#A30 INTEGERS 25 (2025)

ON A GENERALIZATION OF WATSON'S TRIGONOMETRIC SUM (ON DOWKER'S SUM OF ORDER ONE HALF)

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Received: 8/16/24, Accepted: 3/12/25, Published: 3/26/25

Abstract

In this paper we study the finite trigonometric sum $\sum a_l \csc\big(\pi l/n\big)$, where the coefficients a_l are equal to $\cos(2\pi l\nu/n)$, and where the summation index l and the discrete parameter ν both run from 1 to $n-1$. This sum is a generalization of Watson's trigonometric sum, which has been extensively studied in a series of previous papers, and also may be regarded as the so-called Dowker sum of order one half. It occurs in various problems in mathematics, physics and engineering, and plays an important role in some number-theoretic problems. In the paper, we obtain several integral and series representations for the above-mentioned sum, investigate its properties, derive various expansions for it (including asymptotic expansions) and deduce very narrow upper and lower bounds for it. In addition, we obtain two relatively simple approximate formulas containing only a few terms, which are also very accurate and can be particularly useful in applications. Finally, we also derive several advanced summation formulas for the digamma function, which relate the gamma and the digamma functions, the investigated sum, and the product of a sequence of cosecants $\prod\big(\csc(\pi l/n)\big)^{\csc(\pi l/n)}$.

1. Introduction

1.1. A Short Historical Survey

Finite trigonometric sums are an interesting object of study and often appear in analysis, discrete mathematics, combinatorics, number theory, applied statistics, and in many other areas of mathematics. They also often occur in applications, especially in physics, and in a variety of related disciplines, such as, for example, digital signal processing, computer science, information theory, telecommunications, and cryptography; see, for example, [19, Section 1.1]. Albeit many finite summation

DOI: 10.5281/zenodo.15091086

formulas are known and can be found in various handbooks and tables of series; see, for example, [52, 54, 59, 74], such formulas still continue to attract the attention of mathematicians; see, for example, [3, 4, 6, 10, 11, 12, 13, 14, 15, 16, 20, 22, 23, 28, 29, 30, 32, 33, 42, 48, 51, 57, 58, 75, 77, 79]. Indeed, very often, finite trigonometric sums cannot be evaluated in a closed-form at all.[1] In such cases, it may be desirable to have a convenient asymptotic formula; notwithstanding, even the latter may be quite difficult. We, for example, still do not know the asymptotics of many trigonometric sums related to the ζ-function.

In 1916 the famous English mathematician George N. Watson [83] considered the finite sum

$$S_n \equiv \sum_{l=1}^{n-1} \csc\frac{\pi l}{n}\,, \qquad n\in\mathbb{N}\setminus\{1\}\,, \tag{1}$$

which often occurs in mathematics, physics, and in a variety of related disciplines. Watson obtained the complete asymptotic expansion of this sum, which allows one to calculate it quickly and accurately for large n (see Remark 1 hereafter). Watson, of course, was not the first person who dealt with this sum; however, for convenience, throughout the paper we refer to this sum as *Watson's trigonometric sum.* The sum S_n also appeared in many other works, including very recent ones; see, for example, [19, Section 1.1, p. 3] for a nonexhaustive list of references. Furthermore, this sum has been found to be so remarkable, that Chen devoted a whole chapter of his book [22, Chapter 7] to it and to some of its properties. Besides, as noted in [80, Section 1], there exist closed-form expressions for much more complicated sums such as, for example, $\sum_{l=1}^{n-1}\csc^{2p}(\pi l/n)$, $p\in\mathbb{N}$, or $\sum_{l=1}^{n-1}\cos^{2q}(\theta l)\csc^{2p}(\pi l/n)$, $q,p\in\mathbb{N}$; see [22, Chapter 14], [80, Section 1], [74, Volume 1, Section 4.4.6], [75], but still little is known about S_n. In the above-cited paper [19], we partially addressed this issue by examining it in detail and by studying its properties, and also investigated its generalization

$$\sum_{l=1}^{n-1} \csc\!\left(\varphi+\frac{a\pi l}{n}\right)\,, \qquad \varphi+\frac{a\pi l}{n}\neq \pi k\,, \quad k\in\mathbb{Z}\,, \tag{2}$$

where φ and a are some parameters, the initial phase and the scaling factor, respectively. The study of this sum enabled us to obtain, *inter alia*, three advanced summation formulas for the digamma function. It was also discovered that for large n this sum may have qualitatively different behavior, depending on how the initial phase and the scaling factor are chosen. It was established that (2) has four qualitatively different leading terms, which, as φ and a start to vary, appear or dissapear depending on the relationship between φ and a. As a result, as n increases, the sum $\sum\csc\left(\varphi+a\pi l/n\right)$ may become sporadically large [19, Section 2.4.3]. There

[1]By a closed-form expression for a finite sum we mean a compact summation formula with a limited number of terms, which does not depend on the length of the sum.

also are other generalizations and extensions of S_n, which have been treated in the mathematical literature. For instance, in 1922 Hargreaves [56] extended Watson's investigations to the sum of cubes

$$\sum_{l=1}^{n-1} \csc^3 \frac{\pi l}{n}\,, \qquad n \in \mathbb{N} \setminus \{1\}\,,$$

and obtained its dominant term when n becomes large:

$$\sum_{l=1}^{n-1} \csc^3 \frac{\pi l}{n} \;=\; \frac{2}{\pi^3}\left\{n^3\zeta(3) + 3n\zeta(2)\left(\ln n + \gamma - \ln\frac{\pi}{2} - \frac{1}{6}\right)\right\} + \ldots\,.\text{[2]} \tag{3}$$

In 1923, Watson [83] considered more general sums

$$S_n^{(r)} \equiv \sum_{l=1}^{n-1} \csc^r \frac{\pi l}{n}\,, \qquad n, r \in \mathbb{N} \setminus \{1\}\,, \tag{4}$$

extended the result of Hargreaves by finding the complete asymptotics of $S_n^{(3)}$ for large n, proved that

$$\sum_{l=1}^{n-1} \csc^r \frac{\pi l}{n} \;\sim\; \frac{2n^r\zeta(r)}{\pi^r}\,, \qquad n, r \in \mathbb{N} \setminus \{1\}\,, \quad n \to \infty\,, \tag{5}$$

and noted that for an even r the above sum should also have a closed-form (note also the the latter formula is not applicable in the case $r = 1$, which once again indicates that the sum $S_n^{(1)} = S_n$ represents quite a special case).[3] Interestingly, the existence of the closed-form expressions for such sums was noted by Euler over 170 years before Watson. In particular, the formula

$$\sum_{l=0}^{n-1} \csc^2\left(\varphi + \frac{\pi l}{n}\right) \;=\; n^2 \csc^2 n\varphi\,, \qquad n \in \mathbb{N}\,, \quad \varphi \neq \pi(k - {}^{l}\!/_{n})\,,$$

where $l = 0, 1, \ldots, n-1$ and $k \in \mathbb{Z}$, may be credited to Euler; see [19, Section 1.1, pp. 5–6]. Making $\varphi \to 0$, we immediately get

$$\sum_{l=1}^{n-1} \csc^2 \frac{\pi l}{n} \;=\; \lim_{\varphi\to 0}\left\{n^2\csc^2 n\varphi - \csc^2\varphi\right\} \;=\; \frac{n^2-1}{3}\,, \qquad n \in \mathbb{N}\,, \tag{6}$$

[2] Hargreaves did not study the behavior of the remaining terms, but numerical simulations suggest that the rest in Hargreaves' formula (3) should be $o(1)$.

[3] Note that there are two misprints on p. 580 of [83] in the unnumbered formulas defining $S_n^{(2r+1)}$ and $S_n^{(2r)}$: both sums should start with $m = 1$ instead of $m = 0$.

which naturally agrees with Watson's asymptotics (5).[4] For higher even powers r Euler did not explicitly gave closed-form expressions, but indicated a way for obtaining them.[5] It may be also noted that Watson's results on $S_n^{(r)}$ were independently rediscovered several times, including rediscoveries of particular cases of (5). For instance, 40 years later Gardner, Fisher and Carlitz obtained Watson's result (5), but only for even r [49, 46, 23, 24]. Generalizations similar to (4) also appeared in works of Berndt, Alzer, Williams, Apostol, Chu, Marini and of some others; see, for example, [23], [78, p. 267], [48], and the references given therein. A wide class of sums generalizing S_n,

$$\sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n}\cdot\csc^{2q}\frac{\pi l}{n}\,,\qquad \nu=0,1,\ldots,n-1\,,\qquad q\in\mathbb{N}\,,\tag{7}$$

was considered and used in some applications by Dowker [37], [38, p. 772, Equation 14], who, among other things, found a closed-form expression for it in terms of the Bernoulli polynomials of higher order $B_n^{(s)}(x)$ [39, Equations 4–7], and later, in terms of the generalized Bernoulli polynomials [40, Equations 1–2]; see Section 1.2 hereafter. Sum (7) is also sometimes reffered to as *Dowker's sum* [29, Equation 1.1], [58, Equation 1.5], [48, Equation 4.47]. Somewhat similar generalizations were also considered by Chu and by some other authors; see, for example, [23] and the references therein. There exist, of course, many other generalizations of Watson's trigonometric sum, but it is worth noting that sums and series with secants and cosecants are often very difficult to study, even asymptotically. For instance, Chu [23, p. 137], studying sums similar to (7), remarks that "However, the resulting expressions will not be reproduced due to their complexity". Furthermore, one can recall that we still do not know wherever the *Flint Hills series* $\sum l^{-3}\csc^2 l$ converges or not [72, pp. 57–59 and 265–268], [2], [85].

The aim of this paper is to investigate another generalization of Watson's sum (1), namely the trigonometric sum

$$C_n(\nu)\,\equiv\sum_{l=1}^{n-1}\cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n}\,,\qquad \nu=0,1,2,\ldots,n-1\,,\tag{8}$$

and its particular case

$$C_n\,\equiv\sum_{l=1}^{n-1}(-1)^{l+1}\csc\frac{\pi l}{n}\,,\tag{9}$$

which may be obtained from $C_n(\nu)$ by setting $\nu=\frac{1}{2}n$ when n is even. If n is odd, then by symmetry $C_n=0$. Note that the particular case $\nu=0$ is uninteresting for

[4] As far as we know, formula (6) has not been given by Euler explicitly in his works, albeit it is a particular case of his results, so that sometimes it may be attributed to other mathematicians; see, for example, footnote 9 in [19], or Remark 2 in [3] and the references therein.

