ARITHMETIC DENSITY OF CERTAIN ℓ-REGULAR OVERPARTITIONS

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Abstract

For an integer $\ell \geq 2$, let $\overline{A}_\ell(n)$ denote the number of ℓ-regular overpartitions of n. In this article, we investigate the divisibility properties of $\overline{A}_p(n)$, $\overline{A}_{3p}(n)$, and $\overline{A}_{3\cdot 2^\alpha}(n)$ by arbitrary powers of 2, where p is an odd prime and $\alpha \geq 1$. We also prove arithmetic properties of congruences, including some Ramanujan-type congruences satisfied by $\overline{A}_7(n)$ and $\overline{A}_\ell(n)$ for $\ell \equiv 3 \pmod 4$, employing theta function identities, Hecke operators, and the theory of modular forms. Furthermore, leveraging an Ono and Taguchi result on the nilpotency of Hecke operators, we identify infinite families of congruences modulo arbitrary powers of 2 satisfied by $\overline{A}_{3\cdot 2^\alpha}(n)$.

1. Introduction

For $|ab| < 1$, Ramanujan's general theta function $g(a, b)$ [4, p. 35, Entry 19] is defined by

$$g(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} = (-a, ab)_\infty (-b, ab)_\infty (ab, ab)_\infty,$$

where here and throughout this paper, we assume that

$$(a;q)_\infty = \prod_{m=0}^{\infty} (1 - aq^n),\ |q| < 1 \ \text{ and } \ f_k := (q^k; q^k)_\infty.$$

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The following are three special cases of $g(a, b)$:

$$f(-q) := g(-q, -q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = (q;q)_\infty = f_1,$$

$$\varphi(q) := g(q, q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2},$$

and

$$\psi(q) := g(q, q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}.$$

A *partition* of n is a non-increasing sequence of positive integers whose sum is n. An *overpartition* of n is a partition in which the first occurrence of a part may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. The generating function for $\overline{p}(n)$ is defined as follows:

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{f_2}{f_1^2}.$$

Corteel and Lovejoy [6] introduced and developed the theory of overpartitions. Since then, numerous mathematicians studied overpartition functions and have uncovered several interesting properties.

For an integer $\ell \geq 2$, an *ℓ-regular partition* of n is a partition in which none of the parts is divisible by ℓ. Similarly, an *ℓ-regular overpartition* of n is an overpartition in which none of the parts is a multiple of ℓ. The number of ℓ-regular overpartitions of n is denoted by $\overline{A}_\ell\,(n)$. The generating function for $\overline{A}_\ell\,(n)$ is:

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_\ell(n) q^n &= \frac{(-q;q)_\infty (q^\ell;q^\ell)_\infty}{(q;q)_\infty (-q^\ell;q^\ell)_\infty} \\ &= \frac{f_2 f_\ell^2}{f_1^2 f_{2\ell}}. \qquad (1)\end{aligned}$$

Lovejoy studied the ℓ-regular overpartition function $\overline{A}_\ell\,(n)$ in his series of papers [9, 10, 11]. Shen [17] investigated the generating functions of $\overline{A}_3\,(n)$ and $\overline{A}_4\,(n)$. Further, Shen established several arithmetic properties modulo $3, 6$, and 24. Alanzi et al. [2] proved infinite families of Ramanujan-type congruences modulo 3 satisfied by $\overline{A}_{3^j}\,(n)$ for positive integers $j \geq 3$. Adiga and Ranganatha [1] demonstrated various infinite families of congruences for small powers of 2 for $\overline{A}_{2^j}\,(n)$ where $j \geq 2$ is a positive integer. Ray and Barman [3] obtained an infinite family of congruences modulo 4 for $\overline{A}_{2\ell}\,(n)$ and modulo $4, 8$, and 16 for $\overline{A}_{4\ell}\,(n)$. Recently, Ray and Chakraborty [5] studied the divisibility properties of $\overline{A}_\ell\,(n)$ by arbitrary powers of p_i under certain conditions on ℓ and p. They obtained arithmetic properties for $\overline{A}_5\,(n)$ and established a Ramanujan-type congruence modulo 7 for $\overline{A}_7\,(n)$ using properties of modular forms and Hecke operators. In this paper, we study divisibility properties of ℓ-regular overpartition functions $\overline{A}_\ell(n)$ by arbitrary powers of 2 for certain values of ℓ. Let p be an odd prime and j be a fixed positive integer. In our first theorem, we prove that $\overline{A}_p(n)$ is divisible by a fixed but arbitrary power of 2 for almost all n. Specifically, we prove the following.

Theorem 1. *Let p be an odd prime and for any positive integer j, we have*

$$\lim_{X\to\infty} \frac{\#\left\{n \leq X : \overline{A}_p(n) \equiv 0 \pmod{2^j}\right\}}{X} = 1.$$

In the next theorem, we prove that $\overline{A}_{3p}(n)$ is divisible by arbitrary powers of 2 for almost all n.

Theorem 2. *If p be an odd prime, then for any integer positive integer j, we have*

$$\lim_{X\to\infty} \frac{\#\left\{n \leq X : \overline{A}_{3p}(n) \equiv 0 \pmod{2^j}\right\}}{X} = 1.$$

We further prove that the partition function $\overline{A}_{3\cdot 2^\alpha}(n)$ is almost always divisible by arbitrary powers of 2 for all $\alpha > 0$.

Theorem 3. *Let j and α be positive integers with $j \geq \alpha + 1$. We have*

$$\lim_{X\to\infty} \frac{\#\left\{n \leq X : \overline{A}_{3\cdot 2^\alpha}(n) \equiv 0 \pmod{2^j}\right\}}{X} = 1.$$

In the following theorem, using a result of Ono and Taguchi on nilpotency of Hecke operators, we prove that there exists an infinite family of congruences modulo 2 satisfied by $\overline{A}_{3\cdot 2^\alpha}(n)$.

