EXTENDING RECENT PARITY RESULTS OF NYIRENDA AND MUGWANGWAVARI FOR PARTITIONS WITH INITIAL REPETITIONS

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Abstract

Recently, Nyirenda and Mugwangwavari considered several restricted partition functions which they viewed from the perspective of partitions with initial repetitions. Utilizing a number of results from Slater, along with classical generating function manipulations, they proved several Ramanujan–like congruences modulo 2 satisfied by these functions. Our goal in this note is to establish infinite families of Ramanujan-type congruences for these functions that have the aforementioned congruences as special cases.

1. Introduction

A partition of a positive integer n is a sum of the form $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. Each summand λ_i is called a part of the partition. The number of times that a particular summand occurs in the partition is known as the multiplicity of that part. We say that a part is repeated if its multiplicity is greater than one.

In 2009, Andrews [3] introduced the idea of partitions with initial repetitions. His main definition is as follows: A partition with initial k-repetitions is a partition in which if any j appears at least k times as a part, then each positive integer less than j appears at least k times as a part. We highlight two examples of what Andrews proved in his paper, the first of which relates to Glaisher's generalization

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of Euler's famous result on odd-part partitions and distinct-part partitions:

Theorem 1. *The number of partitions of n with initial k-repetitions equals the number of partitions of n into parts not divisible by $2k$ and also equals the number of partitions of n in which no part is repeated more than $2k-1$ times.*

Keith [4] soon followed with a bijective proof of Theorem 1. Andrews [3] also proved the following result which focuses on the parity of the parts in question.

Theorem 2. *Let $F_e(n)$ denote the number of partitions of n in which no odd parts are repeated, and if an even part $2j$ is repeated, then each even positive integer smaller than $2j$ appears in the partition as a repeated part. Finally, no odd integers smaller than $2j$ appear as a part. Then, for all n, $F_e(n)$ equals the number of partitions of n into parts $\equiv 0, \pm 2 \pmod 7$.*

Additional work of this type was also completed by Munagi and Nyirenda [5].

Motivated by Andrews' original work on partitions with initial repetitions, Nyirenda and Mugwangwavari [6] recently defined several functions which they viewed within the realm of partitions with initial repetitions. Relying heavily on the work of Slater [8] and utilizing classical techniques for manipulating generating functions, Nyirenda and Mugwangwavari proved several parity results for these partition functions and identified some examples of Ramanujan-like congruences that follow from them. In this paper, we build on their work by performing the elementary analysis necessary to establish infinite families of Ramanujan–like congruences modulo 2 satisfied by three of their functions. In the process, we will work to "quantify" the number of such congruences that exist.

2. Three Crucial Lemmas

As we mentioned above, part of our work in this note involves quantifying the number of Ramanujan–like congruences modulo 2 that will hold for each function in question. The following lemmas will play a key role in proving such results below.

Lemma 1. *Suppose p is an odd prime and m and n are integers such that $p \nmid mn(n-m)$. Then the number of r with $0 \leq r \leq p-1$ such that $mr+1$ and $nr+1$ are both quadratic nonresidues modulo p is*

$$\left\lfloor \frac{p}{4} \right\rfloor - \frac{1}{4}\left[\left(\frac{m}{p}\right) - \left(\frac{n-m}{p}\right)\right]\left[\left(\frac{n}{p}\right) - \left(\frac{m-n}{p}\right)\right],$$

where $\left(\frac{x}{p}\right)$ is the Legendre symbol of x modulo p.

