m+1$ for s. □

One can derive similar formulae for certain restrictions on partitions. For instance, when considering just those partitions in which each part is odd, one arrives at the following theorem.

Theorem 8. *The generating function for the number of odd partitions of n, with an h-fixed hook in the mth column arising from a part of size $k \geq m$, is given by*

$$\sum_{s=k-h-m+1}^{\infty} \frac{q^{s(k+m)+m(m+h-k-1)}}{(q^2;q^2)_{s-1}(q;q^2)_{(m-1)/2}} \binom{s+h-k+(k-m)/2}{s+h-k}_{q^2}$$

if m is odd, and

$$\sum_{s=k-h-m+1}^{\infty} \frac{q^{s(k+m+1)+m(h-k+m)+h-k+1}}{(q^2;q^2)_{s-1}(q;q^2)_{m/2}} \binom{s+h-k+(k-m-1)/2}{s+h-k}_{q^2}$$

if m is even.

Proof. In the case that $m=1$, one gets the formula

$$\sum_{s=k-h}^{\infty} \frac{q^{k(s-1)+h+s}}{(q^2;q^2)_{s-1}} \binom{s+h-k+(k-1)/2}{s+h-k}_{q^2}.$$

The term $q^{k(s-1)}/(q^2;q^2)_{s-1}$ generates the $s-1$ rows above the part of size k we are considering. The term $q^{h+s} = q^{k+(s+h-k)}$ times the binomial coefficient generates the fixed hook and the $s+h-k$ rows below it. Generalizing to an arbitrary odd m, we add $m-1$ columns to the left of the partition. Each column must have length at least $2s+h-k$ and the parts below the fixed hook must be odd. This is generated by $q^{(m-1)(2s+h-k)}/(q;q^2)_{m-2}$, which gives

$$\sum_{s=k-h}^{\infty} \frac{q^{(m-1)(2s+h-k)+k(s-1)+h+s}}{(q^2;q^2)_{s-1}(q;q^2)_{(m-1)/2}} \binom{s+h-k+(k-1)/2}{s+h-k}_{q^2}.$$

Parts of size k are now of size $k+m-1$. After reindexing by substituting $k+m-1$ for k and simplifying the exponent, we have the first equation. The case that m is even is similar, but with two minor changes. All the parts below the fixed hook must have opposite parity to m and this is taken care of by multiplying in q^{s+h-k} and changing the binomial coefficient to $\binom{s+h-k+(k-2)/2}{s+h-k}_{q^2}$. We also change the second Pochhammer to $(q;q^2)_{m/2}$, which gives

$$\sum_{s=k-h}^{\infty} \frac{q^{(m-1)(2s+h-k)+k(s-1)+h+s+(s+h-k)}}{(q^2;q^2)_{s-1}(q;q^2)_{m/2}} \binom{s+h-k+(k-2)/2}{s+h-k}_{q^2}.$$

After reindexing by substituting $k+m-1$ for k and simplifying the exponent, we have the second equation. □

Despite partitions into odd parts and those into distinct parts being equinumerous, the sets of corresponding hook lengths behave quite differently. To illustrate this we have the following theorem to compare to Theorem 8.

Theorem 9. *The generating function for the number of distinct partitions of n, with an h-fixed hook in the mth column arising from a part of size $k \geq m$, is given by*

$$\sum_{s=0}^{k-m} \frac{q^{s(k+m)+k(k-h-m+1)+\binom{s+k-m+1-h}{2}+\binom{s}{2}}(-q;q)_{m-1}}{(q;q)_{s+k-h-m}} \binom{k-m}{s}_q.$$

Proof. In the case that m = 1, one gets the formula

$$\sum_{s=k-h}^{2k-h-1} \frac{q^{(s-1)k+s+h+\binom{s}{2}+\binom{s+h-k}{2}}}{(q;q)_{s-1}} \binom{k-1}{s+h-k}_q.$$