Theorem 4. *Let $\alpha > 0$ be an integer. Then there exists a non-negative integer c such that for each $d \geq 1$ and distinct primes $p_1, \cdots, p_{c+d}$ coprime to 6, we have*

$$\overline{A}_{3\cdot 2^\alpha}\left(\frac{p_1 \cdots p_{c+d} \cdot n}{24}\right) \equiv 0 \pmod{2^d},$$

whenever n is coprime to $p_1, \ldots, p_{c+d}$.

Let $p_i > 2$ be a prime such that $p_i \not\equiv 1 \pmod 6$. We prove the following infinite family of congruences modulo 4 satisfied by $\overline{A}_\ell(n)$ where $\ell \equiv 3 \pmod 4$, using the theory of Hecke eigenforms.

Theorem 5. *Let $n, k \geq 0$ and $\ell \equiv 3 \pmod 4$ be integers. If $p_i > 2$ is prime such that $p_i \equiv 3 \pmod 4$ for $i \in \{1, 2, ..., k+1\}$, then, for $j \not\equiv 0 \pmod{p_{k+1}}$, we have*

$$\overline{A}_\ell\left(p_1^2 \cdots p_k^2 \cdot p_{k+1}(p_{k+1}(4n+1) + 4j)\right) \equiv 0 \pmod 4.$$

Let $\ell \equiv 3 \pmod 4$ be an integer, and let $p > 2$ be a prime such that $p \not\equiv 1 \pmod 6$. Suppose all the primes $p_1, p_2, \ldots, p_{k+1}$ are equal to the same prime p. We obtain the following corollary as an immediate consequence of Theorem 5.

Corollary 1. *Let $\ell \equiv 3 \pmod 4$ be an integer, and let $p > 2$ be a prime such that $p \not\equiv 1 \pmod 6$. Then we have*

$$\overline{A}_\ell\left(p^{2k+2}(4n+1) + p^{2k+1}4j\right) \equiv 0 \pmod 4,$$

where $j \not\equiv 0 \pmod p$.

Next, we prove the following congruence modulo 4 satisfied by $\overline{A}_\ell(n)$ where $\ell \equiv 3 \pmod 4$, using theta function identities.

Theorem 6. *If $n \geq 0$ and $k \geq 0$ are integers and $\ell \equiv 3 \pmod 4$, then we have*

$$\overline{A}_\ell\left(4\ell n + 4j + 3\right) \equiv 0 \pmod 4,$$

where $j \not\equiv \frac{\ell-3}{4} \pmod \ell$.

In the following theorem, using the theory of Hecke operators, we prove the self-similarity congruences modulo 7 satisfied by $\overline{A}_7(n)$.

Theorem 7. *Let $n \geq 0$ and $k \geq 0$ be integers. Then we have*

$$\overline{A}_7\left(7^{2k}(4n+3)\right) \equiv \overline{A}_7(4n+3) \pmod 7.$$

Finally, using theta function identities and the theory of modular forms, we prove the following Ramanujan-type congruences modulo 7 satisfied by $\overline{A}_7(n)$.

Theorem 8. *Let $n \geq 0$ be integers. Then we have*

$$\overline{A}_7\left(16n+4\right) \equiv 0 \pmod 7, \tag{2}$$

$$\overline{A}_7\left(64n+i\right) \equiv 0 \pmod 7, \tag{3}$$

$$\overline{A}_7\left(256n+j\right) \equiv 0 \pmod 7, \tag{4}$$

$$\overline{A}_7\left(1024n+k\right) \equiv 0 \pmod 7, \tag{5}$$

where $i \in \{4, 36\}$, $j \in \{4, 36, 68, 100, 132, 164, 196, 228\}$ and $k \in \{4, 36, 68, 100, 132, 164, 196, 228, 260, 292, 324, 356, 388, 420, 452, 484, 516, 548, 580, 612, 644, 676, 708, 740, 772, 804, 836, 868, 900, 932, 964, 996\}$.

2. Preliminaries

In this section, we collect certain definitions and results on modular forms. Detailed discussions can be found in [13, 8].

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denote the upper half of the complex plane. The congruence subgroups $\Gamma_0(N), \Gamma_1(N)$, and Γ_∞ of $SL_2(\mathbb{Z})$ of level N are defined as follows:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \,\middle|\, a \equiv d \equiv 1 \pmod{N} \right\},$$

and

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \,\middle|\, h \in \mathbb{Z} \right\},$$

where N is a positive integer. Suppose that k be a positive integer and $M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N))$) represents the complex vector space of modular forms (resp. cusp forms) of weight k with respect to a congruence subgroup $\Gamma_1(N)$.

Definition 1 ([13, Definition 1.15])**.** Let χ be a Dirichlet character modulo N. Then a modular form $f \in M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N))$) has Nebentypus character χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z),$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms (resp. cusp forms) is denoted by $M_k(\Gamma_0(N), \chi)$ $\big($resp. $S_k(\Gamma_0(N), \chi)\big)$.

The Dedekind *eta-function* $\eta(z)$ is given by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24} \prod_{n=1}^{\infty}(1-q^n),$$

where $q = e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is defined as an eta-quotient provided it satisfies

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and $r_\delta \in \mathbb{Z}$.

The following result of Gordon, Huges, and Newman is very useful when dealing with eta-quotients.

Theorem 9 ([13, Theorem 1.64])**.** *Suppose $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$, with the following additional properties,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24},$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then we have $f(z)$ *that satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z),$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. *The character* χ *is formulated as* $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, *where* $s := \prod_{\delta|N} \delta^{r_\delta}$.

Let f be an eta-quotient satisfying the conditions of Theorem 9. Note that if f is also holomorphic at all the cusps of $\Gamma_0(N)$, then $f \in M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$). The following theorem provides the necessary criterion for determining the orders of eta-quotients at the cusps.