[5] The reader interested in such kinds of expressions may find them, for example, in [22, Chapter 14], [74, Volume 1, Section 4.4.6], [75], [48].

us, because $C(n,0) = S_n$, the latter sum being already mentioned here and being extensively studied in a series of previous papers. Thus, everywhere below, except if stated otherwise, we suppose that ν is not congruent to 0 modulo n. We also remark that a similar sum

$$\sum_{l=1}^{n-1} \sin\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n}\,, \qquad\qquad \nu\in\mathbb{Z}\,,$$

equals 0 by virtue of symmetry. It is also worth noting that formally the sum $C_n(\nu)$ is Dowker's sum (7) of order $q = \frac{1}{2}$. Besides, it may also be interesting to notice that formally, the sum $C_n(\nu)$ is also, up to one term and normalizing coefficients, the discrete cosine transform of the sequence of cosecants

$$\left\{\csc\frac{\pi}{n}\,,\ \csc\frac{2\pi l}{n}\,,\ \csc\frac{3\pi l}{n}\,,\ \ldots\,,\ \csc\frac{(n-1)\,\pi l}{n}\right\},$$

and can, therefore, be regarded as the *principal value* of this transform.[6] Furthemore, since the similar sum of sines vanishes identically, $C_n(\nu)$ also gives, up to a coefficient, the principal value of the discrete Fourier transform and that of the discrete Hartley transform. The sums $C_n(\nu)$ and C_n are not only interesting from a theoretical viewpoint, but also occur in applications. For example, the former appears in an important number-theoretic problem related to the *trigonometric Pólya–Vinogradov sum* $f(n,k)$ whose properties still remain little studied.[7] In fact, summing the geometric progression and then using the Fourier series expansion for $|\sin x|$, we see at once that

$$f(n,k) \equiv \sum_{l=1}^{n-1}\left|\sum_{r=m}^{m+k-1}\exp\!\left(\frac{2\pi i l r}{n}\right)\right| = \sum_{l=1}^{n-1}\frac{\big|\sin(\pi l k/n)\big|}{\sin\,(\pi l/n)} = \frac{2S_n}{\pi} - \frac{4}{\pi}\sum_{r=1}^{\infty}\frac{C_n(rk)}{4r^2-1}\,,$$

where $n\in\mathbb{N}\backslash\{1\}$, $m\in\mathbb{N}$, and k is a discrete parameter running through a complete residue system modulo n.

In the present paper, we provide several series and integral representations for $C_n(\nu)$ and C_n, establish their basic properties, obtain their asymptotic expansions (of two different kinds), and derive very accurate upper and lower bounds for them. We also obtain two relatively simple approximate formulas containing only a few terms, which are both quite accurate and can be particularly useful in applications. Much as in our previous research [19], we find that these sums are closely connected with the digamma function and the square of the Bernoulli numbers, which is quite unusual. Finally, we show that there exist summation relations between $C_n(\nu)$, the

[6]The principal value of a sum is defined according to [19, Section 1.2].

[7]For further reading on the trigonometric sum of Pólya and Vinogradov, see [81, pp. 56 and 173–174], [7, pp. 173–174], [25], [71], [60], [26], [5], [73].

cosecant and its logarithm, as well as the gamma and digamma functions, namely

$$\sum_{\nu=1}^{n-1} \Psi\Big(\frac{\nu}{n}\Big) \csc\frac{\pi\nu}{n} \;=\; -\,(\gamma+\ln 2n)\, S_n \;-\sum_{\nu=1}^{n-1} C_n(\nu) \ln\csc\frac{\pi\nu}{n}\,,$$

$$\sum_{\nu=1}^{n-1} \Psi\Big(\frac{\nu}{n}\Big) \csc\frac{\pi\nu}{n} \;=\; -\,(\gamma+\ln 2\pi n)\, S_n \;-\; 2\sum_{\nu=1}^{n-1} \ln\Gamma\Big(\frac{\nu}{n}\Big)\, C_n(\nu)\,, \tag{10}$$

$$\sum_{\nu=1}^{n-1} \Psi\Big(\frac{\nu}{n}\Big)\, C_n(\nu) \;=\; (\gamma+n\ln 2)\, S_n \;-\; n\ln\prod_{\nu=1}^{n-1}\Big(\csc\frac{\pi\nu}{n}\Big)^{\csc\frac{\pi\nu}{n}}\,. \tag{11}$$

These relations complete the advanced summation formulas for the digamma function, which we obtained earlier in [17, Appendix B, Equations (B.6)–(B.11)] and in [19, Equations (12)–(15)]. The two latter formulas are particularly beautiful. Equation (10) relates the values of the digamma function, evaluated at points uniformly distributed over the unit interval, to those of the logarithm of the gamma function evaluated at the same rational points. Formula (11) computes the product of a sequence of the form $a_l^{a_l}$, where each a_l equals the cosecant of a rational part of π.

1.2. Notation and Conventions

By definition, the set of natural numbers $\mathbb{N}$ does not include zero. The symbol $\equiv$ means "is defined as" and should not be confused with the congruence symbol from modular arithmetic. Different special numbers are denoted as follows: $\gamma \equiv \lim_{n\to\infty}\big(H_n - \ln n\big) = 0.5772156649\ldots$ is the Euler constant, $H_n \equiv 1 + {}^1\!/_2 + \ldots + {}^1\!/_n$ stands for the nth harmonic number, and B_n denotes the nth Bernoulli number. In particular $B_0 = +1$, $B_1 = {}^{-1}\!/_2$, $B_2 = {}^{+1}\!/_6$, $B_3 = 0$, $B_4 = {}^{-1}\!/_{30}$, $B_5 = 0$, $B_6 = {}^{+1}\!/_{42}$, $B_7 = 0$, $B_8 = {}^{-1}\!/_{30}$, $B_9 = 0$, $B_{10} = {}^{+5}\!/_{66}$, $B_{11} = 0$, $B_{12} = {}^{-691}\!/_{2730}\,, \ldots$.[8] The Bernoulli numbers are the particular values of the Bernoulli polynomials, which are denoted by $B_n(x)$. These polynomials are defined either implicitly via their generating function

$$\frac{z\,e^{xz}}{e^z-1} \;= \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}\,z^n\,, \qquad\qquad |z|<2\pi\,,$$

or explicitly via the Bernoulli numbers and the binomial coefficients $\binom{n}{k}$

$$B_n(x) \;=\; B_n \;+ \sum_{k=0}^{n-1}\binom{n}{k}\,x^{n-k}B_k\,.$$

[8]For further values and definitions, see [68, Chapter 2, Section 1], [61, Chapter 1, Section 1.1], [1, Section 23], Note also that there exist slightly different definitions for the Bernoulli numbers; see, for example, [53, p. 91], [63, pp. 32, 71], [83, 86], or [8, pp. 3–6].

Both definitions imply that $B_n = B_n(0)$. Furthermore, the $B_n(x)$ are themselves the particular case of the Bernoulli polynomials of higher order $B_n^{(s)}(x)$, namely $B_n(x) = B_n^{(1)}(x)$; see [61, Chapter 1, Section 1.2], [68, Chapters 2 and 6], [65, pp. 127–135], [21, p. 323, Equation (1.4)], [67], [69], [1, Section 23], [9, Volume III, Section 19.7], [31, Equation (4)], [18, p. 16, Equation (52)]. We also make use of numerous abbreviations for the functions and series. In particular, $\lfloor z \rfloor$ denotes the integer part of z and $\delta_{k,l}$ is the Kronecker delta of discrete variables k and l. Furthermore, $\operatorname{tg} z$, $\operatorname{ctg} z$, $\operatorname{ch} z$, and $\operatorname{sh} z$ stand for the tangent of z, the cotangent of z, the hyperbolic cosine of z, and the hyperbolic sine of z, respectively. We also denote by $\Gamma(s)$, $\Psi(s)$, $\Psi_n(s)$ and $\zeta(s)$ the gamma (Γ) function, the digamma (or Ψ) function, the polygamma function of order n and the Euler-Riemann zeta (ζ) function of argument s, respectively. In order to be consistent with the previous notation of Watson [83, 84] and of some other authors, we denote by S_n Watson's trigonometric sum (1), and by $C_n(\nu)$ and C_n the sums (8) and (9), respectively. Occasionally, we make use of divergent series; summability, summations methods and their regularity are defined according to Hardy's monograph [55]. It is also supposed that the reader possesses, up to a certain point, a working knowledge of the theory of asymptotic expansions [27, 36, 43, 44, 45, 70]. The order symbols O, o and the asymptotic equivalence symbol $\sim$ are defined according to Evgrafov's and Erdélyi's books [45, Chapter 1, Section 4], [43, Chapter 1]. By the error between the quantity A and its approximated value B, we mean $A - B$. Finally, all the figures in the paper were exported from CAS Maple. The data used to trace the graphs were calculated with 50-digit "precision", enabled with the help of the command `Digits:=50`. Some graphs and results were also verified independently with the help of MATLAB and CAS Mathematica.[9]

2. Basic Properties

First of all, we note that albeit ν can take only discrete values, if ν was a continuous complex variable, then $C_n(\nu)$ would be analytic in the whole complex plane, except for the case $n \to \infty$. Furthermore, $C_n(\nu)$, defined in (8) only for $\nu = 0, 1, 2, \ldots, n-1$, can easily be extended to any integer ν by means of the formula

$$C_n(\nu + mn) = \begin{cases} S_n\,, & \nu = 0\,, \qquad m \in \mathbb{Z}\,, \\ C_n(\nu)\,, & \nu = 1, 2, \ldots, n-1\,, \qquad m \in \mathbb{Z}\,, \end{cases} \tag{12}$$

[9] It should be noted that such a "precision" (50 digits) is not guaranteed at all by Maple. For instance, by computing the approximation error for the graph in Figure 4 with 8-digit precision (which should, in principle, be largely sufficient to corectly trace a graph), we first obtained completely erroneous values. Writing in Maple 12 `evalf(`ϵ`(300,34),8)`, where $\epsilon(n,\nu)$ is the approximation error, returned the value `-0.3136e-4`, while the correct value is close to `-5e-011`. In other words, in some situations, Maple 12 fails to compute correctly even the order of magnitude when using 8-digit precision.