Each of the $s-1$ rows above the part of size k we are considering have length at least $k+1$ giving the term $q^{(s-1)k}$ and the term $q^{\binom{s}{2}}/(q;q)_{s-1} = q^{1+2+\ldots+s-1}/(q;q)_{s-1}$ guarantees these parts are all distinct. The hook itself is generated by q^{s+h} and the portion of the partition under the hook is generated by $q^{\binom{s+h-k}{2}}\binom{k-1}{s+h-k}_q$. The finite bounds on the sum are due to the limited possible part sizes below a part of size k in a distinct partition. Generalizing to an arbitrary m, we add $m-1$ columns to the left of the partition. Each of these columns must have length at least $2s+h-k$ which gives the term $q^{(m-1)(2s+h-k)}(-q;q)_{m-1}$. We then have,

$$\sum_{s=k-h}^{2k-h-1} \frac{q^{(s-1)k+s+h+\binom{s}{2}+\binom{s+h-k}{2}+(m-1)(2s+h-k)}(-q;q)_{m-1}}{(q;q)_{s-1}} \binom{k-1}{s+h-k}_q.$$

Parts of size k are now of size $k+m-1$, so after reindexing by substituting k for $k+m-1$ and simplifying the exponent, we have

$$\sum_{s=k-m+1-h}^{2k-2m+1-h} q^{s(k+m)+m(m+h-k-1)+\binom{s}{2}+\binom{s+h-k+m-1}{2}}$$

$$\cdot \frac{(-q;q)_{m-1}}{(q;q)_{s-1}} \binom{k-m}{s+h-k+m-1}_q.$$

Lastly, reindexing by substituting $s+k-m+1-h$ for s gives the theorem. □

3. Hook Sizes

The generating functions tend to be simpler when considering fixed hooks arising from a given hook size rather than a given part size, and this section will focus primarily on this interpretation. Previously established (Theorem 4.1 [8]) was a generating function analogous to Theorem 6, but by a given hook length instead of a given part size. Another benefit of counting by hook length is that the generating functions all turn out to be finite sums which allows them to be summed over all h and m to obtain the generating function for the number of hooks of a given length in certain restricted families of partitions easier than may be otherwise possible. We restate Theorem 4.1 from [8] for completeness.

Theorem 10. *The generating function for the number of partitions of n, with an h-fixed hook in the first columm arising from a hook of size k, is given by*

$$\sum_{l=1}^{k} \frac{q^{k+l(k-h-1)}}{(q;q)_{k-h-1}} \binom{k-1}{l-1}_q.$$

We then generalize this to an arbitrary column in the following theorem.

Theorem 11. *The generating function for the number of partitions of n, with an h-fixed hook in the mth column arising from a hook of size k, is given by*

$$\sum_{l=1}^{k} \frac{q^{(m-1)(2k-h-l)+k+l(k-h-1)}}{(q;q)_{k-h-1}(q;q)_{m-1}} \binom{k-1}{l-1}_q.$$

Proof. Using Theorem 10 we extend the Ferrers diagram by adding in $m-1$ columns to the left. Each of these columns must have length at least $k-h-1+k-l+1 = 2k-h-l$. This is generated by $q^{(m-1)(2k-h-l)}/(q;q)_{m-1}$ and gives the theorem. □

Similar to the previous section, we specialize to obtain similar generating functions for certain families of partitions.

Theorem 12. *The generating function for the number of odd partitions of n, with an h-fixed hook in the mth column arising from a hook of size k, is given by*

$$\sum_{\substack{l=1 \\ l\ odd}}^{k} \frac{q^{k+l(k-h-1)+(m-1)(2k-h-l)}}{(q^2;q^2)_{k-h-1}(q;q^2)_{(m-1)/2}} \binom{k-l+(l-1)/2}{k-l}_{q^2}$$

if m is odd, and

$$\sum_{\substack{l=1\\ l\ even}}^{k} \frac{q^{k+l(k-h-1)+(m-1)(2k-h-l)+(k-l)}}{(q^2;q^2)_{k-h-1}(q;q^2)_{m/2}} \binom{k-l+(l-2)/2}{k-l}_{q^2}$$

if m is even.