Theorem 10 ([13, Theorem 1.65])**.** *Let* c, d, *and* N *be positive integers with* $d \mid N$ *and* $\gcd(c, d) = 1$. *If* f *is an eta-quotient satisfying the conditions of Theorem 9 for* N, *then the order of vanishing of* $f(z)$ *at the cusp* $\frac{c}{d}$ *is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

We use the following result of Strum [18] to verify that the coefficients of two modular forms are congruent modulo any given prime.

Theorem 11. *Let* p *be a prime,* $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$, *and* $g(z) = \sum_{n=n_1}^{\infty} b(n)q^n \in M_k(\Gamma_0(N), \psi)$, *where* n_0 *and* n_1 *are non-negative integers. If either* $\chi = \psi$ *and*

$$a(n) \equiv b(n) \pmod{p} \quad \textit{for all} \quad n \leq \frac{kN}{12} \prod_{d \text{ prime}; \; d|N} \left(1 + \frac{1}{d}\right),$$

or $\chi \neq \psi$ *and*

$$a(n) \equiv b(n) \pmod{p} \quad \textit{for all} \quad n \leq \frac{kN^2}{12} \prod_{d \text{ prime}; \; d|N} \left(1 - \frac{1}{d^2}\right),$$

then $f(z) \equiv g(z) \pmod{p}$ (*i.e.,* $a(n) \equiv b(n) \pmod{p}$ *for all* n).

Next, we recall the following definitions of Hecke operators, which are useful for proving our main results.

Definition 2. Let m be a positive integer, and $f(z)=\sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N),\chi)$. Then *action of Hecke operator T_m* on $f(z)$ can be formulated as

$$f(z)|T_m := \sum_{n=0}^{\infty}\left(\sum_{d|\gcd(n,m)} \chi(d)d^{\ell-1}a\left(\frac{nm}{d^2}\right)\right)q^n.$$

Precisely, if $m=p$ is prime, then

$$f(z)|T_p := \sum_{n=0}^{\infty}\left(a(pn)+\chi(p)p^{\ell-1}a\left(\frac{n}{p}\right)\right)q^n. \tag{6}$$

Note that $a(n)=0$ unless n is a non-negative integer.

If f is an eta-quotient that satisfies the presumptions of Theorem 9 and $p|s$ (here s is as defined in Theorem 9), then $\chi(p)=0$ so that the latter term vanishes. In this case, we have the factorization property

$$\left(f\cdot\sum_{n=0}^{\infty} g(n)q^{pn}\right)|T_p = \left(\sum_{n=0}^{\infty} a(pn)q^n\right)\left(\sum_{n=0}^{\infty} g(n)q^n\right). \tag{7}$$

Definition 3. A modular form $f(z)=\sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N),\chi)$ is called a *Hecke eigenform* if for every $m\geq 2$, a complex number $\lambda(m)$ can be found, such that

$$f(z)|T_m = \lambda(m)f(z). \tag{8}$$

To prove the divisibility properties of various partition functions, we use the following result of Serre.

Theorem 12 ([13, Theorem 2.65])**.** *Let m be a positive integer, and suppose $f(z)=\sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N),\chi)$ has Fourier expansion*

$$f(z)=\sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]].$$

Then there exits a constant $\alpha>0$ such that

$$\#\left\{n\leq X : c(n)\not\equiv 0 \pmod{m}\right\} = \mathcal{O}\left(\frac{X}{(logX)^{\alpha}}\right).$$

The following lemma is immediate from the Binomial theorem.

Lemma 1. *For any positive integers s and t, and any prime p, we have*

$$f_s^{p^t} \equiv f_{sp}^{p^{t-1}} \pmod{p^t}. \tag{9}$$

3. Proofs of Theorems 1 - 3

In this section, we give the proofs of our first three main results by applying Theorems 9, 10, and 12.

Proof of Theorem 1. Setting $\ell = p$ in (1), we have

$$\sum_{n=0}^{\infty} \overline{A}_p(n)q^n = \frac{f_2 f_p^2}{f_1^2 f_{2p}}. \tag{10}$$

Let $\mathcal{B}_j(z) = \prod_{n=1}^{\infty} \frac{(1-q^{24n})^2}{(1-q^{48n})} = \frac{\eta^2(24z)}{\eta(48z)}$. By employing Lemma 1. We have

$$\mathcal{B}_j^{2^j}(z) = \frac{\eta^{2^{j+1}}(24z)}{\eta^{2^j}(48z)} \equiv 1 \pmod{2^{j+1}}.$$

Define

$$\begin{aligned}\mathcal{C}_j(z) &= \frac{\eta(48z)\eta^2(24pz)}{\eta^2(24z)\eta(48pz)} \cdot \mathcal{B}_j^{2^j}(z)\\ &= \frac{\eta^{2^{j+1}-2}(24z)\eta^2(24pz)}{\eta^{2^j-1}(48z)\eta(48pz)}.\end{aligned}$$

Taking this equation modulo 2^{j+1}, we obtain

$$\begin{aligned}\mathcal{C}_j(z) &\equiv \frac{\eta(48z)\eta^2(24pz)}{\eta^2(24z)\eta(48pz)}\\ &= \frac{f_{48} f_{24p}^2}{f_{24}^2 f_{48p}}. \end{aligned}\tag{11}$$

From Identities (10) and (11), we obtain

$$\mathcal{C}_j(z) \equiv \sum_{n=0}^{\infty} \overline{a}_p(n)q^{24n} \pmod{2^{j+1}}. \tag{12}$$

Note that $\mathcal{C}_j(z)$ is an eta-quotient with level $N = 192p$. The cusps of $\Gamma_0(192p)$ can be written as fractions $\frac{c}{d}$, where $d \mid 192p$ and $\gcd(c, d) = 1$. Using Theorem 10, we observe that, $\mathcal{C}_j(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\mathcal{D} := 4p\left(2^j - 1\right)\frac{\gcd(d, 24)^2}{\gcd(d, 48p)^2} + 4\frac{\gcd(d, 24p)^2}{\gcd(d, 48p)^2} - p\left(2^j - 1\right)\frac{\gcd(d, 48)^2}{\gcd(d, 48p)^2} - 1 \geq 0.$$