In addition, $C_n(\nu)$ also has the following basic properties:

$$C_n(\nu) = C_n(n-\nu)\,, \qquad C_{2n-1} = 0\,, \tag{13}$$

$$-C_{2n} \leqslant C_{2n}(\nu)\,, \qquad C_n(\nu) \leqslant S_n\,, \qquad 0 < C_{2n} < S_{2n}\,, \tag{14}$$

$$C_n(\nu+1) = C_n(\nu) - 2\operatorname{ctg}\frac{(2\nu+1)\,\pi}{2n}\,, \tag{15}$$

$$C_n(\nu+\kappa) = C_n(\nu) - 2\sum_{l=1}^{\kappa}\operatorname{ctg}\frac{(2\nu+2l-1)\,\pi}{2n}\,, \qquad \kappa\in\mathbb{N}\,, \tag{16}$$

$$C_n(\nu+\kappa) = C_n(\kappa) - 2\sum_{l=1}^{\nu}\operatorname{ctg}\frac{(2\kappa+2l-1)\,\pi}{2n}\,, \qquad \kappa\in\mathbb{N}\,, \tag{17}$$

$$C_n(2\nu) = C_n(\nu) - 2\sum_{l=1}^{\nu}\operatorname{ctg}\frac{(2\nu+2l-1)\,\pi}{2n}\,, \tag{18}$$

$$C_n(\nu) - C_n(\kappa) = -2\sum_{l=\kappa+1}^{\nu}\operatorname{ctg}\frac{(2l-1)\,\pi}{2n}\,, \qquad \nu>\kappa\,, \tag{19}$$

$$\sum_{\nu=1}^{n-1} C_n(\nu) = -S_n\,, \qquad \sum_{\nu=1}^{n-1} C_n^2(\nu) = \frac{n\left(n^2-1\right)}{3} - S_n^2\,, \tag{20}$$

$$\sum_{\nu=1}^{n-1} C_n(\nu)\cos\frac{2\pi\nu k}{n} = n\csc\frac{\pi k}{n} - S_n\,, \qquad \sum_{\nu=1}^{n-1} C_n(\nu)\sin\frac{2\pi\nu k}{n} = 0\,, \tag{21}$$

$$\sum_{\nu=1}^{n-1} C_n(\nu)\operatorname{ctg}\frac{\pi\nu}{n} = 0\,, \qquad \sum_{\nu=1}^{n-1} \nu\, C_n(\nu) = -\frac{n\,S_n}{2}\,, \tag{22}$$

$$\sum_{\nu=1}^{n-1} C_n(\nu)\ln\sin\frac{\pi\nu}{n} = (\gamma+\ln 2n)\,S_n + \sum_{r=1}^{n-1}\Psi\!\left(\frac{r}{n}\right)\csc\frac{\pi r}{n}\,, \tag{23}$$

$$\sum_{\nu=1}^{n-1} C_n(\nu)\ln\Gamma\!\left(\frac{\nu}{n}\right) = -\frac{(\gamma+\ln 2\pi n)\,S_n}{2} - \frac{1}{2}\sum_{\nu=1}^{n-1}\Psi\!\left(\frac{\nu}{n}\right)\csc\frac{\pi\nu}{n}\,, \tag{24}$$

$$\sum_{\nu=1}^{n-1} C_n(\nu)\,\Psi\!\left(\frac{\nu}{n}\right) = (\gamma+n\ln 2)\,S_n - n\sum_{\nu=1}^{n-1}\csc\frac{\pi\nu}{n}\cdot\ln\csc\frac{\pi\nu}{n}\,, \tag{25}$$

$k = 1, 2, \ldots, n-1$, which can be obtained without much difficulty from the definition of $C_n(\nu)$. For instance, the recurrence relationship (15) is obtained as follows:

$$\begin{aligned}
C_n(\nu+1) &= \sum_{l=1}^{n-1} \cos\left(\frac{2\pi\nu l}{n} + \frac{2\pi l}{n}\right) \csc\frac{\pi l}{n} \\
&= \sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n} \cdot \csc\frac{\pi l}{n} \cdot \cos\frac{2\pi l}{n} - \sum_{l=1}^{n-1} \sin\frac{2\pi\nu l}{n} \cdot \csc\frac{\pi l}{n} \cdot \sin\frac{2\pi l}{n} \\
&= \sum_{l=1}^{n-1} \left(1 - 2\sin^2\frac{\pi l}{n}\right) \cos\frac{2\pi\nu l}{n} \cdot \csc\frac{\pi l}{n} - \sum_{l=1}^{n-1} \sin\frac{2\pi\nu l}{n} \cdot \csc\frac{\pi l}{n} \cdot \sin\frac{2\pi l}{n} \\
&= \sum_{l=1}^{n-1} \left(1 - 2\sin^2\frac{\pi l}{n}\right) \cos\frac{2\pi\nu l}{n} \cdot \csc\frac{\pi l}{n} - 2\sum_{l=1}^{n-1} \sin\frac{2\pi\nu l}{n} \cdot \cos\frac{\pi l}{n} \\
&= C_n(\nu) - 2\sum_{l=1}^{n-1} \sin\frac{(2\nu+1)\pi l}{n} = C_n(\nu) - 2\operatorname{ctg}\frac{(2\nu+1)\pi}{2n}\,,
\end{aligned} \tag{26}$$

where at the final step we accounted for the well-known result

$$\sum_{l=1}^{n-1} \sin\frac{\pi r l}{n} = \begin{cases} \operatorname{ctg}\dfrac{\pi r}{2n}\,, & r = 1, 3, 5, \ldots \\ 0\,, & r = 2, 4, 6, \ldots\,. \end{cases} \tag{27}$$

Repeatedly using (15) for $C_n(\nu+2)$, $C_n(\nu+3)$ and so on, we obtain (16). Since the sum is commutative, we also have (17). The duplication formula (18) is obtained from (16) by setting $\kappa = \nu$. Identities (20)–(25) follow from various summation and orthogonality properties of the cosine. For instance, the second of properties (20) is obtained by the orthogonality formula

$$\sum_{\nu=0}^{n-1} \cos\frac{2\pi\nu k}{n} \cdot \cos\frac{2\pi\nu l}{n} = \frac{n}{2}\Big\{\delta_{k,l} + \delta_{k,n-l}\Big\},$$

where both discrete variables k and l run from 1 to $n-1$, as well as with the help of (6). The summation formulas involving the gamma and the digamma functions are obtained as follows. Gauss' digamma theorem states that

$$\Psi\left(\frac{l}{n}\right) = -\gamma - \ln 2n - \frac{\pi}{2}\operatorname{ctg}\frac{\pi l}{n} + \sum_{\nu=1}^{n-1} \cos\frac{2\pi\nu l}{n} \cdot \ln\sin\frac{\pi\nu}{n}\,, \tag{28}$$

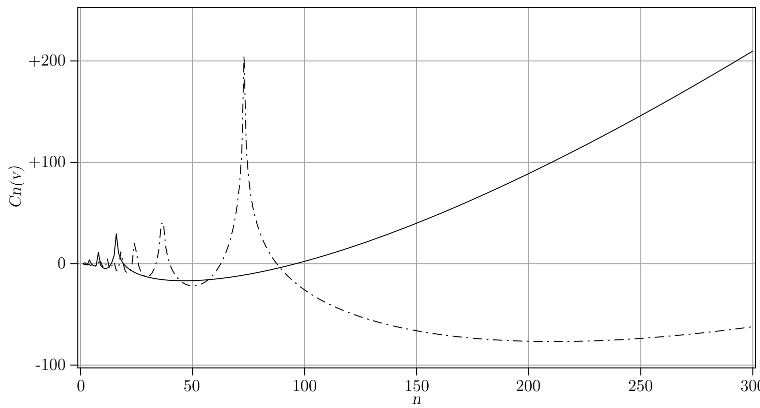

Figure 1: The sum $C_n(\nu)$ as a function of n, where $n \in [2, 300]$, for two different values of argument: $\nu = 16$ (solid line) and $\nu = 73$ (dash-dotted line).

where $l = 1, 2, \ldots, n-1$, $n = 2, 3, 4, \ldots$; see [17, Equations (B.4b)] or [9, Volume I, Section 1.7.3, Equation (29)]. Therefore,

$$\begin{aligned}\sum_{\nu=1}^{n-1} C_n(\nu) \ln\sin\frac{\pi\nu}{n} &= \sum_{l=1}^{n-1} \csc\frac{\pi l}{n} \sum_{\nu=1}^{n-1} \cos\frac{2\pi\nu l}{n} \cdot \ln\sin\frac{\pi\nu}{n} \\ &= \sum_{l=1}^{n-1} \left\{ \Psi\left(\frac{l}{n}\right) + \gamma + \ln 2n + \frac{\pi}{2}\operatorname{ctg}\frac{\pi l}{n} \right\} \csc\frac{\pi l}{n} \\ &= (\gamma + \ln 2n)\, S_n + \sum_{l=1}^{n-1} \Psi\left(\frac{l}{n}\right) \csc\frac{\pi l}{n} + \frac{\pi}{2}\sum_{l=1}^{n-1} \cos\frac{\pi l}{n} \csc^2\frac{\pi l}{n}\,.\end{aligned}$$

Remarking that the last sum equals 0 by virtue of symmetry, we arrive at (23). Formula (24) is derived analogously by using Malmsten's variant of Gauss' digamma theorem [17, Equations (B.4c)] instead of (28). Finally, identity (25) follows from (22) and Gauss' digamma theorem (28). Note, by the way, that many properties are related to Watson's trigonometric sum.

We conclude this section with two graphs of $C_n(\nu)$, depicted in Figures 1 and 2. The former figure displays the graph of $C_n(\nu)$ as a function of n for two values of ν.[10] It is quite remarkable to observe, see Figure 1, that when $n < \nu$, as n approaches the values having common divisors with ν, the sums $C_n(\nu)$ reach a local maximum. It can also be clearly observed, see Figure 2, that $C_n(\nu)$ reaches the minimum when $\nu = \frac{1}{2}n = 150$, see property (14) and remark that $-C_{300} \approx -132$, and the maximum at the endpoints, at which $C_{300}(\nu)$ becomes equal to $S_{300} \approx 1113$

[10] We deliberately take values of ν, which lie outside the interval defined earlier, in order to observe the overall behavior of $C_n(\nu)$.

(we do not show them on the graph, since they are too high). The explanation of these interesting phenomena is given in Section 4 and follows from Theorem 2. We also clearly observe that $C_n(\nu)$ verifies the first property from (13) and can take both positive and negative values. Furthermore, it can be seen that the minimum of $|C_n(\nu)|$ occurs at $\nu = 50 = \frac{1}{6}n$ and $\nu = 250 = \frac{5}{6}n$ (empirical studies show that at these points $C_n(\nu)$ is always very small in absolute value, never equal to zero, but tends to zero as $n \to \infty$, n being a multiple of 6; see also Corollary 4 in Section 6).

3. Integral Representation

Below, we derive an important integral representation for the sum $C_n(\nu)$, which is useful to establish some further properties.