Proof. Note that in all cases we need the arm length, l, and m to have the same parity in order to guarantee that the fixed hook is arising from an odd part. In the case that $m = 1$, one gets the formula

$$\sum_{\substack{l=1\\ l\text{ odd}}}^{k} \frac{q^{k+l(k-h-1)}}{(q^2;q^2)_{k-h-1}} \binom{k-l+(l-1)/2}{k-l}_{q^2},$$

where the sum is taken over possible arm lengths that the hook can take on. The term $q^{l(k-h-1)}/(q^2;q^2)_{k-h-1}$ generates the $k-h-1$ rows above the hook of size k we are considering. The term q^k generates the fixed hook and the binomial coefficient generates the portion of the partition under the hook. Generalizing to an arbitrary odd m, we add $m-1$ columns to the left of the partition. Each column must have length at least $2k-h-l$ and the parts below the fixed hook must be odd. This is generated by $q^{(m-1)(2k-h-l)}/(q;q^2)_{(m-1)/2}$, which gives the first equation. The case that m is even is similar. We need all parts to be odd, so we insert a q^{k-l} under the hook and change the binomial coefficient to $\binom{k-l+(l-2)/2}{k-l}_{q^2}$. We also change the second Pochhammer to $(q;q^2)_{m/2}$. Summing over even l instead of odd gives the second equation. □

Theorem 13. *The generating function for the number of distinct partitions of n, with an h-fixed hook in the mth column arising from a hook of size k, is given by*

$$\sum_{l=\lceil (k+1)/2\rceil}^{k} \frac{q^{k+l(k-h-1)+(m-1)(2k-h-l)+\binom{k-h}{2}+\binom{k-l}{2}}(-q;q)_{m-1}}{(q;q)_{k-h-1}} \binom{l-1}{k-l}_q .$$

Proof. In the case that $m = 1$, one gets the formula

$$\sum_{l=\lceil (k+1)/2\rceil}^{k} \frac{q^{k+l(k-h-1)+\binom{k-h}{2}+\binom{k-l}{2}}}{(q;q)_{k-h-1}} \binom{l-1}{k-l}_q .$$

The term $q^{k+l(k-h-1)+\binom{k-h}{2}}/(q;q)_{k-h-1}$ generates the hook and the $k-h-1$ distinct rows above the hook. The term $q^{\binom{k-l}{2}}\binom{l-1}{k-l}_q$ generates the distinct parts below the hook. Generalizing to an arbitrary m, we add $m-1$ columns to the left of the partition. Each column has length at least $2k-h-l$ and still distinct which is generated by $q^{(m-1)(2k-h-l)}(-q;q)_{m-1}$, giving the theorem. □

Remark 1. For a fixed k, summing the generating functions from Theorems 12 and 13 over all m and h gives a way to count the number of hooks of length k in the corresponding sets of partitions. This generates $a_k(n)$ and $b_k(n)$ as defined in [4], which is equivalent to Theorem 2.1 in [7].

Theorem 14. *The generating function for the number of odd and distinct partitions of n, with an h-fixed hook in the mth column arising from a hook of size k, is given by*

$$\sum_{\substack{l=\lceil \frac{2k-1}{3}\rceil \\ l \text{ odd}}}^{k} \frac{q^{(m-1)(2k-h-l)+k+l(k-h-1)+2\binom{k-h}{2}+2\binom{k-l}{2}}(-q,q^2)_{\frac{m-1}{2}}}{(q^2;q^2)_{k-h-1}} \binom{k-l+\frac{3l-2k}{2}}{k-l}_{q^2}$$

if m is odd and

$$\sum_{\substack{l=\lceil \frac{2k}{3}\rceil \\ l \text{ even}}}^{k} \frac{q^{k-l+(m-1)(2k-h-l)+k+l(k-h-1)+2\binom{k-h}{2}+2\binom{k-l}{2}}(-q,q^2)_{\frac{m}{2}}}{(q^2;q^2)_{k-h-1}} \binom{k-l+\frac{3l-2k-2}{2}}{k-l}_{q^2}$$

if m is even.

Proof. Note that l and m must have the same parity since $m-1+l$ must be an odd part size. Considering first m odd, we have the term $q^{(m-1)(2k-h-l)}(-q,q^2)_{(m-1)/2}$ generating the first $m-1$ columns. The term

$$q^{l(k-h-1)+2\binom{k-h}{2}}/(q^2;q^2)_{k-h-1} = q^{l(k-h-1)+2+4+\ldots+2(k-h-1)}/(q^2;q^2)_{k-h-1}$$

generates the rest of the first $k-h-1$ rows of the partition. The last factor is

$$q^{2\binom{k-l}{2}}\binom{k-l+\frac{3l-2k}{2}}{k-l}_{q^2} = q^{2+4+\ldots+2(k-l-1)}\binom{k-l+\frac{3l-2k}{2}}{k-l}_{q^2}$$

which generates the portion under the hook while guaranteeing each part is still distinct and odd. The case that m is even is almost identical. There is an extra term of q^{k-l} that is inserted under the hook adding 1 to each part to make them all odd. The Pochhammer is now $(-q,q^2)_{m/2}$ and the binomial coefficient is now $\binom{k-l+\frac{3l-2k-2}{2}}{k-l}_{q^2}$. □