Proof. For convenience, let $\phi(x)$ represent the Legendre symbol $\left(\frac{x}{p}\right)$. Note that $\phi(0) = 0, \phi(xy) = \phi(x)\phi(y)$, and for $x \not\equiv 0 \pmod p$ we have $\phi(x^2) = 1$ and $\phi(x^{-1}) = \phi(x)$. Let M be the number of r with $0 \le r \le p-1$ for which $mr+1$ is a quadratic residue and $nr+1$ is a nonresidue and let N be the number of r where both $mr+1$ and $nr+1$ are nonresidues. Note that $M+N$ will be $\frac{p-1}{2}$ unless $mr+1 \equiv 0 \pmod p$ for some r with $nr+1$ a nonresidue, in which case $M+N = \frac{p-1}{2} - 1$. Equivalently,

$$M + N = \frac{p-1}{2} - \frac{1}{2}(1 - \phi(-nm^{-1} + 1)) = \frac{1}{2}(p - 2 + \phi(m)\phi(m-n)).$$

Note also that

$$M - N = \sum_{ni+1 \text{ is a nonresidue}} \phi(mi+1).$$

Fix a nonresidue a. Then $ni+1$ is a nonresidue if and only if $ni+1 \equiv ax^2 \pmod p$ for some x with $1 \le x \le \frac{p-1}{2}$. Moreover, $mi + 1 = \frac{1}{n}[m(ni+1) + (n-m)]$. It follows that

$$\begin{aligned}
M - N &= \sum_{x=1}^{(p-1)/2} \phi\left(\frac{1}{n}(max^2 + n - m)\right) \\
&= \frac{1}{2}\phi(n)\sum_{x=1}^{p-1} \phi(max^2 + n - m) \\
&= -\frac{1}{2}\phi(n)\phi(n-m) + \frac{1}{2}\phi(n)\sum_{x=0}^{p-1}\phi(max^2 + n - m).
\end{aligned}$$

Now $\sum_{x=0}^{p-1} \phi(ux^2 + v) = \sum_{x=0}^{p-1} \phi(ux+v)(1+\phi(x))$, and $\sum_{x=0}^{p-1}\phi(ux+v) = 0$ for $u \not\equiv 0 \pmod p$. Using these and changing variables, we find

$$\begin{aligned}
M - N &= -\frac{1}{2}\phi(n)\phi(n-m) + \frac{1}{2}\phi(n)\sum_{x=0}^{p-1}\phi(max + n - m)\phi(x) \\
&= -\frac{1}{2}\phi(n)\phi(n-m) + \frac{1}{2}\phi(n)\sum_{x=0}^{p-1}\phi(max + 1)\phi(x) \\
&= -\frac{1}{2}\phi(n)\phi(n-m) + \frac{1}{2}\phi(n)\phi(ma)\sum_{x=0}^{p-1}\phi(x+1)\phi(x).
\end{aligned}$$

Since a is a quadratic nonresidue modulo p, $\phi(a) = -1$. Also,

$$\sum_{x=0}^{p-1}\phi(x+1)\phi(x) = \sum_{x=1}^{p-1}\phi(x+1)\phi(x^{-1}) = \sum_{x=1}^{p-1}\phi(1+x^{-1}) = \sum_{x=1}^{p-1}\phi(1+x),$$

and $\sum_{x=1}^{p-1}\phi(1+x)=\sum_{x=0}^{p-1}\phi(1+x)-1=-1$. Then

$$M-N=-\frac{1}{2}\phi(n)\phi(n-m)+\frac{1}{2}\phi(n)\phi(m),$$

and subtracting this from our expression for $M+N$ gives

$$2N=\frac{1}{2}(p-2+\phi(m)\phi(m-n))+\frac{1}{2}\phi(n)\phi(n-m)-\frac{1}{2}\phi(n)\phi(m).$$

Finally, using the facts that $\left\lfloor\frac{p}{4}\right\rfloor=\frac{p-2+\phi(-1)}{4}$ and

$$\begin{aligned}&-\phi(-1)+\phi(m)\phi(m-n)+\phi(n)\phi(n-m)-\phi(n)\phi(m)\\&=-\left[\left(\frac{m}{p}\right)-\left(\frac{n-m}{p}\right)\right]\left[\left(\frac{n}{p}\right)-\left(\frac{m-n}{p}\right)\right],\end{aligned}$$

the result follows. □

In Corollary 1 and Corollary 3 below we provide congruences modulo 2 for arithmetic progressions of the form 2^kn+r. The proofs of these corollaries require knowledge about squares modulo powers of 2. We provide the needed information in the following lemmas.