Theorem 1 (Integral Representations for $C_n(\nu)$)**.** *The sum $C_n(\nu)$, as defined in (8), can be represented via the following integral, containing the discrete Poisson kernel:*

$$\sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n} \cdot \csc\frac{\pi l}{n} = \frac{2n}{\pi} \int_0^1 \frac{(1+x^n)\cos\dfrac{2\pi\nu}{n} - x^{n-1} - x}{1+x^n} \cdot \frac{dx}{x^2 - 2x\cos\dfrac{2\pi\nu}{n} + 1} \, .$$

This integral can also be written in several alternative forms, for example,

$$\sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n} \cdot \csc\frac{\pi l}{n} = (n-2\nu)\operatorname{ctg}\frac{2\pi\nu}{n} - \frac{n}{\pi}\int_0^\infty \frac{\operatorname{ch}\left[x\left(\dfrac{n}{2}-1\right)\right]}{\operatorname{ch}\dfrac{xn}{2}\left(\operatorname{ch}x - \cos\dfrac{2\pi\nu}{n}\right)}\, dx \, .$$

Proof. Let x and φ be real variables such that $xe^{i\varphi} \neq 1$. On summing the geometric progression

$$\sum_{l=1}^{n-1} x^l e^{i\varphi l} = \sum_{l=1}^{n-1} \left(xe^{i\varphi}\right)^l = \frac{x^n e^{i\varphi n} - xe^{i\varphi}}{xe^{i\varphi} - 1} \, , \qquad n = 2, 3, 4, \ldots$$

and then on separating the real and the imaginary parts, we obtain

$$\sum_{l=1}^{n-1} x^l \cos\varphi l = \frac{x^{n+1}\cos\varphi(n-1) - x^n\cos\varphi n + x\cos\varphi - x^2}{x^2 - 2x\cos\varphi + 1} \tag{29}$$

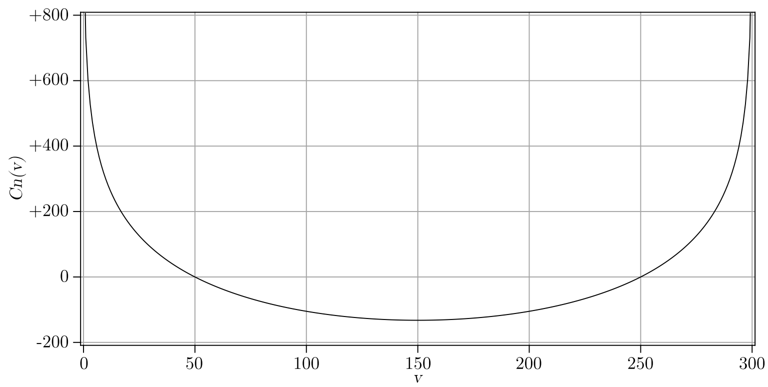

Figure 2: The sum $C_{300}(\nu)$ as a function of ν, where $\nu \in [0, 300]$.

and

$$\sum_{l=1}^{n-1} x^l \sin\varphi l \;=\; \frac{x^{n+1}\sin\varphi(n-1) - x^n \sin\varphi n + x\sin\varphi}{x^2 - 2x\cos\varphi + 1}\,, \tag{30}$$

respectively. Now consider the following well-known result due to Euler[11]

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx \;=\; \pi\csc p\pi\,, \qquad 0 < \operatorname{Re} p < 1\,,$$

and let $p = l/n$, where $n \in \mathbb{N} \setminus \{1\}$ and $0 < l < n$. Multiplying both sides by $\cos\varphi l$, where $\varphi = 2\pi\nu/n \equiv \theta$, and then summing the result from $l = 1$ to $l = n-1$, we have, by virtue of (29),

$$\begin{aligned}
\sum_{l=1}^{n-1} \cos\theta l \cdot \csc\frac{\pi l}{n} \;&=\; \frac{1}{\pi}\int_0^\infty \frac{1}{x(1+x)} \sum_{l=1}^{n-1} x^{\frac{l}{n}} \cos\theta l\,dx \\
&=\; \frac{1}{\pi}\int_0^\infty \frac{x^{\frac{1}{n}+1}\cos\theta - x + x^{\frac{1}{n}}\cos\theta - x^{\frac{2}{n}}}{x^{\frac{2}{n}} - 2x^{\frac{1}{n}}\cos\theta + 1} \cdot \frac{dx}{x(1+x)} \\
&=\; \frac{n}{\pi}\int_0^\infty \frac{y^n\cos\theta - y^{n-1} + \cos\theta - y}{\big(1+y^n\big)\big(y^2 - 2y\cos\theta + 1\big)}\,dy \\
&=\; \frac{2n}{\pi}\int_0^1 \frac{\big(1+y^n\big)\cos\theta - y^{n-1} - y}{\big(1+y^n\big)\big(y^2 - 2y\cos\theta + 1\big)}\,dy\,,
\end{aligned} \tag{31}$$

[11]See, for example, [52, no. 3.222-2], [41, no. 856.02], [66, p. 170, no. 5.3.4.15, and p. 172, no. 5.3.4.20-1], [82, p. 125, no. 878].

where we made a change of variable $y = x^{\frac{1}{n}}$ and then split the interval of integration into two parts $[0,1]$ and $[1,\infty)$, the integral over $[1,\infty)$ being equal to that over $[0,1]$. We, thus, have arrived at the first result of the theorem.

Making once again a change of variable $y = e^{-t}$ in the last integral in (31), we get

$$\begin{aligned} C_n(\nu) &= \frac{2n}{\pi}\int_0^1 \frac{\big(1+y^n\big)\cos\theta - y^{n-1} - y}{\big(1+y^n\big)\big(y^2 - 2y\cos\theta + 1\big)}\,dy \\ &= \frac{n}{\pi}\int_0^\infty \frac{\operatorname{ch}\frac{1}{2}tn\cdot\cos\theta - \operatorname{ch}\big[t\big(\frac{1}{2}n - 1\big)\big]}{\big(\operatorname{ch}t - \cos\theta\big)\operatorname{ch}\frac{1}{2}tn}\,dt\,. \end{aligned}$$

The final step, of accounting for the elementary integral[12]

$$\int_0^\infty \frac{dt}{\operatorname{ch}t - \cos\varphi} = \begin{cases} \dfrac{\pi-\varphi}{\sin\varphi}\,, & 0<\varphi<2\pi\,, \quad \varphi\neq\pi\,, \\ 1\,, & \varphi = \pi\,, \end{cases}$$

evaluated at $\varphi = \theta$, yields the second result of the theorem. Note that both results of the theorem hold for integer ν only, although the above method can be extended to the continuous values of ν as well, in which case the resulting expression is more complicated.[13] □

4. Series Representations

In this section, we obtain several series representations for $C_n(\nu)$. We begin with four alternative finite series representations.

Theorem 2 (Finite Series Representations for $C_n(\nu)$)**.** *The finite sum $C_n(\nu)$ and Watson's trigonometric sum S_n are directly related to each other by means of these four non-exhaustive formulas:*

$$\sum_{l=1}^{n-1}\cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n} = S_n - 2\sum_{l=1}^{\nu}\operatorname{ctg}\frac{(2l-1)\,\pi}{2n}\,,$$

$$\sum_{l=1}^{n-1}\cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n} = S_n - 2\sum_{l=1}^{n-1}\sin^2\frac{\pi\nu l}{n}\cdot\csc\frac{\pi l}{n}\,,$$

$$\sum_{l=1}^{n-1}\cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n} = -S_n + 2\sum_{l=1}^{n-1}\cos^2\frac{\pi\nu l}{n}\cdot\csc\frac{\pi l}{n}\,,$$

[12] See, for example, [52, no. 2.444-2] or [41, no. 859.163].

[13] The assumption of discrete ν enables us to simplify our calculations in the second line of (31).

$$\sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n} \;=\; S_n - \operatorname{ctg}\frac{\pi}{2n} + \operatorname{ctg}\frac{\pi\,(2\nu+1)}{2n}$$

$$-2\sum_{l=1}^{n-1}\operatorname{ctg}\frac{\pi l}{n}\cdot\sin\frac{\pi l\nu}{n}\cdot\sin\frac{\pi l\,(\nu+1)}{n}\,.$$

Proof. The first formula is deduced from property (16). Setting $\nu = 0$, using (12) and then writing ν instead of κ, yields this formula. The second and third equalities follow from the elementary trigonometric identity $\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$. Finally, the fourth equality is obtained from the first one as follows. In view of (27), the cotangent sum can be written in the form

$$\sum_{l=1}^{\nu}\operatorname{ctg}\frac{(2l-1)\,\pi}{2n} \;=\; \sum_{k=1}^{n-1}\sum_{l=1}^{\nu}\sin\!\left(\frac{2\pi kl}{n}-\frac{\pi k}{n}\right)$$

$$= \sum_{k=1}^{n-1}\left\{\left[-\frac{\cos\!\left(\frac{2\pi k\nu}{n}+\frac{\pi k}{n}\right)}{2\sin\frac{\pi k}{n}} + \frac{1}{2}\operatorname{ctg}\frac{\pi k}{n}\right]\times\cos\frac{\pi k}{n}\right.$$

$$\left. - \left[\frac{\sin\!\left(\frac{2\pi k\nu}{n}+\frac{\pi k}{n}\right)}{2\sin\frac{\pi k}{n}} - \frac{1}{2}\right]\times\sin\frac{\pi k}{n}\right\}, \qquad (32)$$

by virtue of (29) and (30) evaluated at $x = 1$ and $\varphi = 2\pi k/n$. Then, employing twice (27), we obtain

$$\sum_{k=1}^{n-1}\sin\frac{\pi k}{n} \;=\; \operatorname{ctg}\frac{\pi}{2n} \qquad\text{and}\qquad \sum_{k=1}^{n-1}\sin\frac{\pi k\,(2\nu+1)}{n} \;=\; \operatorname{ctg}\frac{(2\nu+1)\,\pi}{2n}\,.$$

Inserting latter formulas into (32) yields, after some algebra, the fourth formula of the theorem. □

The last three formulas obtained in the preceding theorem are, in some sense, similar. As to the first formula $C_n(\nu) = S_n - 2\sum_{l=1}^{\nu}\operatorname{ctg}\frac{(2l-1)\pi}{2n}$, it is clearly of different type and deserves to be briefly discussed. First of all, it immediately implies that $C_n(\nu)$ cannot be greater than S_n and lesser than $-C_n$, properties that we already gave in Section 2. They follow from the fact that $\sum\operatorname{ctg}\frac{(2l-1)\pi}{2n}$ increases while ν remains below $\frac{1}{2}n$, decreases when $\nu > \frac{1}{2}n$ (when $l > \frac{1}{2}n$, the lth term sums with the $(n-l)$th term, which has the same magnitude and the opposite sign, leading thus to the overall decrease of $\sum\operatorname{ctg}\frac{(2l-1)\pi}{2n}$), and vanishes as ν reaches n; see, for example, Figure 2. It also explains why as long as $n < \nu$, the sum $C_n(\nu)$ reaches local maxima as n approaches common divisors with ν (see Figure 1). This is due to the sum of cotangents, which not only vanishes at $n = \nu$, but as long as $n < \nu$, is also relatively small whenever $\gcd(n,\nu) > 1$. Furthermore, as noted before, S_n is very well investigated; the study of $C_n(\nu)$ can, therefore, be reduced to that of $\sum \operatorname{ctg} \frac{(2l-1)\pi}{2n}$. Note, however, that the investigation of the cotangent sums often faces considerable difficulties.[14] Furthermore, some of such sums even appear to be directly associated to the study of the Riemann hypothesis [10, 15, 16, 20, 28, 30, 33, 34, 35, 42, 47, 50, 62, 64, 76, 78].

We now obtain an infinite series representation for the function $C_n(\nu)$, which is useful for the derivation of the asymptotic expansion of $C_n(\nu)$ at large n. In order to derive it, we first need to prove the following Lemma.

