SOME PERPENDICULARITIES OF ARITHMETICAL FUNCTIONS

Pentti Haukkanen
Faculty of Information Technology and Communication Sciences, Tampere University, Finland
pentti.haukkanen@tuni.fi

Timo Tossavainen
Department of Health, Education and Technology, Lulea University of Technology, Sweden
timo.tossavainen@ltu.se

Received: 11/25/24, Accepted: 3/21/25, Published: 4/25/25

Abstract

We introduce the notion of perpendicularity of arithmetical functions and discuss a few concrete examples of perpendicularities. Further, we show that the set of arithmetical functions f with $f(1) = 1$ forms a real vector space with the Dirichlet convolution as addition and real power under the Dirichlet convolution as scalar multiplication. Moreover, we prove that multiplicative functions are perpendicular to antimultiplicative functions with respect to the natural perpendicularity.

1. Introduction

The following quotation from Graham, Knuth, and Patashnik [5] has already become famous: "Hear us, O Mathematicians of the World! Let us not wait any longer! We can make many formulas clearer by adopting a new notation now! Let us agree to write '$m \perp n$', and to say "m is prime to n," if m and n are relatively prime." This announcement was motivated by noting that "Like perpendicular lines don't have a common direction, perpendicular numbers don't have common factors" [5, p. 115].

Relating perpendicularity of lines to the property of numbers having no common factors was not only notationally useful but also prophetical. Indeed, let M be a module over a ring R. A *perpendicularity* in M is a binary relation $\perp$ satisfying for all $x, y, y_1, y_2 \in M$, and $\gamma \in R$,

(A1) $x \neq 0 \implies x \not\perp x$;

(A2) $x \perp y \implies y \perp x$;

DOI: 10.5281/zenodo.15283662

(A3) $x \perp z$ for some $z \in M$;

(A4) $x \perp y_1, y_2 \implies x \perp (y_1 + y_2)$;

(A5) $x \perp y \implies x \perp \gamma y$.

The axioms A1–A5 define perpendicularity also in a vector space, given that the scalars γ now establish a field. Further, because an Abelian group is a $\mathbb{Z}$-module, the above definition makes sense in this context, too. Yet the fifth axiom reads simply

$$x \perp y \implies x \perp -y;$$

cf. [6, 10].

Now, the above-mentioned interpretation of perpendicularity of numbers can be refined. Graham, Knuth, and Patashnik seem to have meant that, for positive integers, there is a binary relation $\perp$ defined via

$$a \perp b \iff \forall p : (\nu_p(a) = 0 \vee \nu_p(b) = 0). \tag{1}$$

Here $\nu_p(a)$ is the exponent of p in the canonical factorization of a. In monoid $(\mathbb{Z}_+, \cdot)$, this relation satisfies the axioms A1–A4. This observation can be expressed by saying that $\perp$ is a *pre-perpendicularity* in $(\mathbb{Z}_+, \cdot)$.

If we replace $\mathbb{Z}_+$ with the set of positive rational numbers, then $\perp$ in (1) satisfies A1–A5 in this Abelian group [6, Example 10]. Similarly, given a vector space V and an inner product $\langle \cdot, \cdot \rangle$ in V, the above axiom system is satisfied in V by the binary relation $\perp$ defined via

$$x \perp y \iff \langle x, y \rangle = 0.$$

In this paper, we introduce another kind of perpendicularity arising from number theory. We show that there are perpendicularities in the sets of arithmetical functions, i.e., real-valued functions of positive integers. We introduce some examples of them and focus especially on the one that can be called a natural perpendicularity. In this case, we show that multiplicative functions are perpendicular to antimultiplicative functions.