Lemma 2. *For each $k\geq 1$, there are exactly two values which are squares modulo 2^{2k-1} but not squares modulo 2^{2k}, namely 2^{2k-1} and $2^{2k-1}+2^{2k-2}$.*

Proof. We prove our assertion by induction on k. For $k=1$, we note that $2=2^1$ and $3=2^1+2^0$ are both squares modulo 2 but not squares modulo $4=2^2$. Hence, the base case holds. Now let $k>1$ and assume the result holds for $k-1$. Let r be a square modulo 2^{2k-1} but not a square modulo 2^{2k}. Then $r\equiv s^2 \pmod{2^{2k-1}}$ and $r\equiv s^2+2^{2k-1} \pmod{2^{2k}}$ for some integer s. If s is odd, then

$$(s+2^{2k-2})^2=s^2+2^{2k-1}s+2^{4k-4}\equiv s^2+2^{2k-1}\equiv r \pmod{2^{2k}},$$

violating the condition that r not be a square modulo 2^{2k}. Thus, s is even, say $s=2t$ for some integer t. If $s^2+2^{2k-1}=4(t^2+2^{2k-3})$ is not a square modulo 2^{2k}, then t^2+2^{2k-3} is not a square modulo 2^{2k-2}. It follows from the inductive hypothesis that $t^2\equiv 0 \pmod{2^{2k-2}}$ or $t^2\equiv 2^{2k-4} \pmod{2^{2k-2}}$. Thus, $s^2\equiv 0 \pmod{2^{2k}}$ or $s^2\equiv 2^{2k-2} \pmod{2^{2k}}$, completing the induction. □

Lemma 3. *For each $k\geq 1$, the number $2^{2k}+2^{2k-2}$ is the unique value which is a square modulo 2^{2k} but not a square modulo 2^{2k+1}.*

The proof of Lemma 3 is nearly identical to the proof of Lemma 2 and is omitted here.

Remark 1. For any $n \geq 1$, the number of squares modulo 2^n is given in [7, A023105]. From this, we can deduce that the number of non–squares modulo 2^n satisfies the recurrence

$$a_n = 2a_{n-1} + \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

3. Main Results

The first function from [6] that we consider is defined as follows:

Let $c_2(n)$ be the number of partitions of n in which there exists $j \geq 1$ such that j appears exactly j times and it is the only part less than $2j+1$, even parts $\geq 2j+2$ are distinct, odd parts $\geq 2j+1$ appear unrestricted.

Theorem 3 ([6], Theorem 2)**.** *For all $n \geq 0$, $c_2(5n+2) \equiv 0 \pmod 2$.*

We now prove our main result on $c_2(n)$, which includes Theorem 3 as a special case.

Theorem 4. *Let $p \geq 5$ be prime. Then for any $1 \leq r \leq p-1$ such that neither $16r+1$ nor $8r+1$ is a quadratic residue modulo p we have*

$$c_2(pn+r) \equiv 0 \pmod 2$$

for all $n \geq 0$.

Proof. We note that Nyirenda and Mugwangwavari prove that

$$\sum_{n=0}^{\infty} c_2(n)q^n \equiv \sum_{a=-\infty}^{\infty} q^{4a^2+a} + \sum_{b=0}^{\infty} q^{b(b+1)/2} \pmod 2.$$

Thus, if n cannot be represented as $4a^2+a$ and n also cannot be represented as $b(b+1)/2$, then we immediately know that $c_2(n) \equiv 0 \pmod 2$. Via completing the square, this means we want to know whether $16n+1$ is representable as $(8a+1)^2$ and whether $8n+1$ is representable as $(2b+1)^2$. It follows that $c_2(n) \equiv 0 \pmod 2$ if neither $16n+1$ nor $8n+1$ is a quadratic residue modulo p. $\square$

We next determine the number of Ramanujan–like congruences that Theorem 4 provides modulo a prime $p \geq 5$.