Lemma 1. *If* $\operatorname{Re}(\alpha + b - \beta) > 0$, *then the following improper integral converges and can be evaluated via a combination of four digamma functions:*

$$\int_0^\infty e^{-\alpha x} \frac{\operatorname{ch} \beta x}{\operatorname{ch} bx}\, dx = \frac{1}{4b}\left\{\Psi\left(\frac{3}{4} + \frac{\alpha+\beta}{4b}\right) - \Psi\left(\frac{1}{4} + \frac{\alpha-\beta}{4b}\right) + \Psi\left(\frac{3}{4} + \frac{\alpha-\beta}{4b}\right) - \Psi\left(\frac{1}{4} + \frac{\alpha+\beta}{4b}\right)\right\}.$$

Proof. Consider the integral

$$\int_0^\infty e^{-\alpha x} \frac{\operatorname{ch} \beta x}{\operatorname{ch} bx}\, dx\,.$$

Multiplying the numerator and the denominator of the integrand by $2 \operatorname{sh} bx$ yields

$$\int_0^\infty e^{-\alpha x} \frac{\operatorname{ch} \beta x}{\operatorname{ch} bx}\, dx = 2\int_0^\infty e^{-\alpha x} \frac{\operatorname{sh} bx \operatorname{ch} \beta x}{2 \operatorname{ch} bx \operatorname{sh} bx}\, dx = 2\int_0^\infty e^{-\alpha x} \frac{\operatorname{sh}[x(b+\beta)] + \operatorname{sh}[x(b-\beta)]}{\operatorname{sh} 2bx}\, dx\,.$$

Using the well-known formula[15]

$$\int_0^\infty e^{-\alpha x} \frac{\operatorname{sh} \mu x}{\operatorname{sh} mx}\, dx = \frac{1}{2m}\left\{\Psi\left(\frac{1}{2} + \frac{\alpha+\mu}{2m}\right) - \Psi\left(\frac{1}{2} + \frac{\alpha-\mu}{2m}\right)\right\},$$

twice, which holds for $\operatorname{Re}(\alpha + m - \mu) > 0$ with $m = 2b$ and $\mu = b \pm \beta$, respectively, we immediately arrive at the required result. □

[14] Despite the fact that they have been regularly studied at least since Euler's time; see, for example, [19, Sections 1.1 and 4].

[15] See [52, no. 3.541-2], [9, Volume I, Section 1.7.2, Equations (14)–(15)].

Theorem 3 (Infinite Series Representations for $C_n(\nu)$)**.** *The finite sum $C_n(\nu)$, defined by (8), admits the series representation*

$$\sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n} = \frac{2n}{\pi}\ln\Bigl(2\sin\frac{\pi\nu}{n}\Bigr) - \frac{2}{\pi}\left\{\Psi\left(\frac{2}{n}\right)-\Psi\left(\frac{1}{n}\right)\right\} - \frac{2}{\pi}\csc\frac{2\pi\nu}{n}$$
$$\times\sum_{l=2}^{\infty}\left\{\Psi\left(\frac{l-1}{2n}\right)-\Psi\left(\frac{l+1}{2n}\right)-\Psi\left(\frac{l-1}{n}\right)+\Psi\left(\frac{l+1}{n}\right)\right\}\sin\frac{2\pi\nu l}{n}\,, \tag{33}$$

where the last infinite series converges at the same rate as $\sum l^{-2}\sin(2\pi\nu l/n)$ *.*

Proof. Consider the second formula of Theorem 1 and let θ denote $2\pi\nu/n$ for the purpose of brevity. Expanding $(\operatorname{ch}x-\cos\theta)^{-1}$ into the uniformly convergent series

$$\frac{1}{\operatorname{ch}x-\cos\theta} = \frac{2}{\sin\theta}\sum_{l=1}^{\infty}e^{-xl}\sin\theta l\,, \qquad \operatorname{Re}x>0\,,$$

and using Lemma 1, we see that

$$\begin{aligned} C_n(\nu) &= \bigl(n-2\nu\bigr)\operatorname{ctg}\theta - \frac{2n}{\pi\sin\theta}\sum_{l=1}^{\infty}\sin\theta l\int\limits_0^{\infty}e^{-xl}\,\frac{\operatorname{ch}\bigl[x\bigl(\tfrac12 n-1\bigr)\bigr]}{\operatorname{ch}\tfrac12 xn}\,dx \\ &= \bigl(n-2\nu\bigr)\operatorname{ctg}\theta - \frac{1}{\pi\sin\theta}\sum_{l=1}^{\infty}\sin\theta l\left\{\Psi\left(1+\frac{l-1}{2n}\right)-\Psi\left(\frac{l+1}{2n}\right)\right. \\ &\qquad\left.+\Psi\left(\frac12+\frac{l+1}{2n}\right)-\Psi\left(\frac12+\frac{l-1}{2n}\right)\right\} \\ &= \bigl(n-2\nu\bigr)\operatorname{ctg}\theta - \frac{1}{\pi}\left\{2\ln 2+\Psi\left(\frac12+\frac1n\right)-\Psi\left(\frac1n\right)\right\} \\ &\qquad - \frac{1}{\pi\sin\theta}\sum_{l=2}^{\infty}\sin\theta l\left\{\frac{2n}{l-1}+\Psi\left(\frac{l-1}{2n}\right)-\Psi\left(\frac{l+1}{2n}\right)\right. \\ &\qquad\left.+\Psi\left(\frac12+\frac{l+1}{2n}\right)-\Psi\left(\frac12+\frac{l-1}{2n}\right)\right\}. \end{aligned} \tag{34}$$

Using two well-known Fourier series in order to evaluate

$$\begin{aligned} \sum_{l=2}^{\infty}\frac{\sin\theta l}{l-1} &= \cos\theta\sum_{l=1}^{\infty}\frac{\sin\theta l}{l} + \sin\theta\sum_{l=1}^{\infty}\frac{\cos\theta l}{l} \\ &= \frac{\pi-\theta}{2}\cos\theta - \sin\theta\cdot\ln\Bigl(2\sin\frac{\theta}{2}\Bigr)\,, \end{aligned} \tag{35}$$

and employing thrice the duplication formula for the digamma function

$$\Psi\left(\frac{1}{2}+\frac{1}{n}\right)-\Psi\left(\frac{1}{n}\right)=2\Psi\left(\frac{2}{n}\right)-2\Psi\left(\frac{1}{n}\right)-2\ln 2\,,$$

$$\Psi\left(\frac{l-1}{2n}\right)-\Psi\left(\frac{1}{2}+\frac{l-1}{2n}\right)=2\Psi\left(\frac{l-1}{2n}\right)-2\Psi\left(\frac{l-1}{n}\right)+2\ln 2\,,$$

$$\Psi\left(\frac{1}{2}+\frac{l+1}{2n}\right)-\Psi\left(\frac{l+1}{2n}\right)=-2\Psi\left(\frac{l+1}{2n}\right)+2\Psi\left(\frac{l+1}{n}\right)-2\ln 2\,,$$

the last expression in (34) immediately reduces to (33).

Finally, in order to study the behavior of the general term of series (33) at large index l, we use the Stirling formula

$$\Psi(x)\;=\;\ln x\;-\;\frac{1}{2x}\;-\;\frac{1}{2}\sum_{r=1}^{N-1}\frac{B_{2r}}{r\,x^{2r}}\;-\;\frac{\lambda\,B_{2N}}{2Nx^{2N}}\,,\qquad x>0\,,$$

where, as usual, $0<\lambda<1$ and $N=2,3,4,\ldots,\ N<\infty$. For a sufficiently large l, we, therefore, have:

$$\Psi\left(\frac{l-1}{2n}\right)-\Psi\left(\frac{l+1}{2n}\right)-\Psi\left(\frac{l-1}{n}\right)+\Psi\left(\frac{l+1}{n}\right)\;\sim\;-\frac{n}{l^2}\,.$$

Thus, series (33) converges slightly better than Euler's series $\sum l^{-2}$. □

5. Preliminary Asymptotic Studies for Large n

In this part we study the asymptotic behavior of $C_n(\nu)$ at large n. The results are mostly based on the series representations obtained earlier in Theorem 3.

Theorem 4 (Almost Asymptotic Expansion of $C_n(\nu)$)**.** *For any $n=2,3,4,\ldots$ and $N=2,3,4,\ldots,\ N<\infty$, the sum $C_n(\nu)$ admits the expansion*

$$\sum_{l=1}^{n-1}\cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n}=-\frac{2n}{\pi}\ln\Big(2\sin\frac{\pi\nu}{n}\Big)+2\sum_{r=1}^{N-1}\frac{\big(1-2^{1-2r}\big)\pi^{2r-1}B_{2r}\,\mathscr{H}_{2r-1}(\nu/n)}{(2r)!\,n^{2r-1}}$$
$$+\,2\lambda\frac{\big(1-2^{1-2N}\big)\pi^{2N-1}B_{2N}\,\mathscr{H}_{2N-1}(\nu/n)}{(2N)!\,n^{2N-1}}\,,\qquad(36)$$

$0<\lambda<1$, *where we denoted for the sake of brevity*

$$\mathscr{H}_{2r-1}\Big(\frac{\nu}{n}\Big) \;\equiv\; \frac{d^{2r-1}}{d\varphi^{2r-1}}\,\operatorname{ctg}\varphi\bigg|_{\varphi=\pi\nu/n} = -\frac{1}{\pi^{2r}}\left\{\Psi_{2r-1}\left(\frac{\nu}{n}\right)+\Psi_{2r-1}\left(1-\frac{\nu}{n}\right)\right\}$$

$$= (-1)^r\,\frac{(2n)^{2r-1}}{r}\sum_{s=1}^{n} B_{2r}\Big(\frac{s}{n}\Big)\cos\frac{2\pi s\,\nu}{n}\,, \tag{37}$$

here the $B_r(x)$ *are the Bernoulli polynomials and all* $\mathscr{H}_{2r-1}(\nu/n)$ *are negative.*

We remark that the above theorem provides the expansion of $C_n(\nu)$ via the derivatives of the cotangent. In fact, not only does the tail contain such derivatives, but also the dominant term which is, up to some coeffcients, the antiderivative of the cotangent (the only term which does not contain the derivatives of the cotangent is $-\frac{2n\ln 2}{\pi}$). The above theorem also leads to several important corollaries and has useful applications. In particular, under some conditions, one can readily deduce from it several asymptotic formulas for $C_n(\nu)$ and C_n at large n.

Corollary 1 (Asymptotic Representation of $C_n(\nu)$ at Large n)**.** *At large* n *the following asymptotic representation holds for the sum* $C_n(\nu)$*:*

$$\sum_{l=1}^{n-1}\cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n}\;\sim\;-\frac{2n}{\pi}\,\ln\Big(2\sin\frac{\pi\nu}{n}\Big)\,,$$

where $\nu\neq\frac{1}{6}n$ *and* $\nu\neq\frac{5}{6}n$.