Summing Theorem 14 over all m and h gives a way to count the number of hooks of a given length in all odd and distinct partitions of n. While not the cleanest representation, the following proposition does give an explicit generating function for the number of hooks of a given length in all odd and distinct partitions.

Proposition 1. *The generating function for the number of hooks of length k in all odd and distinct partitions of n is given by*

$$(-q;q^2)_\infty q^k \sum_{\substack{l=\lceil \frac{2k-1}{3} \rceil \\ l \text{ odd}}}^{k} q^{2\binom{k-l}{2}} \binom{k-l+\frac{3l-2k}{2}}{k-l}_{q^2} \sum_{m=0}^{\infty} \frac{q^{2m(k-l+1)}}{(-q^{2m+1};q^2)_{\frac{l-1}{2}}}$$

$$+ (-q;q^2)_\infty q^k \sum_{\substack{l=\lceil \frac{2k}{3} \rceil \\ l \text{ even}}}^{k} q^{2\binom{k-l}{2}+k-l} \binom{k-l+\frac{3l-2k-2}{2}}{k-l}_{q^2} \sum_{m=0}^{\infty} \frac{q^{2m(k-l+1)}}{(-q^{2m+1};q^2)_{\frac{l}{2}}}.$$

Proof. We begin by summing the odd case of Theorem 14 over all m and h, and simplifying from there. For a fixed k, we have

$$q^k \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} (-q,q^2)_{\frac{m-1}{2}} \sum_{h=-\infty}^{k-1} \frac{q^{2\binom{k-h}{2}}}{(q^2;q^2)_{k-h-1}}$$

$$\cdot \sum_{\substack{l=\lceil \frac{2k-1}{3} \rceil \\ l \text{ odd}}}^{k} q^{(m-1)(2k-h-l)+l(k-h-1)+2\binom{k-l}{2}} \binom{k-l+\frac{3l-2k}{2}}{k-l}_{q^2}.$$

Reordering the sums and reindexing by substituting h for $k-h-1$ gives

$$q^k \sum_{\substack{l=\lceil \frac{2k-1}{3} \rceil \\ l \text{ odd}}}^{k} q^{2\binom{k-l}{2}} \binom{k-l+\frac{3l-2k}{2}}{k-l}_{q^2}$$

$$\cdot \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} q^{(m-1)(k-l+1)} (-q;q^2)_{\frac{m-1}{2}} \sum_{h=0}^{\infty} \frac{q^{h^2+h(l+m)}}{(q^2;q^2)_h}.$$

Using the identity (Corollary 2.2 in [2])

$$(-z;q)_\infty = \sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)/2}}{(q;q)_n},$$

the inner sum becomes precisely $(-q^{l+m+1};q^2)_\infty$. Since the sum on m is taken over odd values, we can reindex by substituting $2m+1$ for m and take the sum over all m to get

$$q^k \sum_{\substack{l=\lceil \frac{2k-1}{3} \rceil \\ l \text{ odd}}}^{k} q^{2\binom{k-l}{2}} \binom{k-l+\frac{3l-2k}{2}}{k-l}_{q^2} \sum_{m=0}^{\infty} q^{2m(k-l+1)} (-q;q^2)_m (-q^{l+2m+2};q^2)_\infty.$$

Using the fact that

$$(-q;q^2)_m (-q^{l+2m+2};q^2)_\infty = (-q;q^2)_\infty / (-q^{2m+1};q^2)_{\frac{l-1}{2}}$$

we get the first term of the proposition. The case that l is even follows similarly. □

Remark 2. In [6] Cossaboom used their Theorem 2.2, which is equivalent to Proposition 1, and Theorem 1.2 in [1] in order to settle Conjecture 5.3 in [7]. This made explicit the different asymptotic behaviors of hook lengths in self-conjugate partitions and in partitions that are odd and distinct.