From Table 1, we find that $\mathcal{D} \geq 0$ for all $d \mid 192p$. Hence, $\mathcal{C}_j(z)$ is holomorphic at every cusp $\frac{c}{d}$. Using Theorem 9, we find that the weight of $\mathcal{C}_j(z)$ is $k = 2^{j-1}$, and the associated character, $\chi_1(\bullet)$, equals $(((-1)^{2^{j-1}} \cdot 2^{2^{j+1}\cdot 3^{2^j}\cdot p})/\bullet)$. Hence, by Theorems 9 and 10, we have $\mathcal{C}_j(z) \in M_{2^{j-1}}\left(\Gamma_0(192p), \chi_1(\bullet)\right)$. Thus, by Serre's Theorem 12, the Fourier coefficients of $\mathcal{C}_j(z)$ are almost always divisible by 2^j for all $j > 0$. By employing (12), we complete the proof of Theorem 1. $\square$

$d \mid 192p$	$\frac{\gcd(d,24)^2}{\gcd(d,48p)^2}$	$\frac{\gcd(d,24p)^2}{\gcd(d,48p)^2}$	$\frac{\gcd(d,48)^2}{\gcd(d,48p)^2}$	$\mathcal{D}$
$1, 2, 3, 4, 6, 8, 12, 24$	1	1	1	$3+3p(2^k-1)$
$16, 32, 48, 64, 96, 192$	$1/4$	$1/4$	1	0
$p, 2p, 3p, 4p, 6p, 8p, 12p, 24p$	$1/p^2$	1	$1/p^2$	$3+3(\frac{2^k-1}{P})$
$16p, 32p, 48p, 64p, 96p, 192p$	$1/4p^2$	$1/4$	$1/p^2$	0

Table 1: Table to find the possible values of $\mathcal{D}$.

The proofs of Theorems 2 and 3 are similar to Theorem 1. Thus we present a modified proof.

Proof of Theorem 2. Setting $\ell = 3p$ in (1), we have

$$\sum_{n=0}^{\infty} \overline{A}_{3p}(n)q^n = \frac{f_2 f_{3p}^2}{f_1^2 f_{6p}}. \tag{13}$$

Define

$$\mathcal{F}_j(z) = \frac{\eta^{2^{j+1}-2}(24z)\eta^2(72pz)}{\eta^{2^j-1}(48z)\eta(144pz)}.$$

It is easy to obtain

$$\mathcal{F}_j(z) \equiv \sum_{n=0}^{\infty} \overline{A}_{3p}(n)q^{24n} \pmod{2^{j+1}}. \tag{14}$$

Clearly, $\mathcal{F}_j(z)$ is an eta-quotient with level $N = 576p$. The cusps of $\Gamma_0(576p)$ can be written as fractions $\frac{c}{d}$, where $d \mid 576p$ and $\gcd(c, d) = 1$. Using Theorem 10, we observe that, $\mathcal{F}_j(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\begin{aligned}\mathcal{G} := 12p\left(2^j - 1\right)\frac{\gcd(d,24)^2}{\gcd(d,144p)^2} + 4\frac{\gcd(d,72p)^2}{\gcd(d,144p)^2} \\ - 3p\left(2^j - 1\right)\frac{\gcd(d,48)^2}{\gcd(d,144p)^2} - 1 \geq 0.\end{aligned}$$

From Table 2, we see that $\mathcal{G} \geq 0$ for all $d \mid 576p$. Thus, $\mathcal{F}_j(z)$ is holomorphic at every cusp $\frac{c}{d}$. Using Theorem 9, the weight of $\mathcal{F}_j(z)$ is $k = 2^{j-1}$, and the associated character, $\chi_2(\bullet)$, equals $(((-1)^{2^{j-1}} \cdot 2^{2^{j+1}\cdot 3^{2^j+1}\cdot p})/\bullet)$. Again, by Theorems 9 and 10, $\mathcal{F}_j(z) \in M_{2^{j-1}}\left(\Gamma_0(576p), \chi_2(\bullet)\right)$. Therefore, by Serre's Theorem 12, the Fourier coefficients of $\mathcal{F}_j(z)$ are almost always divisible by 2^j for all $j > 0$. By employing Identity (14), we complete the proof of Theorem 2. ☐

$d \mid 576p$	$\frac{\gcd(d,24)^2}{\gcd(d,144p)^2}$	$\frac{\gcd(d,48)^2}{\gcd(d,144p)^2}$	$\frac{\gcd(d,72p)^2}{\gcd(d,144p)^2}$	$\mathcal{G}$
$1, 2, 3, 4, 6, 8, 12, 24$	1	1	1	$3 + 9p(2^k - 1)$
$16, 32, 48, 64, 96, 192$	$1/4$	1	$1/4$	0
$9, 18, 36, 72$	$1/9$	$1/9$	1	$2^k + 2$
$144, 288, 576$	$1/36$	$1/9$	$1/4$	0
$p, 2p, 3p, 4p, 6p, 8p, 12p, 24p$	$1/p^2$	1	$1/p^2$	$3 + 9(\frac{2^k-1}{p})$
$16p, 32p, 48p, 64p, 96p, 192p$	$1/4p^2$	$1/p^2$	$1/4$	0
$9p, 18p, 36p, 72p$	$1/9p^2$	$1/9p^2$	1	$3 + \frac{2^k-1}{p}$
$144p, 288p, 576p$	$1/36p^2$	$1/9p^2$	$1/4$	0

Table 2: Table to find the possible values of $\mathcal{G}$.

We now present a proof of the Theorem 3.