Corollary 2 (Complete Asymptotics of C_n at Large n)**.** *Let* $N=2,3,4,\dots$, $N<\infty$ *. Then, for* $n=2,4,6,\dots$, *the sum* C_n*, admits the following expansion:*

$$\sum_{l=1}^{n-1}(-1)^{l+1}\csc\frac{\pi l}{n} \;=\; \frac{2n\ln 2}{\pi} + 2\sum_{r=1}^{N-1}\frac{(-1)^{r+1}\big(2^{2r-1}-1\big)\big(2^{2r}-1\big)\pi^{2r-1}B_{2r}^2}{r\,(2r)!\,n^{2r-1}}$$

$$+\,O\big(n^{1-2N}\big)\,,$$

which, as $n\to\infty$*, becomes its complete (or full) asymptotics. Writing down the first few terms, we have*

$$\sum_{l=1}^{n-1}(-1)^{l+1}\csc\frac{\pi l}{n} = \frac{2n\ln 2}{\pi}+\frac{\pi}{12\,n}-\frac{7\pi^3}{1440\,n^3}+\frac{31\pi^5}{30\,240\,n^5}-\frac{2159\pi^7}{4\,838\,400\,n^7}+\dots\,.$$

If n *is odd, then*

$$\sum_{l=1}^{n-1}(-1)^{l+1}\csc\frac{\pi l}{n}\;=\;0\,.$$

Remark 1. The above asymptotic expansion can be compared to that of

$$\sum_{l=1}^{n-1} \csc\frac{\pi l}{n} \;=\; \frac{2n}{\pi}\left(\ln\frac{2n}{\pi}+\gamma\right) - \frac{2}{\pi}\sum_{r=1}^{N-1}\frac{(-1)^{r+1}\big(2^{2r-1}-1\big)\pi^{2r}B_{2r}^2}{r\,(2r)!\,n^{2r-1}} + O\big(n^{1-2N}\big)\,,$$

which is probably due to Watson, see [19, Theorems 6a,b], and which also contains the square of the Bernoulli numbers. It is also interesting that unlike $\sum\csc(\pi l/n)$ the asymptotic expansion of $\sum(-1)^{l+1}\csc(\pi l/n)$ does not contain Euler's constant.

Proof. Consider (29) and (30) at $x=1$:

$$\sum_{l=1}^{n-1}\cos\varphi l \;=\; \frac{\sin\big(n\varphi-\frac12\varphi\big)}{2\sin\frac12\varphi} \;-\; \frac12\,,$$

$$\sum_{l=1}^{n-1}\sin\varphi l \;=\; -\frac{\cos\big(n\varphi-\frac12\varphi\big)}{2\sin\frac12\varphi} \;+\; \frac12\operatorname{ctg}\frac{\varphi}{2}\,.$$

As n tends to infinity, both series diverge, but if φ is not congruent to $0 \pmod{2\pi}$, then they still remain Cesàro summable:

$$\sum_{l=1}^{\infty}\cos\varphi l \;=\; -\frac12 \qquad (\mathrm{C},1)\,, \qquad\qquad \sum_{l=1}^{\infty}\sin\varphi l \;=\; \frac12\operatorname{ctg}\frac{\varphi}{2} \qquad (\mathrm{C},1)\,, \tag{38}$$

since

$$\sum_{n=1}^{N}\sin\big(n\varphi-\tfrac12\varphi\big) \;=\; O(1) \qquad \text{and} \qquad \sum_{n=1}^{N}\cos\big(n\varphi-\tfrac12\varphi\big) \;=\; O(1)\,,$$

as $N\to\infty$, and hence

$$\lim_{N\to\infty}\left\{\frac1N\sum_{n=1}^{N}\frac{\sin\big(n\varphi-\frac12\varphi\big)}{2\sin\frac12\varphi}\right\} \;=\; 0$$

and similarly for the cosine, respectively. Moreover, formulas (38) can also be obtained by employing other regular summation methods, such as, for example, Euler summations ($\mathfrak{E}$) and (E, 1), Abel summation (A), Borel summation methods (B) and (B'), etc.[16]

Let us now examine the series expansion for the digamma function given in [9, Volume I, Section 1.17, Equation (5)]. This series possesses the property that the

[16] Readers interested in a deeper study of the divergent series are kindly invited to refer to monograph [55]. Note that the use of divergent series for the derivation of asymptotic series is a frequent practice [27, 36, 43, 44, 70].

error, due to stopping at any term, is numerically less than the first term neglected. This means that the same series can also be written in the form

$$\Psi(x) \;=\; -\frac{1}{x} - \gamma + \sum_{m=2}^{M-1} (-1)^m x^{m-1} \zeta(m) + \lambda (-1)^M x^{M-1} \zeta(M)\,, \tag{39}$$

where $0<\lambda<1$, $M = 3, 4, 5, \ldots$, and which has the advantage of holding for any $x>0$, while [9, Volume I, Section 1.17, Equation (5)] holds in the unit disc only. Using this expansion, as well as (38) and their formal derivatives with respect to φ, the second line of (33) becomes:

$$\begin{aligned}
&\sum_{l=2}^{\infty} \left\{ \Psi\left(\frac{l-1}{2n}\right) - \Psi\left(\frac{l+1}{2n}\right) - \Psi\left(\frac{l-1}{n}\right) + \Psi\left(\frac{l+1}{n}\right) \right\} \sin\theta l \\
&\qquad = -2n \sum_{l=2}^{\infty} \frac{\sin\theta l}{l^2-1} \;-\; \sin\theta \cdot \sum_{m=2}^{2N-2} (-1)^m \big(1-2^{1-m}\big)\zeta(m) \\
&\qquad\qquad \times \frac{2\mathscr{C}_{m-1}(\theta) + 2^{m-1}}{n^{m-1}} \;+\; R_N(\lambda, n, \theta)\,,
\end{aligned} \tag{40}$$

where again for brevity we put $\theta \equiv 2\pi\nu/n$, where

$$\mathscr{C}_{m-1}(\theta) \;=\; \begin{cases} 0\,, & m = 2r-1\,, \quad r\in\mathbb{N}\,, \\ \dfrac{(-1)^{\frac{m}{2}-1}}{2} \left(\operatorname{ctg}\dfrac{\varphi}{2}\right)^{(m-1)}_{\varphi=\theta}\,, & m = 2r\,, \quad r\in\mathbb{N}\,, \end{cases} \tag{41}$$

and where $R_N(\lambda, n, \theta)$ stands for the remainder. Proceeding with the first sum similarly to (35), one can easily show that

$$\sum_{l=2}^{\infty} \frac{\sin\theta l}{l^2-1} \;=\; \left[\frac{1}{4} - \ln\!\left(2\sin\frac{\theta}{2}\right)\right] \sin\theta\,. \tag{42}$$

Furthermore,

$$\left(\operatorname{ctg}\frac{\varphi}{2}\right)^{(2r-1)}_{\varphi=\theta} \;=\; \frac{1}{2^{2r-1}} \cdot \frac{d^{2r-1}\operatorname{ctg}\varphi}{d\varphi^{2r-1}}\bigg|_{\varphi=\frac{1}{2}\theta} \;\equiv\; \frac{\mathscr{H}_{2r-1}({}^{\nu}/_{n})}{2^{2r-1}}\,. \tag{43}$$

Now, employing again (39), we see that a part of the last sum in (40) reduces to

$$\sum_{m=2}^{2N-2} (-1)^m \big(1-2^{1-m}\big)\zeta(m) n^{1-m} 2^{m-1}$$

$$= \sum_{m=2}^{2N-2} (-1)^m \big(2^{m-1}-1\big)\zeta(m) n^{1-m}$$

$$= \Psi\!\left(\frac{2}{n}\right) - \Psi\!\left(\frac{1}{n}\right) - \frac{n}{2} + \lambda\big(2^{2N-2}-1\big)\zeta(2N-1)n^{2-2N}\,, \tag{44}$$

$0<\lambda<1$. Substituting (43) into (41) and (42) into (40), as well as accounting for (44), formula (33) from Theorem 3 becomes

$$C_n(\nu) = -\frac{2n}{\pi}\ln\!\left(2\sin\frac{\theta}{2}\right) + \frac{2}{\pi}\sum_{r=1}^{N-1}\frac{(-1)^{r-1}\big(1-2^{1-2r}\big)\zeta(2r)}{(2n)^{2r-1}}\mathscr{H}_{2r-1}(\nu/n)$$

$$+ 2\lambda\frac{(-1)^{N-1}\big(1-2^{1-2N}\big)\zeta(2N)}{\pi(2n)^{2N-1}}\mathscr{H}_{2N-1}(\nu/n)\,, \tag{45}$$

where again $0<\lambda<1$. Finally, using the famous result

$$\zeta(2r) = \frac{(-1)^{r+1}(2\pi)^{2r}B_{2r}}{2\,(2r)!}\,, \qquad r\in\mathbb{N}\,,$$

established by Euler in the first half of the XVIIIth century, we immediately arrive at the expansion of Theorem 4. The second representation of $\mathscr{H}_{2r-1}(\nu/n)$ in (37), that via two polygamma functions, is obtained by differentiating the reflection formula of the digamma function $\pi\operatorname{ctg}\varphi = \Psi(1-\varphi/\pi)-\Psi(\varphi/\pi)$ with respect to φ $(2r-1)$ times, and then by setting $\varphi=\theta/2=\pi\nu/n$. The third representation of $\mathscr{H}_{2r-1}(\nu/n)$ directly follows from the relationship between the derivatives of the cotangent at rational multiples of π and the Bernoulli polynomials; see, for example, [31, p. 218].

Now, retaining only the dominant term in the asymptotic expansion from Theorem 4, we obtain the formula given in Corollary 1. Note that it is valid neither for $\nu=\frac{1}{6}n$, nor for $\nu=\frac{5}{6}n$, because at these points the argument of the logarithm equals 1 independently of n. However, the definition of the asymptotic equivalence does not imply the possibility of dividing by a quantity, which is identically equal to 0.

Finally, setting $\nu=\frac{1}{2}n$, n is even, in the asymptotic expansion from Theorem 4 and remarking that

$$\frac{d^{2r-1}}{d\varphi^{2r-1}}\operatorname{ctg}\varphi\bigg|_{\varphi=\frac{1}{2}\pi} = (-1)^r\frac{2^{2r-1}\left(2^{2r}-1\right)B_{2r}}{r}\,, \qquad r\in\mathbb{N}\,,$$ [17]

as well as bearing in mind that $C_n=0$ for odd n, we arrive at Corollary 2. □

[17] See, for example, [31, Corollary 1].