Theorem 5. *Let $p \geq 5$ be prime and define $f_2(p)$ to be the number of values of r, $1 \leq r \leq p-1$, such that neither $16r+1$ nor $8r+1$ is a quadratic residue modulo p. Then $f_2(p) = \lfloor \frac{p}{4} \rfloor$.*

Proof. Thanks to Lemma 1 with $m = 8, n = 16$, we have

$$f_2(p) = \left\lfloor \frac{p}{4} \right\rfloor - \frac{1}{4}\left[\left(\frac{8}{p}\right) - \left(\frac{8}{p}\right)\right]\left[\left(\frac{16}{p}\right) - \left(\frac{-8}{p}\right)\right] = \left\lfloor \frac{p}{4} \right\rfloor.$$

□

Next, let $c_3(n)$ be the number of partitions of n in which there is a positive integer j such that 1 appears with multiplicity j^2 or j^2+1, odd parts larger than 1 are distinct, all even parts are distinct and those greater than $2j$ are at least $4j+4$ in size and divisible by 4.

Theorem 6 ([6], Theorem 4)**.** *For all $n \geq 0$,*

$$\begin{aligned} c_3(11n+5) &\equiv 0 \pmod 2, \\ c_3(11n+7) &\equiv 0 \pmod 2, \quad \textit{and} \\ c_3(11n+9) &\equiv 0 \pmod 2. \end{aligned}$$

We now prove our main result on $c_3(n)$, which includes Theorem 6 as a special case.

Theorem 7. *Let $p \geq 5$ be prime. Then for any $1 \leq r \leq p-1$ such that neither $12r+1$ nor $8r+1$ is a quadratic residue modulo p we have*

$$c_3(pn+r) \equiv 0 \pmod 2$$

for all $n \geq 0$.

Proof. In [6] the authors prove that

$$\sum_{n=0}^{\infty} c_3(n)q^n \equiv \sum_{a=-\infty}^{\infty} q^{a(3a+1)} + \sum_{b=0}^{\infty} q^{b(b+1)/2} \pmod 2.$$

Thus, if n cannot be represented as $a(3a+1)$ and n also cannot be represented as $b(b+1)/2$, then $c_3(n) \equiv 0 \pmod 2$. Via completing the square, this means we want to know whether $12n+1$ is representable as $(6a+1)^2$ and whether $8n+1$ is representable as $(2b+1)^2$. It follows that $c_3(n) \equiv 0 \pmod 2$ if neither $12r+1$ nor $8r+1$ is a quadratic residue modulo p. □

Theorem 8. *Let $p \geq 5$ be prime and define $f_3(p)$ to be the number of values of r, $1 \leq r \leq p-1$, such that neither $12r+1$ nor $8r+1$ is a quadratic residue modulo p. Then*

$$f_3(p) = \begin{cases} \lfloor \frac{p}{4} \rfloor - 1 & \textit{if } \ p \equiv 5 \pmod{24} \\ \lfloor \frac{p}{4} \rfloor + 1 & \textit{if } \ p \equiv 11 \pmod{24} \\ \lfloor \frac{p}{4} \rfloor & \textit{otherwise} \end{cases}.$$

Proof. Thanks to Lemma 1 with $m = 8, n = 12$, we have

$$\begin{aligned} f_3(p) &= \left\lfloor \frac{p}{4} \right\rfloor - \frac{1}{4}\left[\left(\frac{8}{p}\right) - \left(\frac{4}{p}\right)\right]\left[\left(\frac{12}{p}\right) - \left(\frac{-4}{p}\right)\right] \\ &= \left\lfloor \frac{p}{4} \right\rfloor - \frac{1}{4}\left[\left(\frac{2}{p}\right) - 1\right]\left[\left(\frac{3}{p}\right) - \left(\frac{-1}{p}\right)\right]. \end{aligned}$$

Checking cases modulo 24, this count agrees with the count in Theorem 8. □

Remark 2. Note that, in the case of Theorem 4 in [6], Nyirenda and Mugwangwavari provided $\lfloor \frac{11-1}{4} \rfloor + 1 = 2 + 1 = 3$ such congruences, and one can verify that the case $p = 11$ of Theorem 7 yields precisely their three congruences. We also highlight that $f_3(5) = 0$.