Proof of Theorem 3. Setting $\ell = 3 \cdot 2^\alpha$ in Identity (1), we have

$$\sum_{n=0}^{\infty} \overline{A}_{3\cdot 2^\alpha}(n) q^n = \frac{f_2 f_{3\cdot 2^\alpha}^2}{f_1^2 f_{6\cdot 2^\alpha}}.$$

Define

$$\mathcal{L}_j(z) = \frac{\eta(48z)\eta^{2^{j+1}+2}(3^2 \cdot 2^{\alpha+3}z)}{\eta^2(24z)\eta^{2^j+1}(3^2 \cdot 2^{\alpha+4}z)}.$$

It is easy to obtain

$$\mathcal{L}_j(z) \equiv \sum_{=0}^{\infty} \overline{A}_{3\cdot 2^\alpha}(n) q^{24n} \pmod{2^{j+1}}. \tag{15}$$

On using Theorem 9, the level of the eta-quotient $\mathcal{L}_j(z) = 3^2 \cdot 2^{\alpha+4}u$, where u is the smallest integer such that

$$u\left[3 \cdot 2^k - 3(3 \cdot 2^\alpha - 1)\right] \equiv 0 \pmod{24}.$$

Hence, the level of $\mathcal{L}_j(z)$ is $3^2 \cdot 2^{\alpha+7}$. The cusps of $\Gamma_0(3^2 \cdot 2^{\alpha+7})$ can be written as fractions $\frac{c}{d}$, where $d \mid 3^2 \cdot 2^{\alpha+7}$ and $\gcd(c, d) = 1$. Using Theorem 10, $\mathcal{L}_j(z)$ is holomorphic at a cusp $\frac{c}{d}$ if and only if

$$\mathcal{M} := 3 \cdot 2^\alpha \frac{\gcd(d, 48)^2}{\gcd(d, 3^2 \cdot 2^{\alpha+4})^2} - 3 \cdot 2^{\alpha+2} \frac{\gcd(d, 24)^2}{\gcd(d, 3^2 \cdot 2^{\alpha+4})^2} + 4\left(2^j + 1\right) \frac{\gcd(d, 3^2 \cdot 2^{\alpha+3})^2}{\gcd(d, 3^2 \cdot 2^{\alpha+4})^2} - \left(2^j + 1\right) \geq 0.$$

$d = 2^r \cdot 3^s \mid 3^2 \cdot 2^{\alpha+7}$	$\frac{\gcd(d,24)^2}{\gcd(d,3^2\cdot 2^{\alpha+4})^2}$	$\frac{\gcd(d,48)^2}{\gcd(d,3^2\cdot 2^{\alpha+4})^2}$	$\frac{\gcd(d,3^2\cdot 2^{\alpha+3})^2}{\gcd(d,3^2\cdot 2^{\alpha+4})^2}$	$\mathcal{M}$
$0 \le r \le 3$, $0 \le s \le 1$	1	1	1	$3(2^j+1)-9\cdot 2^\alpha$
$0 \le r \le 3$, $s = 2$	1/9	1/9	1	$3(2^j+1)-2^\alpha$
$4 \le r \le \alpha+3$, $0 \le s \le 1$	2^{6-2r}	2^{8-2r}	1	$3(2^j+1)$
$4 \le r \le \alpha+3$, $s = 2$	$2^{6-2r}/9$	$2^{8-2r}/9$	1	$3(2^j+1)$
$\alpha+4 \le r \le \alpha+7$, $0 \le s \le 1$	$2^{-2-2\alpha}$	$2^{-2\alpha}$	1/4	0
$\alpha+4 \le r \le \alpha+7$, $s = 2$	$2^{-2-2\alpha}/9$	$2^{-2\alpha}/9$	1/4	0

Table 3: Table to find the possible values of $\mathcal{M}$.

From Table 3, we find that $\mathcal{M} \ge 0$ for all $d \mid 3^2\cdot 2^{\alpha+7}$, and $j \ge 2\alpha+1$. Hence, $\mathcal{L}_j(z)$ is holomorphic at every cusp $\frac{c}{d}$. Using Theorem 9, the weight of $\mathcal{L}_j(z)$ is 2^{j-1}, and the associated character, $\chi_4(\bullet)$, equals $(((-1)^{2^{j-1}} \cdot 3^{2^{j+1}+1} \cdot 2^{2^j(\alpha+2)+\alpha})/\bullet)$. Hence, by Theorems 9 and 10, $\mathcal{L}_j(z) \in M_{2^{j-1}}\left(\Gamma_0(3^2\cdot 2^{\alpha+7}), \chi_4(\bullet)\right)$ for all $j \ge 2\alpha+1$. Thus, by Serre's Theorem 12, the Fourier coefficients of $\mathcal{L}_j(z)$ are almost always divisible by 2^j. By employing Identity (15), we complete the proof of Theorem 3. □

4. Proof of Theorems 4 - 7

In order to prove our Theorem 5, we recall the following result on nilpotency of the Hecke operators given by Ono and Taguchi [14].

Theorem 13 ([14, Theorem 1.3])**.** . *Let $n \ge 0$, $k > 0$ be integers, and let χ be a quadratic Dirichlet character with conductor $9\cdot 2^n$. There exists a non-negative integer c such that for every $f(z) \in M_k(\Gamma(9\cdot 2^n), \chi) \cap \mathbb{Z}[[q]]$ and for every $t \ge 1$,*

$$f(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_{c+t}} \equiv 0 \pmod{2^t},$$

whenever the primes p_1, p_2,...,p_{c+t} are coprime to 6.

We now present the proof of Theorems 4.