6. Bounds, Inequalities, Approximate Equalities, and Asymptotic Expansion

We already have bounds (14) obtained earlier; however, they are very rough and for many problems may be too inaccurate. In Theorems 5–6 and in Corollary 3, we provide both upper and lower bounds for $C_n(\nu)$ and C_n, respectively, which are much more accurate. It may also be useful in some cases to have a suitable approximation for $C_n(\nu)$. In Theorem 7, we give a simple approximation for $C_n(\nu)$, which can be useful for applications. Besides, it is well known that one and the same function may have asymptotic expansions involving different asymptotic sequences [43, Chapter 1]; see also [27, 36, 44, 45, 70]. Theorems 8 and 9 give, in this sense, alternative asymptotic expansions for $C_n(\nu)$, which in some situations, may be more desirable than the expansion obtained in Theorem 4, and which may be used as a very good approximation for $C_n(\nu)$ as well. Since Theorems 7–9 are closely related to each other, their proofs are given together.

Theorem 5 (Bounds and Inequalities for $C_n(\nu)$)**.** *For $n = 2, 3, 4\ldots$ and $1 < \nu < n$, the sums $C_n(\nu)$ are bounded from below and from above as follows:*

$$-\frac{2n}{\pi}\ln\Big(2\sin\frac{\pi\nu}{n}\Big)+A(n,\nu)<\sum_{l=1}^{n-1}\cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n}<-\frac{2n}{\pi}\ln\Big(2\sin\frac{\pi\nu}{n}\Big)+B(n,\nu),$$

where $A(n,\nu) \equiv -\dfrac{\pi}{12n}\csc^2\dfrac{\pi\nu}{n}$ and

$$B(n,\nu) \equiv -\frac{\pi}{12n}\csc^2\frac{\pi\nu}{n} + \frac{7\pi^3}{1440\,n^3}\left\{1 + 2\cos^2\frac{\pi\nu}{n}\right\}\csc^4\frac{\pi\nu}{n}\,.$$

Theorem 6 (Bounds and Inequalities for C_n)**.** *For $n = 2, 4, 6, \ldots,$ the values of the alternating finite sum $\sum(-1)^{l+1}\csc\big(\pi l/n\big)$ always lie in the interval*

$$\frac{2n\ln 2}{\pi} + \frac{\pi}{12\,n} - \frac{7\pi^3}{1440\,n^3} < \sum_{l=1}^{n-1}(-1)^{l+1}\csc\frac{\pi l}{n} < \frac{2n\ln 2}{\pi} + \frac{\pi}{12\,n}\,.$$

For $n = 3, 5, 7, \ldots,$ this finite sum vanishes.

Corollary 3 (Simple Bounds and Inequalities for C_n)**.** *The finite trigonometric sum $\sum(-1)^{l+1}\csc\big(\pi l/n\big)$ obeys the following bounds:*

$$\frac{2n\ln 2}{\pi} + \frac{0.252}{n} < \sum_{l=1}^{n-1}(-1)^{l+1}\csc\frac{\pi l}{n} < \frac{2n\ln 2}{\pi} + \frac{0.262}{n}\,, \qquad n = 4, 6, 8, \ldots\,,$$

which are slightly less accurate that those provided by Theorem 6.

Proof. Differentiating $m-1$ times [9, Volume I, Section 1.9, Equation (10)] with respect to z, we get

$$\Psi_m(x) \;=\; (-1)^{m+1}\, m! \sum_{l=0}^{\infty} \frac{1}{(x+l)^{m+1}} \;=\; (-1)^{m+1}\, m!\, \zeta(m+1,x)\,, \qquad m \in \mathbb{N}\,.$$

Hence, $\Psi_{2r-1}(x) > 0$ for $x > 0$. Therefore $\Psi_{2r-1}(x) + \Psi_{2r-1}(1-x) > 0$ for $0 < x < 1$. Considering now the definition of $\mathscr{H}_{2r-1}(\nu/n)$ given in Theorem 4, we see at once that

$$\mathscr{H}_{2r-1}\left(\frac{\nu}{n}\right) < 0\,,$$

whence

$$\operatorname{sgn}\left[\frac{\left(1-2^{1-2r}\right)\pi^{2r-1} B_{2r}}{(2r)!\, n^{2r-1}} \mathscr{H}_{2r-1}\left(\frac{\nu}{n}\right)\right] \;=\; (-1)^r\,.$$

Thus, the series on the right-hand side of (36) possesses the usual property concerning the magnitude and sign of the remainder. Setting $N=2$ and $N=3$ into (36) and accounting for the sign yields both inequalities stated in Theorem 5. By a similar line of reasoning, we deduce the result announced in Theorem 6. Finally, simpler bounds given in Corollary 3 are obtained as follows. We can replace the lower bound from Theorem 6 by a/n, with $a = O(1)$, if for some n_0,

$$\frac{a}{n} \;<\; \frac{\pi}{12\,n} \;-\; \frac{7\pi^3}{1440\,n^3}\,, \qquad n > n_0\,.$$

Solving this quadratic inequality for $n_0 = 4$, we obtain

$$a \;=\; \frac{\pi}{12} \;-\; \frac{7\pi^3}{23\,040} \;=\; 0.2523\ldots\,,$$

that gives the lower bound. Accounting for the numerical value of $\pi/12$, we obtain the upper bound. Note, lastly, that bounds obtained in Theorem 6 and in Corollary 3 are quite "sharp" in the sense that in both cases as $n \to \infty$ lower and upper bounds tend to the same value. □

Theorem 7 (A Simple Approximation for $C_n(\nu)$)**.** *A simple and relatively good approximation for the sum $C_n(\nu)$ is given by the following expression:*

$$\sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n} \cdot \csc\frac{\pi l}{n} \;\approx\; -\frac{2n}{\pi}\ln\!\left(2\sin\frac{\pi\nu}{n}\right) \;-\; \frac{\pi}{12n}\csc^2\frac{\pi\nu}{n} \;+\; \frac{7n}{480\pi\nu^4}\,.$$

The abve approximation is quite accurate, the right-hand side is asymptotically equivalent to $C_n(\nu)$ at large n, but one should bear in mind that the approximation error does not tend to zero as $n \to \infty$.

Theorem 8 (An Asymptotic Expansion and Accurate Approximation for $C_n(\nu)$)**.** *The sum $C_n(\nu)$, $1<\nu<n$, admits the asymptotic expansion*

$$\sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n} \;=\; -\frac{2n}{\pi}\ln\Bigl(2\sin\frac{\pi\nu}{n}\Bigr) \;-\; \frac{\pi}{12n}\csc^2\frac{\pi\nu}{n} \;+\; nf(\nu) + \frac{7\pi^3}{21\,600\,n^3} + o\bigl(n^{-3}\bigr)\,, \qquad n\to\infty\,,$$

where

$$f\,(\nu) = -\frac{1}{\pi}\left\{4H_{2\nu} - 2H_\nu - 2\ln\nu - 4\ln 2 - 2\gamma - \frac{1}{12\nu^2}\right\}.$$

The expansion provided by this theorem may also be used as a very accurate approximation for $C_n(\nu)$, whose error rapidly tends to zero as $n\to\infty$.

Theorem 9 (A Cosecant-Free Asymptotic Expansion of $C_n(\nu)$)**.** *The sum $C_n(\nu)$, $1<\nu<n$, admits the following cosecant-free asymptotic expansion*

$$\sum_{l=1}^{n-1} \cos\frac{2\pi\nu l}{n}\cdot\csc\frac{\pi l}{n} \;=\; -\frac{2n}{\pi}\ln\Bigl(2\sin\frac{\pi\nu}{n}\Bigr) \;+\; ng(\nu) \;-\; \frac{\pi}{36\,n} - \frac{\bigl(120\nu^2-7\bigr)\,\pi^3}{21\,600\,n^3} + o\bigl(n^{-3}\bigr)\,, \qquad n\to\infty\,,$$

where

$$g\,(\nu) = -\frac{1}{\pi}\Bigl\{4H_{2\nu} - 2H_\nu - 2\ln\nu - 4\ln 2 - 2\gamma\Bigr\}.$$

Similarly to the previous case, this expansion may also be used as an approximation for $C_n(\nu)$, whose error tends to zero as $n\to\infty$.

Corollary 4. *Let n be a multiple of 6. If $\nu=\frac{1}{6}n$ or $\nu=\frac{5}{6}n$, then as $n\to\infty$ the sum $C_n(\nu)$ tends to zero:*

$$\lim_{n\to\infty}\sum_{l=1}^{n-1}\cos\frac{\pi l}{3}\cdot\csc\frac{\pi l}{n} \;=\; 0\,.$$

Proof. From the inequalities established in the previous theorem, it follows that

$$C_n(\nu) \;=\; -\frac{2n}{\pi}\ln\Bigl(2\sin\frac{\pi\nu}{n}\Bigr) \;-\; \frac{\pi}{12n}\csc^2\frac{\pi\nu}{n} + \frac{7\lambda\pi^3}{1440\,n^3}\left\{1 \;+\; 2\cos^2\frac{\pi\nu}{n}\right\}\csc^4\frac{\pi\nu}{n}\,, \tag{46}$$

where $0<\lambda<1$. Expanding the last term into the power series in n, $n\to\infty$, we have

$$\frac{7\pi^3}{1440\,n^3}\left\{1 \;+\; 2\cos^2\frac{\pi\nu}{n}\right\}\csc^4\frac{\pi\nu}{n} \;=\; \frac{7n}{480\pi\nu^4} + \frac{7\pi^3}{21\,600\,n^3} + O\bigl(n^{-5}\bigr)\,,$$

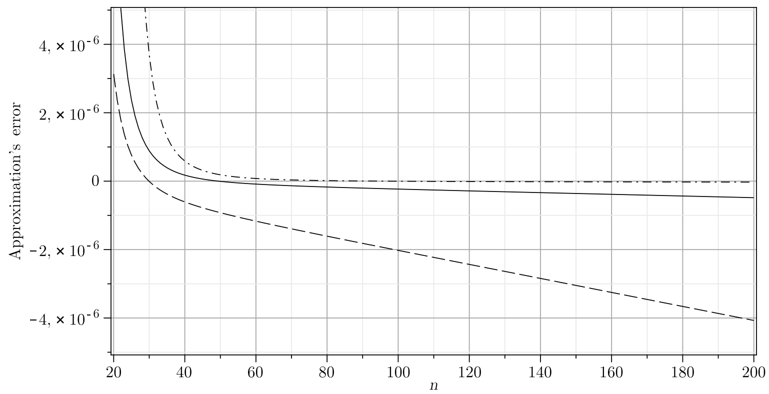

Figure 3: The difference between $C_n(\nu)$ and its approximation, provided by Theorem 7, as a function of n for $\nu = 7$ (dashed line), $\nu = 10$ (solid line) and $\nu = 16$ (dash-dotted line).

where the expression on the left is clearly positive. Substituting this result into (46), we obtain

$$C_n(\nu) \approx -\frac{2n}{\pi}\ln\Bigl(2\sin\frac{\pi\nu}{n}\Bigr) - \frac{\pi}{12n}\csc^2\frac{\pi\nu}{n} + nf(\nu)\,,$$

where $f(\nu)$ is a bounded function

$$0 < f(\nu) < \frac{7}{480\pi\nu^4}\,.$$

Proceeding analogously with the term corresponding to $r = 3$ in (36), whose asymptotics is

$$\sim -\frac{31}{4032\pi\nu^6}\,, \qquad n \to \infty\,,$$

we see that at sufficiently large n and fixed ν, the function $f(\nu)$ obeys the inequalities

$$\frac{7}{480\pi\nu^4} - \frac{31}{4032\pi\nu^6} < f(\nu) < \frac{7}{480\pi\nu^4}\,.$$

Numerically, such a correction from below (due to the term with $r = 3$) is almost negligible (for example, for $\nu \geqslant 8$ it is less than 1%). Hence, $f(\nu)$ is practically equal to its upper bound, whence we get the approximation stated in Theorem 7.