Finally, let $c_8(n)$ be the number of partitions of n in which either

(a) all parts are even and distinct, or

(b) the largest odd part $2j-1$ appears once, all positive odd integers $\leq j$ appear once or twice, all positive odd integers $> j$ appear once, all even parts are distinct, and those even parts $> j$ must be $\geq 2j+2$ in size.

Theorem 9 ([6], Theorem 9)**.** *For all $n \geq 0$, if $r \in \{6, 20, 27, 34, 41, 48\}$, then*

$$c_8(49n + r) \equiv 0 \pmod 2.$$

Nyirenda and Mugwangwavari prove Theorem 9 after showing that

$$\sum_{n=0}^{\infty} c_8(n) q^n \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} \sum_{n=-\infty}^{\infty} q^{n(5n+1)/2} \pmod 2.$$

We now show that the generating function for $c_8(n)$ is also congruent modulo 2 to a single theta series, which will allow us to identify many more arithmetic progressions $An + B$ on which c_8 will always be even.

We begin by recalling the q–Pochhammer notation: For $n \geq 1$ and $|q| < 1$, we define

$$\begin{aligned} (A;q)_0 &:= 1, \\ (A;q)_n &:= (1-A)(1-Aq)(1-Aq^2)\dots(1-Aq^{n-1}), \quad \text{and} \\ (A;q)_\infty &:= \lim_{n\to\infty} (A;q)_n. \end{aligned}$$

We now state a theorem of Andrews [2] which is key to our analysis of $c_8(n)$ modulo 2.

Theorem 10 ([2], (6.22))**.** *We have*

$$(q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j} =$$

$$\sum_{\lambda=-\infty}^{\infty} \left(q^{60\lambda^2+4\lambda} - q^{60\lambda^2+44\lambda+8} + q^{60\lambda^2+16\lambda+1} - q^{60\lambda^2+64\lambda+17} \right). \qquad (1)$$

Some remarks are in order before we move forward. First, note that the sum on the left–hand side of the equation above appears in the celebrated first Rogers–Ramanujan identity [1, p. 104]. Second, note that the right–hand side of the equation above provides a lacunary 2–dissection of the expression in question.

This now leads us to the following theorem with which we can extend the original parity results of Nyirenda and Mugwangwavari for $c_8(n)$.

Theorem 11. *Let N be an integer such that $15N + 1$ is not a square. Then*

$$c_8(N) \equiv 0 \pmod{2}.$$

Proof. From the work of Nyirenda and Mugwangwavari, we know

$$\sum_{n=0}^{\infty} c_8(n) q^n \equiv (q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} \pmod{2}.$$

Moreover, it is straightforward to prove that the exponents of q in the sum on the right–hand side of (1) in Theorem 10 are exactly the integers of the form $(n^2-1)/15$. (See [7, A204221] and [7, A204542] for more information.) We then see that

$$\sum_{n=0}^{\infty} c_8(n) q^n \equiv \sum_{n=0}^{\infty}{}' q^{(n^2-1)/15} \pmod{2}$$

where the sum on the right–hand side above is taken only over those integers n wherein $(n^2 - 1)/15$ is an integer. Thus,

$$\sum_{n=0}^{\infty} c_8(n) q^{15n+1} \equiv \sum_{n=0}^{\infty} q^{n^2} \pmod{2}.$$

The result follows. □

Remark 3. For each prime $p > 5$ there are $p - 1$ values of r such that $0 \le r < p^2$ and $15r+1$ is divisible by p but not by p^2. By Theorem 11, for every $n \ge 0$ and each such r, $c_8(p^2 n + r) \equiv 0 \pmod{2}$. When $p = 7$, the corresponding values of r are exactly those given in Theorem 9. Thus, Theorem 11 is a significant generalization of Theorem 9.