Proof of Theorem 4. From Identity (15), we have

$$\mathcal{L}_j(z) \equiv \sum_{n=0}^{\infty} \overline{A}_{3\cdot 2^\alpha}(n) q^{24n} \pmod{2^{j+1}}.$$

This implies

$$\mathcal{L}_j(z) := \sum_{n=0}^{\infty} \mathcal{C}_j(n) q^n \equiv \sum_{n=0}^{\infty} \overline{A}_{3\cdot 2^\alpha} \left(\frac{n}{24}\right) q^n \pmod{2^{j+1}}. \tag{16}$$

Clearly, $\mathcal{L}_j(z) \in M_{2^{\beta-1}}\left(\Gamma_0(9\cdot 2^{\beta+7}), \chi_4\right)$. By Theorem 13, there exists a non-negative integer c such that for every $d \geq 1$,

$$\mathcal{L}_j(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_{c+d}} \equiv 0 \pmod{2^d},$$

whenever the primes p_1, p_2,...,p_{c+d} are coprime to 6. By Definition 1 of Hecke operators, if p_i are distinct primes and $(n, p_i) = 1$ for $1 \leq i \leq c+d$, then

$$\mathcal{C}_j(p_1 \cdots p_{c+d} \cdot n) \equiv 0 \pmod{2^d}. \tag{17}$$

Using Identities (16) and (17), we complete the proof of Theorem 4. □

Next, we present the proof of Theorem 5

Proof of Theorem 5. We recall the following even-odd dissection formulas from Berndt [4, p. 40, Entry 25]:

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8},$$
$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}.$$

Employing the above even-odd dissection formulas in Identity (1) and drawing out the common terms of q^{2n+1}, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_\ell(2n+1) q^n &= 2\frac{f_2^2 f_2^8 f_{4\ell}^5}{f_1^4 f_4 f_{2\ell}^2 f_{8\ell}^2} - 2q^{\frac{\ell-1}{2}} \frac{f_4^5 f_{8\ell}^2}{f_1^4 f_8^2 f_{4\ell}} \\ &\equiv 2f_2^6 + 2q^{\frac{\ell-1}{2}} f_{2\ell}^6 \pmod{4}. \end{aligned} \tag{18}$$

Since $\ell \equiv 3 \pmod 4$, on extracting the coefficient of q^{2n} from Identity (18), we have

$$\sum_{n=0}^{\infty} \overline{A}_\ell(4n+1) q^n \equiv 2f_1^6 \pmod{4}, \tag{19}$$

which implies

$$\sum_{n=0}^{\infty} \overline{A}_\ell(4n+1) q^{4n+1} \equiv 2\eta^6(4z) \pmod{4}.$$

Let $\eta^6(4z)=\sum_{n=0}^{\infty}c(n)q^n$. Note that $c(n)=0$ if $n\not\equiv 1 \pmod 4$ for all $n\geq 0$ and

$$\overline{A}_\ell(6n+1)\equiv c(4n+1) \pmod 4. \tag{20}$$

By Theorems 9 and 10, we have $\eta^6(4z)\in S_3\big(\Gamma_0(16),\big(\frac{-1}{\bullet}\big)\big)$. Since $\eta^6(4z)$ is a Hecke eigenform (see [12]), using identities (6) and (8), we obtain

$$\eta^6(4z)\mid T_p=\sum_{n=1}^{\infty}\left[c(pn)+p^2\left(\frac{-1}{p}\right)c\left(\frac{n}{p}\right)\right]q^n=\lambda(p)\sum_{n=1}^{\infty}c(n)q^n,$$

which implies

$$c(pn)+p^2\left(\frac{-1}{p}\right)c\left(\frac{n}{p}\right)=\lambda(p)c(n). \tag{21}$$

For $c(1)=1$, $c\big(\frac{1}{p}\big)=0$, and for $n=1$ in Identity (21), we obtain $c(p)=\lambda(p)$. Since $c(p)=0$ for all $p\not\equiv 1 \pmod 4$, we obtain $\lambda(p)=0$. Thus, Identity (13) yields

$$c(pn)+p^2\left(\frac{-1}{p}\right)c\left(\frac{n}{p}\right)=0, \tag{22}$$

which is true for all primes $p\equiv 3 \pmod 4$. Replacing n by $pn+r$ in Identity (22), we obtain

$$c(p^2n+pr)=0, \tag{23}$$

for all $n\geq 0$ and $p\nmid r$. Similarly, replacing n by pn in Identity (22), we obtain

$$c(p^2n)=-p^2c(n)\equiv c(n) \pmod 2. \tag{24}$$

Replacing n by $4n-pr+1$ in Identity (23) and using Identity (20), we obtain

$$\overline{A}_\ell\left(4\left(p^2n+\frac{(p^2-1)}{4}+pr\frac{(1-p^2)}{4}\right)+1\right)\equiv 0 \pmod 4. \tag{25}$$

Similarly, replacing n by $4n+1$ in Identity (24) and using Identity (20), we obtain

$$\overline{A}_\ell\left(4\left(p^2n+\frac{(p^2-1)}{4}\right)+1\right)\equiv \overline{A}_\ell(4n+1) \pmod 4. \tag{26}$$

For a prime $p>2$, we have $4\mid(1-p^2)$ and $\gcd(\frac{1-p^2}{4},p)=1$. It is easy to see that, when r runs over a residue system excluding the multiples of p, so does $\frac{1-p^2}{4}r$. Therefore, Identity (25) can be rewritten as

$$\overline{A}_\ell\left(4\left(p^2n+\frac{(p^2-1)}{4}+pj\right)+1\right)\equiv 0 \pmod 4, \tag{27}$$

where $p \nmid j$. For odd primes p_i satisfying $p_i \equiv 3 \pmod 4$, we have

$$p_1^2 \dots p_k^2 n + \frac{p_1^2 \dots p_k^2 - 1}{4} = p_1^2 \left(p_2^2 \dots p_k^2 n + \frac{p_2^2 \dots p_k^2 - 1}{4} \right) + \frac{p_1^2 - 1}{4},$$

and applying Identity (26) repeatedly on the above identity, we obtain

$$\overline{A}_\ell\left(4\left(p_1^2 \cdots p_k^2 n + \frac{(p_1^2 \cdots p_k^2 - 1)}{4}\right) + 1\right) \equiv \overline{A}_\ell(4n+1) \pmod 4. \tag{28}$$

For $j \not\equiv 0 \pmod{p_{k+1}}$, set $n = p_{k+1}^2 n + \frac{p_{k+1}^2 - 1}{4} + p_{k+1} j$ in Identity (28). Then, employing Identity (27), we complete the proof of Theorem 5. □

Next, we present the proof of Theorem 6.