Obviously, we can also take into account the contribution of higher terms in the asymptotic expansion from Theorem 4. Remarking that for a sufficiently large n

$$\frac{d^{2r-1}}{d\varphi^{2r-1}}\operatorname{ctg}\varphi\bigg|_{\varphi=\frac{\pi\nu}{n}} = -\frac{n^{2r}(2r-1)!}{(\pi\nu)^{2r}} + K_r + O\bigl(n^{-2}\bigr)\,, \qquad r \in \mathbb{N}\,, \qquad (47)$$

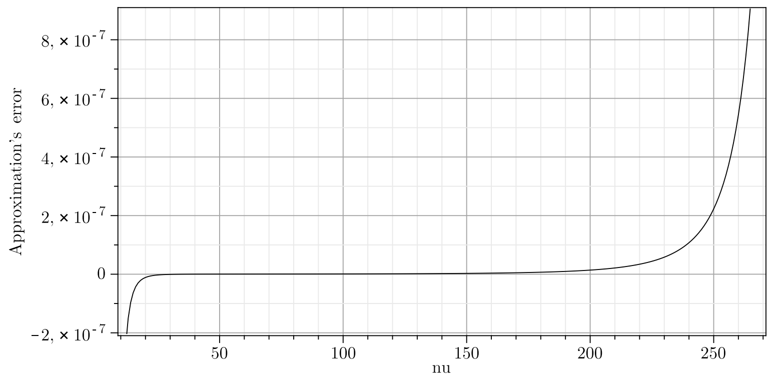

Figure 4: The approximation error for $C_{300}(\nu)$ as a function of ν, where $\nu \in [10, 270]$ (the approximation is given by Theorem 7). For the sake of comparison: $C_{300}(10) \approx +299$, $C_{300}(20) = C_{300}(280) \approx +168$, $C_{300}(50) = C_{300}(250) \approx -3 \times 10^{-3}$, $C_{300}(100) = C_{300}(200) \approx -105$, $C_{300}(150) \approx -132$ (see Figure 2 for the graph of $C_{300}(\nu)$).

where K_r is a constant term, depending neither on n nor on ν, we obtain

$$f(\nu) \;=\; -\frac{1}{\pi}\sum_{r=2}^{N-1}\frac{\left(1-2^{1-2r}\right)B_{2r}}{r\,\nu^{2r}} + \ldots\,. \tag{48}$$

Recalling the Stirling formula for the harmonic numbers

$$H_n \;= \ln n \,+\, \gamma \,+\, \frac{1}{2n} \,-\, \frac{1}{2}\sum_{r=1}^{N-1}\frac{B_{2r}}{r\,n^{2r}} \,-\, \frac{\varepsilon\, B_{2N}}{2Nn^{2N}}\,, \qquad 0<\varepsilon<1\,,$$

we see that (48) is the asymptotic expansion of the difference of two harmonic numbers with some additional terms, namely

$$\begin{aligned} f(\nu) \;&=\; -\frac{1}{\pi}\left\{\sum_{r=2}^{N-1}\frac{B_{2r}}{r\,\nu^{2r}} - 2\sum_{r=2}^{N-1}\frac{B_{2r}}{r\,(2\nu)^{2r}}\right\} + \ldots \\ &=\; -\frac{1}{\pi}\left\{4H_{2\nu} - 2H_{\nu} - 2\ln\nu - 4\ln 2 - 2\gamma - \frac{1}{12\nu^2}\right\} \approx\; \frac{7}{480\pi\nu^4}\,. \end{aligned}$$

Note that since $\frac{7}{480\pi\nu^4}$ is just an approximation for $f(\nu)$, the overall approximation error, given by Theorem 7, linearly grows with n, but with a very small slope, which roughly is inversely proportional to ν^6. In contrast, the approximation error of the alternative asymptotic expansion given by Theorem 8 does tend to 0 as $n \to \infty$. In fact, the latter asymptotic expansion does not contain terms $O(1)$, nor $O(n^{-1})$ nor even $O(n^{-2})$. As to the order n^{-3}, this term is obtained from the constant term

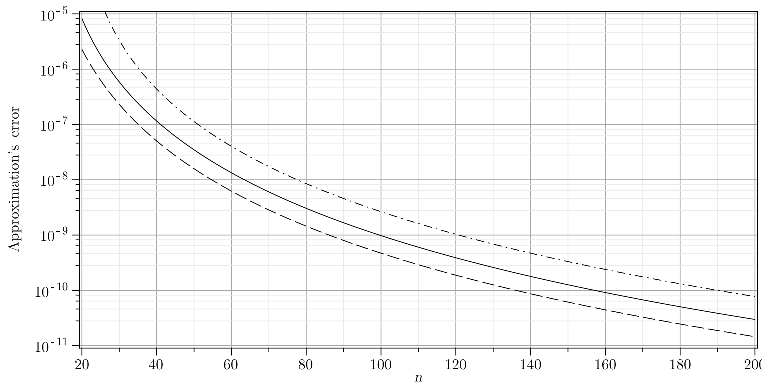

Figure 5: The difference between $C_n(\nu)$ and its approximation, provided by the asymptotics from Theorem 8, as a function of n for $\nu = 7$ (dashed line), $\nu = 10$ (solid line), and $\nu = 16$ (dash-dotted line).

in (47) $K_2 = -2/15$, and from the corresponding contribution of the sum on the right-hand side of (36). Note that this term does not depend on ν at all, and thus, may be regarded as a small bias if we study $C_n(\nu)$ at large fixed n. Furthermore, it can be reasonably expected that by virtue of (47), the remaining terms in the asymptotics of $C_n(\nu)$ should be $O(n^{-5})$, which can readily be verified empirically.

Finally, in some cases, the presence of the square of the cosecant in the asymptotic expansion of $C_n(\nu)$ may be undesirable. In such a situation, we may get rid of it by expanding the cosecant term into power series

$$\csc^2\frac{\pi\nu}{n} = \frac{n^2}{\pi^2\nu^2} + \frac{1}{3} + \frac{\pi^2\nu^2}{15\,n^2} + \frac{2\pi^4\nu^4}{189\,n^4} + O(n^{-6})\,, \qquad n \to \infty.$$

Inserting this expansion into Theorem 8 yields the cosecant-free expansion stated in Theorem 9. □

We conclude this Section with several graphs, showing how well the approximate formula, as well as the alternative asymptotics, represent the sum $C_n(\nu)$. The difference between $C_n(\nu)$ and its approximation, provided by Theorem 7, is shown in Figures 3 and 4. Figure 3 displays the approximation error as a function of n for three different values of the argument ν. Note that the greater the argument ν, the better the approximation (since the approximation error is proportional roughly to ν^{-6}). One should also bear in mind that $C_{100}(7) \approx 53$, $C_{100}(10) \approx 31$ and $C_{100}(16) \approx 2$, so that in all these cases the approximation is rather accurate (and it remains such at least for moderate values of n[18]). Thus, we see that the approximation error is very small, and hence this approximation can be used for many

[18]For example, for $n = 10\,000$ and $\nu = 7$ the approximation error is about -2×10^{-4}, while $C_{10\,000}(7) \approx 34\,541$.

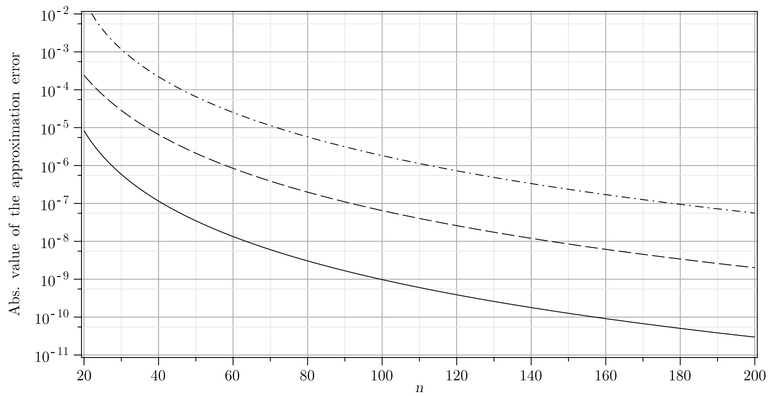

Figure 6: The absolute difference between $C_n(\nu)$ and its approximation, offered by the cosecant-free asymptotics from Theorem 9, as a function of n for $\nu = 7$ (dashed line), $\nu = 10$ (solid line), and $\nu = 16$ (dash-dotted line).

versatile purposes and applications. At the same time, as expected, we observe that it does not tend to zero as n increases; however, even in the worst case, that of $C_n(7)$, the approximation error is still very small even for large values of n (note that $\nu^6 = 7^6 = 117\,649$). Figure 4 shows the approximation error as a function of ν at fixed n. Figure 5 displays the error between $C_n(\nu)$ and its alternative asymptotics, offered by Theorem 8 under the same conditions as in Figure 3, i.e., for the same three arguments ν and in the same interval of n. We see that the error is extremely small and quickly tends to zero as n increases (compare this graph to Figure 3). In other words, the asymptotics from Theorem 8 may also be used as a very accurate approximation for $C_n(\nu)$, but the other side of the coin is the complexity of calculations, which may considerably grow in size or even become prohibitive. Lastly, as to the less complex approximation for $C_n(\nu)$, provided by the cosecant-free asymptotics from Theorem 9, we see, Figure 6, that it is less accurate than that provided by Theorem 8, but the error still tends, though not very quickly, to zero as $n \to \infty$. At the same time, it remains more accurate than the approximation provided by Theorem 7.

Finally, as to the result given in Corollary 4, it is a simple consequence of Theorem 8. Setting $\nu = \frac{1}{6}n$ or $\nu = \frac{5}{6}n$, we see that the leading term in the asymptotics, provided in Theorem 8, identically vanishes. The remaining terms are $o(1)$, whence we obtain the stated result. These two cases are the only ones in which the sum $C_n(\nu)$ converges as $n \to \infty$ (in our case, it converges trivially to zero, since there are no constant terms in the asymptotic expansion of $C_n(\nu)$).

Acknowledgments. We would like to warmly thank Yuri Matiyasevich, Ilia Ponomarenko, and Donal Connon for their comments, which helped to improve the paper.

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