Corollary 1. *For all $k \ge 1$ and all $n \ge 0$,*

$$c_8(2^{2k} n + r_{k,1}) \equiv 0 \pmod{2},$$

$$\begin{aligned} c_8(2^{2k}n + r_{k,2}) &\equiv 0 \pmod{2}, \quad \textit{and} \\ c_8(2^{2k+1}n + r_{k,3}) &\equiv 0 \pmod{2} \end{aligned}$$

where $r_{k,1}$*,* $r_{k,2}$*, and* $r_{k,3}$ *are the least nonnegative integers which satisfy*

$$\begin{aligned} 15_{2^k}^{-1}(2^{2k-1} - 1) &\equiv r_{k,1} \pmod{2^{2k}}, \\ 15_{2^k}^{-1}(2^{2k-1} + 2^{2k-2} - 1) &\equiv r_{k,2} \pmod{2^{2k}}, \quad \textit{and} \\ 15_{2k+1}^{-1}(2^{2k} + 2^{2k-2} - 1) &\equiv r_{k,3} \pmod{2^{2k+1}}, \end{aligned}$$

respectively, and 15_A^{-1} *is the inverse of 15 modulo* 2^A*.*

Proof. If $N = 2^{2k}n + r_{k,1}$ then $15N + 1 \equiv 15 \cdot 2^{2k}n + 2^{2k-1} \equiv 2^{2k-1} \pmod{2^{2k}}$. By Lemma 2, $15N + 1$ is not a square modulo 2^{2k}, proving the first congruence. The other congruences follow by similar arguments, the last making use of Lemma 3. □

Before moving to our next corollary, we pause to share an example to put Corollary 1 in context. When $k = 2$, Corollary 1 tells us that, for all $n \geq 0$,

$$\begin{aligned} c_8(16n + 9) &\equiv 0 \pmod{2}, \\ c_8(16n + 5) &\equiv 0 \pmod{2}, \quad \text{and} \\ c_8(32n + 29) &\equiv 0 \pmod{2}. \end{aligned}$$

Corollary 2. *Let* $p \geq 7$ *be prime. Then for any* $1 \leq r \leq p - 1$ *such that* $15r + 1$ *is not a quadratic residue modulo* p*, we have* $c_8(pn + r) \equiv 0 \pmod{2}$ *for all* $n \geq 0$*.*

4. Closing Thoughts

As we close this work, we note that Andrews [2, (6.23)] provides a companion result to Theorem 10.

Theorem 12 ([2], (6.23))**.** *We have*

$$\begin{aligned} &(q^2; q^2)_\infty \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q;q)_j} = \\ &\qquad \sum_{\lambda=-\infty}^{\infty} \left(q^{60\lambda^2+8\lambda} - q^{60\lambda^2+32\lambda+4} + q^{60\lambda^2+28\lambda+3} - q^{60\lambda^2+52\lambda+11} \right). \end{aligned}$$

Note that the sum in the left–hand side above is the series that appears in the analytic statement of the second Rogers–Ramanujan identity. Moreover, in a fashion similar to that above, it is straightforward to prove that the values which appear in the exponents on the right–hand side of the equation above are exactly the integers for which the quantity $(n^2-4)/15$ are integers.

Motivated by our work above related to $c_8(n)$, as well as the existence of Andrews' Theorem 12, we now define the following function $c_8'(n)$ which is a somewhat natural companion to $c_8(n)$.

Let $c_8'(n)$ be the number of partitions of n in which either

(a) all parts are even and distinct, with no distinguished part present, or

(b) there is a distinguished even part $2n$ with the property that $2n$ and all smaller even parts must appear in the partition, every even part up to n might appear a second time (and no more), all odd parts of size $\leq n$ must be distinct (if they appear at all), and every part greater than n must be even and distinct.

It is important to note that the "distinguished part" in the definition of $c_8(n)$ is simply the largest odd part in the partition (if it exists). So there really is no need to add any distinguishing mark in order to avoid ambiguity. In contrast, we must provide for the possibility of distinguishing a particular even part in the context of $c_8'(n)$ in order to avoid confusion as to which even part is playing the role of the special part in question. Consider the partition $\pi = 10+8+6+4+2$. We could avoid distinguishing a particular part, which would mean that we simply have a partition into distinct even parts, as in part (a) of the definition above. Or, for example, we could have $\pi^* = 10+8+6^*+4+2$, where we are considering the case $n=3$ in part (b) of the definition above. These "modified" partitions are counted separately by $c_8'(n)$.