Proof of Theorem 6. Since $\ell \equiv 3 \pmod 4$, on extracting the common terms of q^{2n+1} from Identity (18), we have

$$\sum_{n=0}^{\infty} \overline{A}_\ell(4n+3) q^n \equiv 2 q^{\frac{\ell - 3}{4}} f_\ell^6 \pmod 4. \tag{29}$$

Now, extracting the common terms of $q^{\ell n + j}$ from Identity (29), where $j \not\equiv \frac{\ell-3}{4} \pmod \ell$, completes the proof of Theorem 6. □

We now present the proof of Theorem 7.

Proof of Theorem 7. Setting $\ell = 7$ in Identity (1), we have

$$\sum_{n=0}^{\infty} \overline{A}_\ell(n) q^n \equiv \frac{f_1^{12}}{f_2^6} = \varphi^6(-q) \pmod 7. \tag{30}$$

From a result of Hirschhorn [7, (1.9.4)], we have

$$\varphi(-q) = \varphi(q^4) - 2q\psi(q^8). \tag{31}$$

Using Identity (31) in (30) and extracting the common terms of q^{4n} and q^{4n+3} from the resulting identity, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_7(4n) q^n \equiv \varphi^6(q) + 2q\varphi^2(q)\psi^4(q^2) \pmod 7, \tag{32}$$

$$\sum_{n=0}^{\infty} \overline{A}_7(4n+3) q^n \equiv \varphi^3(q)\psi^3(q^2) = \frac{f_2^{12}}{f_1^6} \pmod 7. \tag{33}$$

Consider the following *eta*-quotient functions:

$$H_1(z) := \frac{\eta^{12}(2z)}{\eta^6(z)} \eta^6(z) E_6^{48}(z)$$

and

$$H_2(z) := \frac{\eta^{12}(2z)}{\eta^6(z)}\eta^{294}(z),$$

where $E_6(z)$ is the weight 6 normalized Eisenstein series, given by $E_6(z) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n$. The function $E_6(z)$ is a modular form on $\Gamma_0(1)$ with trivial character and $E_6 \equiv 1 \pmod 7$. By Theorems 9 and 10, we observe that H_1 and H_2 are modular forms in the space $M_{312}(\Gamma_0(4), \chi_7)$ with character $\chi_7(\bullet) = \left(\frac{2^{12}}{\bullet}\right)$. Considering identities $H_1(2)$ and $H_2(z)$ modulo 7, and using Identity (21), we have

$$H_1(z) \equiv \sum_{n=0}^{\infty} \overline{A}_7(4n+3)q^{n+1} \cdot f_1^6 \pmod 7$$

and

$$H_2(z) \equiv \sum_{n=0}^{\infty} \overline{A}_7(4n+3)q^{n+13} \cdot f_1^{294} \pmod 7.$$

Applying the Hecke operator T_7 to $H_2(z)$ and employing Identity (7), we have

$$(H_2|T_7)(z) \equiv \sum_{n=0}^{\infty} \overline{A}_7(4(7n+1)+3)q^{n+2} \cdot f_1^{42} \pmod 7.$$

Again, applying the Hecke operator T_7 to $H_2(z)$ and using Identity (7), we have

$$(H_2|T_7^2)(z) \equiv \sum_{n=0}^{\infty} \overline{A}_7(4(49n+36)+3)q^{n+1} \cdot f_1^6 \pmod 7.$$

Since the Hecke operator is an endomorphism on $M_{312}(\Gamma_0(4), \chi_7)$, we have $(H_2|T_7^2)(z) \in M_{312}(\Gamma_0(4), \chi_7)$. Since the modular forms are in the same space $M_{312}(\Gamma_0(4), \chi_7)$ and have the same character, the Sturm bound is 156. Upon using Mathematica, we confirm that all coefficients up to the desired bound are congruent modulo 7. Hence, by Sturm's Theorem 11,

$$H_2(z) \equiv (H_2|T_7^2)(z) \pmod 7,$$

which implies

$$\overline{A}_7(4(49n+36)+3) \equiv \overline{A}_7(4n+3) \pmod 7.$$

This is the $k = 0$ case of Theorem 5. Upon using mathematical induction on k, we complete the proof of Theorem 5. □

5. Proof of Theorem 8

In [5], Ray and Chakraborty proved Identity (2) using the approach developed by Radu and Sellers [15, 16]. Here, we present an alternative proof of Identity (2) using Ramanujan's theta function identities. Additionally, we extend this methodology to prove identities (3)–(5) by employing the approach developed by Radu and Sellers. We recall for prime p, the index of $\Gamma_0(N)$ in Γ is given by

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} \left(1 + p^{-1}\right).$$

For a positive integer M, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of M. Let $r \in R(M)$, let $1 = \delta_1 < \cdots < \delta_k = M$ be the positive divisors of M, and write $r = (r_\delta) = (r_{\delta_1}, \ldots, r_{\delta_k})$. Define $c_r(n)$ by

$$\prod_{\delta|M} \left(q^\delta; q^\delta\right)_\infty^{r_\delta} := \sum_{n=0}^{\infty} c_r(n) q^n.$$

The approach was developed by Radu and Sellers [15, 16] for proving congruences for $c_r(n)$. For a positive integer m, an integer s, $[s]_m$ denotes the set of all elements congruent to s modulo m. Let $\mathbb{Z}_m^*$ denote the set of all invertible elements in $\mathbb{Z}_m$ and $\mathbb{S}_m^*$ denote the set of all squares in $\mathbb{Z}_m^*$. For $t \in \{0, 1, \ldots, m-1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subset \{0, 1, \ldots, m-1\}$ by

$$P_{m,r}(t) := \left\{ t' : \text{there exits } [s]_{24m} \in \mathbb{S}_{24m} \text{ such that } t' \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m} \right\}.$$

Definition 4. Let m, M, and N be positive integers, let $r = r_\delta \in R(M)$, and let $t \in \{0, 1, \ldots, m-1\}$. Let $k = k(m) := \gcd(m^2 - 1, 24)$ and write

$$\prod_{\delta|M} \delta^{|r_\delta|} = 2^s \cdot j,$$

where s and j are non-negative integers with j odd. The set Δ^* consists of all tuples $(m, M, N, (r_\delta), t)$ satisfying these conditions and all of the following.