4. Proofs for Theorems in Section 1

In this section we provide proofs for each of the theorems from Section 1.

Proof of Theorem 1. Setting $h = 0$, fixing m, and then summing Theorem 11 over all k, we get

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{q^{(m-1)(2k-l)+k+l(k-1)}}{(q;q)_{m-1}(q;q)_{k-1}}\binom{k-1}{l-1}_q \\
&\quad = \frac{1}{(q;q)_{m-1}}\sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\frac{q^{l(k-m)+k(2m-1)}}{(q;q)_{l-1}(q;q)_{k-l}} \\
&\quad = \frac{1}{(q;q)_{m-1}}\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}\frac{q^{l(k+l-m)+(k+l)(2m-1)}}{(q;q)_{l-1}(q;q)_{k}} \\
&\quad = \frac{1}{(q;q)_{m-1}}\sum_{l=1}^{\infty}\frac{q^{l(l+m-1)}}{(q;q)_{l-1}}\sum_{k=0}^{\infty}\frac{q^{k(l+2m-1)}}{(q;q)_{k}} \\
&\quad = \frac{1}{(q;q)_{m-1}}\sum_{l=1}^{\infty}\frac{q^{l(l+m-1)}}{(q;q)_{l-1}}\cdot\frac{1}{(q^{l+2m-1};q)_{\infty}} \\
&\quad = \frac{1}{(q;q)_{m-1}(q;q)_{\infty}}\sum_{l=1}^{\infty}q^{l(l+m-1)}(q^{l};q)_{2m-1}.
\end{aligned}$$

The first line and the last line give the combinatorial descriptions in the theorem. □

Proof of Theorem 2. Set $k = m$ in Theorem 6 to obtain

$$\begin{aligned}
\sum_{s=1-h}^{\infty}\frac{q^{m(h-1)+s(2m)}}{(q;q)_{m-1}(q;q)_{s-1}}\binom{s+h-1}{0}_q &= \frac{q^{mh+m}}{(q;q)_{m-1}}\sum_{s=-h}^{\infty}\frac{q^{2ms}}{(q;q)_s} \\
&= \frac{q^{mh+m}}{(q;q)_{m-1}}\left(\frac{1}{(q^{2m};q)_{\infty}}-\sum_{s=0}^{-h-1}\frac{q^{2ms}}{(q;q)_s}\right).
\end{aligned}$$

Interpreting the first and last lines gives the theorem. □

Proof of Theorem 3. Using Theorem 7 we have

$$\sum_{s=0}^{\infty} \frac{q^{s(k+m)+k(k-h-m+1)}}{(q;q)_{s+k-h-m}(q;q)_{m-1}} \binom{s+k-m}{k-m}_q$$
$$= \frac{q^{k(k-h-m+1)-\binom{k-m+1}{2}}}{(q;q)_{m-1}} \sum_{s=0}^{\infty} \frac{q^{s(k+m)+\binom{k-m+1}{2}}}{(q;q)_{s+k-h-m}} \binom{s+k-m}{k-m}_q.$$

Interpreting the first and second lines gives the theorem. □

Proof of Theorem 4. For a fixed m and k, summing Theorem 11 over all h we have,

$$\sum_{h=-\infty}^{k-1} \sum_{l=1}^{k} \frac{q^{(m-1)(2k-h-l)+k+l(k-h-1)}}{(q;q)_{k-h-1}(q;q)_{m-1}} \binom{k-1}{l-1}_q.$$

Reindexing by substituting $k-h-1$ for h gives

$$\frac{q^{km}}{(q;q)_{m-1}} \sum_{l=1}^{k} q^{-(l-1)(m-1)} \binom{k-1}{l-1}_q \sum_{h=0}^{\infty} \frac{q^{h(m+l-1)}}{(q;q)_h}$$
$$= \frac{q^{km}}{(q;q)_{m-1}} \sum_{l=1}^{k} q^{-(l-1)(m-1)} \frac{(q;q)_{k-1}}{(q;q)_{k-l}(q;q)_{l-1}(q^{m+l-1};q)_\infty}.$$