In parallel to the generating function that was supplied by Nyirenda and Mugwangwavari [6] for $c_8(n)$, we then see that the generating function for $c_8'(n)$ can be written as

$$\sum_{n=0}^{\infty} c_8'(n)q^n = (-q^2;q^2)_\infty + \sum_{n\geq 1} q^{2+4+\cdots+2n}(1+q)(1+q^2)\cdots(1+q^n)(-q^{2n+2};q^2)_\infty.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} c_8'(n)q^n &= \sum_{n\geq 0} q^{n(n+1)}(-q;q)_n(-q^{2n+2};q^2)_\infty \\ &\equiv \sum_{n\geq 0} q^{n^2+n}\frac{(q^2;q^2)_n}{(q;q)_n}(q^{2n+2};q^2)_\infty \pmod 2 \\ &= (q^2;q^2)_\infty \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} \\ &= \sum_{\lambda=-\infty}^{\infty}\left(q^{60\lambda^2+8\lambda} - q^{60\lambda^2+32\lambda+4} + q^{60\lambda^2+28\lambda+3} - q^{60\lambda^2+52\lambda+11}\right)\end{aligned}$$

thanks to Theorem 12. With Theorem 11 in mind, as well as the remarks shared after the statement of Theorem 12, we can now write the following parity result.

Theorem 13. *Let N be an integer such that $15N+4$ is not a square. Then*

$$c_8'(N) \equiv 0 \pmod{2}.$$

It is clear that the following corollaries hold (in parallel to Corollaries 1 and 2).

Corollary 3. *For all $k \geq 1$ and all $n \geq 0$,*

$$\begin{aligned} c_8'(2^{2k}n + r_{k,1}') &\equiv 0 \pmod{2}, \\ c_8'(2^{2k}n + r_{k,2}') &\equiv 0 \pmod{2}, \quad \textit{and} \\ c_8'(2^{2k+1}n + r_{k,3}') &\equiv 0 \pmod{2} \end{aligned}$$

where $r_{k,1}'$, $r_{k,2}'$, and $r_{k,3}'$ are the least nonnegative integers which satisfy

$$\begin{aligned} 15_{2k}^{-1}(2^{2k-1} - 4) &\equiv r_{k,1}' \pmod{2^{2k}}, \\ 15_{2k}^{-1}(2^{2k-1} + 2^{2k-2} - 4) &\equiv r_{k,2}' \pmod{2^{2k}}, \quad \textit{and} \\ 15_{2k+1}^{-1}(2^{2k} + 2^{2k-2} - 4) &\equiv r_{k,3}' \pmod{2^{2k+1}}, \end{aligned}$$

respectively, and 15_A^{-1} is the inverse of 15 modulo 2^A.

Proof. For the second congruence, if $N = 2^{2k}n + r_{k,2}'$ then

$$15N + 4 \equiv 15 \cdot 2^{2k}n + 2^{2k-1} + 2^{2k-2} \equiv 2^{2k-1} + 2^{2k-2} \pmod{2^{2k}}.$$

By Lemma 2, $15N+4$ is not a square modulo 2^{2k}, proving the congruence. The first and third congruences follow for similar reasons, the third making use of Lemma 3. □

Similar computations to those provided after the proof of Corollary 1 allow us to see that, for $k=2$ and all $n \geq 0$, we have

$$\begin{aligned} c_8'(16n+12) &\equiv 0 \pmod{2}, \\ c_8'(16n+8) &\equiv 0 \pmod{2}, \quad \text{and} \\ c_8'(32n+16) &\equiv 0 \pmod{2}. \end{aligned}$$

We close with an obvious companion result to Corollary 2.

Corollary 4. *Let $p \geq 7$ be prime, and let r, $1 \leq r \leq p-1$, be such that $15r+4$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$, $c_8'(pn+r) \equiv 0 \pmod{2}$.*

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