(1) Each prime divisor of m is also a divisor of N.

(2) $\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$.

(3) $kN \sum_{\delta|M} r_\delta mN/\delta \equiv 0 \pmod{24}$.

(4) $kN \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$.

(5) $\dfrac{24m}{\gcd\left(-24kt - k\sum_{\delta|M} \delta r_\delta, 24m\right)}$ divides N.

(6) If $2|m$, then either $4|kN$ and $8|sN$ or $2|s$, and $8|(1-j)N$.

For positive integers m, M, and N, $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $r \in R(M)$, and $r' \in R(N)$, set

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,1,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta a + \delta k \lambda c, mc)}{\delta m}$$

and

$$p^*_{r'}(\gamma) := \frac{1}{24} \sum_{\delta|N} r'_\delta \frac{\gcd^2(\delta, c)}{\delta}.$$

Lemma 2 ([15, Lemma 4.5])**.** *Let u be a positive integer, $(m, M, N, (r_\delta), t) \in \Delta^*$, and $r' = (r'_\delta) \in R(N)$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ be a complete set of representatives of the double cosets of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p^*_{r'}(\gamma_i) \geq 0$ for all $1 \leq i \leq n$. Let $t_{min} = \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24}\left\{\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta r'_\delta\right\} - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{min}}{m}.$$

If the congruence $c_r(mn + t') \equiv 0 \pmod{u}$ holds for all $t' \in P_{m,r}(t)$ and $0 \leq n \leq \lfloor \nu \rfloor$, then it holds for all $t' \in P_{m,r}(t)$ and $n \geq 0$.

To apply Lemma 2, we use the following result, which provides a complete set of representatives of the double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$.

Lemma 3 ([19, Lemma 4.3])**.** *If N or $\frac{1}{2}N$ is a square-free integer, then*

$$\bigcup_{\delta|N} \Gamma_0(N) \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \Gamma_\infty = \Gamma.$$

Proof of Theorem 8. To prove Identity (2), we first recall the following result from Berndt [4, p. 40 Entry 25(v) and 25(vi)]:

$$\varphi^2(-q) = \varphi^2(q^2) - 4q\psi^2(q^4). \tag{34}$$

Using Identity (34) in Identity (32) and then extracting the common terms of q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_7(8n+4)q^n \equiv 5\varphi^4(q)\psi^2(q^2) + 2\varphi^2(q)\psi^4(q) + q\psi^6(q^2) \pmod{7}. \tag{35}$$

From a result of Berndt [4, p. 40 Entry 25(iv)], we have

$$\varphi(q)\psi(q^2) = \psi^2(q). \tag{36}$$

Using Identity (36) in Identity (35), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_7(8n+4)q^n \equiv q\psi^6(q^2) \pmod 7. \tag{37}$$

Hence Identity (2) follows from Identity (37) by extracting the common terms of q^{2n}.

The proofs of Identities (3) - (5) are similar, thus we prove only Identity (5). Setting $\ell = 7$ in (1) and using Lemma 1, we have

$$\sum_{n=0}^{\infty} \overline{A}_7(n)q^n \equiv \frac{f_1^{12}}{f_2^6} \pmod 7.$$

Let $(m, M, N, r, t) = (1024, 2, 8, (12, -6), 4)$. By applying Conditions (1) - (6) of Definition 4, it is clear that $(1024, 2, 8, (12, -6), 4) \in \Delta^*$ and $P_{m,r}(t) = \{4, 36, 68, 100, 132, 164, 196, 228, 260, 292, 324, 356, 388, 420, 452, 484, 516, 548, 580, 612, 644, 676, 708, 740, 772, 804, 836, 868, 900, 932, 964, 996\}$. By Lemma 3, we know that

$$\left\{ \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} : \delta | 8 \right\}$$

forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. With $r' = (0, 0, 0, 0) \in R(8)$, on using *Mathematica* we obtain

$$p_{m,r}\left(\begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}\right) + p^*_{r'}\left(\begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}\right) \geq 0, \quad \text{for each} \quad \delta | 8.$$

From Lemma 2, $\lfloor \nu \rfloor = 2$. Upon using Mathematica, we verify that $\overline{A}_7(1024n + t') \equiv 0 \pmod 7$, for $t' \in P_{m,r}(t)$, and for $0 \leq n \leq 2$. Again using Lemma 2, $\overline{A}_7(1024n + t') \equiv 0 \pmod 7$, for $t' \in P_{m,r}(t)$, is true for all $n \geq 0$. This completes the proof of Theorem 8. □

6. Concluding Remarks

In the previous sections, we have demonstrated the application of Ramanujan's theta function identities and the methods developed by Radu and Sellers to establish congruences for partition functions. These techniques have been instrumental in proving specific modular properties of $\overline{A}_7(n)$ under various arithmetic progressions. In particular, the congruences for $\overline{A}_7(8n + 4)$ modulo 7 suggest a more general pattern that holds for larger powers of 4. Motivated by these results and extensive computations, we propose the following conjecture, which extends the modular properties of $\overline{A}_7(n)$ to higher powers of 4. This conjecture, if true, reveals a deeper arithmetic structure underlying the partition function $\overline{A}_7(n)$, specifically in relation to powers of 4 and certain residue classes modulo 32. We now state the conjecture.

Conjecture 1. Let $k \geq 2$ and $0 < j < 4^k$ be integers. We have

$$\overline{A}_7\left(4^k n + j\right) \equiv 0 \pmod 7,$$

where $j \equiv 4 \pmod{32}$.

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