The theorem follows by multiplying by $(q^m;q)_{l-1}/(q^m;q)_{l-1}$ and by the fact that

$$\frac{(q;q)_{k-1}}{(q;q)_{m-1}(q^m;q)_{l-1}(q^{m+l-1};q)_\infty} = \frac{1}{(q^k;q)_\infty}$$

□

Proof of Theorem 5. For a fixed m, we have

$$\lim_{k\to\infty} \frac{q^{m-1}}{(q^k;q)_\infty} \sum_{l=1}^{k} \frac{q^{(m-1)(k-l)}(q^m;q)_{l-1}}{(q;q)_{l-1}(q;q)_{k-l}} = q^{m-1} \lim_{k\to\infty} \sum_{l=0}^{k-1} \frac{q^{(m-1)(l+1)}(q^m;q)_{k-l-1}}{(q;q)_{k-l-1}(q;q)_l}$$
$$= \frac{q^{m-1}(q^m;q)_\infty}{(q;q)_\infty} \sum_{l=0}^{\infty} \frac{q^{(m-1)(l+1)}}{(q;q)_l}$$
$$= \frac{q^{m-1}}{(q^{m-1};q)_\infty (q;q)_{m-1}}$$
$$= \frac{q^{m-1}}{(1-q^{m-1})(q;q)_\infty}.$$

This gives the theorem. □

5. Future Work

There are many directions that future work could be taken. The same methods required to produce generating functions for the sets of restricted partitions studied here could be applied to more exotic families and potentially get similar generating functions; for example, analogous results for self-conjugate partitions to mirror those of odd distinct partitions could be obtained. As mentioned in Remarks 1 and 2, asymptotic analysis on some of these theorems could lend insight into the relative distributions of hook lengths between different sets of equinumerous partitions. Another relatively straightforward extension would be to establish similar combinatorial results to those of the theorems in Section 1. All of these theorems use the generating functions established specifically for unrestricted partitions, but very similar analogous results could readily be established by looking at the generating functions that are associated with the hook lengths in any form of restricted partitions. A less straightforward, but nonetheless interesting, continuation would be to further delve into the connection that the author, Hemmer, Hopkins, and Keith in [8] saw with the truncated pentagonal number theorem of Andrews and Merca [3]. Lastly, Theorem 5 seems to suggest the truth of the following conjecture.

Conjecture 1. For $k \geq m \geq 2$, the difference

$$\frac{q^{m-1}}{(1-q^{m-1})(q;q)_\infty} - \frac{q^{m+k-1}}{(q^k;q)_\infty} \sum_{l=1}^{k} \frac{q^{(m-1)(k-l)}(q^m;q)_{l-1}}{(q;q)_{l-1}(q;q)_{k-l}}$$

is positive.

If true, this would imply that there are more parts of size $m-1$ in partitions of $n+k$ than there are hooks of size k in the mth column of partitions of n and a combinatorial proof of this may be interesting.

References

[1] T. Amdeberhan, G. E. Andrews, K. Ono, and A. Singh, Hook lengths in self-conjugate partitions, *Proc. Amer. Math. Soc. Ser. B* **11** (2024) 345–357.

[2] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, Cambridge, 1998.

[3] G. E. Andrews and M. Merca, The truncated pentagonal number theorem, *J. Combin. Theory Ser. A* **119** (8) (2012), 1639–1643.

[4] C. Ballantine, H. E. Burson, W. Craig, A. Folsom, and B. Wen, Hook length biases and general linear partition inequalities, *Res. Math. Sci.* **10**, (2023) 41.

[5] A. Blecher and A. Knopfmacher, Fixed points and matching points in partitions, *Ramanujan J.* **58** (2022), 23–41.

[6] C. Cossaboom, Hook length biases for self-conjugate partitions and partitions with distinct odd parts, preprint, `arXiv: 2406.18480`.

[7] W. Craig, M. L. Dawsey, and G.-N. Han, Inequalities and asymptotics for hook numbers in restricted partitions, preprint, `arXiv: 2311.15013`.

[8] P. Cuthbertson, B. Hopkins, D. Hemmer, and W. J. Keith, Partitions with Fixed Points in the Sequence of First-Column Hook Lengths, *Ann. Comb.* (2024).

[9] B. Hopkins and J. Sellers, On Blecher and Knopfmacher's fixed points for integer partitions, *Discrete Math.* **347** (5) (2024).

[10] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, 2023. Published electronically at http://oeis.org.