Let us first consider the usual pointwise addition of arithmetical functions $(f + g)(n) = f(n) + g(n)$ and the usual pointwise scalar multiplication $(af)(n) = a(f(n))$. If we equip the set of all arithmetical functions with the above operations, we end up with an infinite dimensional vector space which is isomorphic to $\mathbb{R}^\infty$. Now, an example of perpendicularities in this vector space can be constructed via

$$f \perp g \iff \forall n \in \mathbb{Z}_+ : (f(n) \neq 0 \implies g(n) = 0).$$

Another and more concrete example can be derived from (1). Namely, for every positive rational a, there is an arithmetical function

$$f_a(n) = \nu_{p_n}(a),$$

where p_n is the n-th prime. Then

$$f_a \perp f_b \iff \forall n : (f_a(n) = 0 \vee f_b(n) = 0)$$

is a perpendicularity in the Abelian group consisting of functions of this kind with the binary operation

$$f_a(n) + f_b(n) = f_{ab}(n).$$

There is another and, perhaps, a more novel approach to construct perpendicularities of arithmetical functions. If we exclude those arithmetical functions f for which $f(1) \neq 1$, then we can build on the Dirichlet convolution as addition and real power under the Dirichlet convolution as scalar multiplication. We proceed via two steps: we first study this set of arithmetical functions as an additive Abelian group (Section 3) and, thereafter, we turn it into a vector space by inserting the above scalar multiplication (Section 4).

To contextualize the present study, we summarize shortly some axiomatic studies of perpendicularity in algebraic structures. In the 1970s, Davis [3] investigated orthogonal relations on Abelian groups. His approach aimed at studying a concept of disjointness in the context of groups. Four decades later, Haukkanen, Mattila, Merikoski, and Tossavainen [6] introduced the above axiomatization of perpendicularity on Abelian groups and explored, e.g., the maximality of various perpendicularities. In more recent articles, they have investigated, among other things, the cardinality of the set of maximal perpendicularities in certain groups and, on the other hand, conditions on the existence of a unique maximal perpendicularity. Another question they have touched upon is whether all maximal perpendicularities arise from an inner product or not, and how this question is related to the dimension of an Abelian group or a vector space. For more information about recent results, see [9, 10].

2. Multiplicative and Antimultiplicative Functions

An arithmetical function f is said to be *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. Multiplicative functions is perhaps the most important subclass of arithmetical function. For example, Euler's totient function ϕ and divisor functions σ_m are multiplicative.

The *Dirichlet convolution* of two arithmetical functions f and g is defined as

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

The function δ, defined as $\delta(1) = 1$ and $\delta(n) = 0$ otherwise, serves as the identity under the Dirichlet convolution. An arithmetical function f possesses a Dirichlet inverse $f^{(-1)}$ if and only if $f(1) \neq 0$. A function of this kind is referred to as a *unit* (under the Dirichlet convolution). The Dirichlet inverse of a unit is unique and is given recursively as

$$f^{(-1)}(1) = \frac{1}{f(1)},\ f^{(-1)}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d>1}} f(d) f^{(-1)}(n/d)\ (n > 1).$$

An arithmetical function f is said to be *completely multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers m and n. The Dirichlet inverse of a completely multiplicative is given as $f^{(-1)} = \mu f$, where μ is the Möbius function (defined as the Dirichlet inverse of the constant function 1). The Möbius function μ is the multiplicative function such that $\mu(p) = -1$ and $\mu(p^k) = 0$ for all primes p and integers $k \geq 2$. The *power function* N_m defined as $N_m(n) = n^m$ is completely multiplicative. In particular, we let N denote the function N_1. Then $N_m^{(-1)} = \mu N_m$ and $N^{(-1)} = \mu N$. For further material on multiplicative functions we refer to [1, 8, 14].

Antimultiplicative functions have not been studied much in the literature [4]. They, however, seem to play a substantial role in the linear algebra of arithmetical functions. An arithmetical function f is said to be *antimultiplicative* if $f(1) = 1$ and $f(p^k) = 0$ whenever p^k is a prime power with $k > 0$. For example, the function $f(n) = |\omega(n) - 1|$ is antimultiplicative, where $\omega(n)$ is the number of distinct prime divisors of n with $\omega(1) = 0$. The identity function δ is a trivial example of an antimultiplicative function. Further examples are given in Examples 5 and 6 below.

3. Perpendicularity in the Abelian Group $(U_1, \star)$

Let A denote the set of all arithmetical functions, and let U denote the set of units in A under the Dirichlet convolution, that is, the set of arithmetical functions f with $f(1) \neq 0$. Then U is an Abelian group under the Dirichlet convolution. The set of units f with $f(1) = 1$ is denoted by U_1, and the sets of multiplicative and antimultiplicative functions are denoted by U_M and U_A, respectively.

In this section, we introduce some perpendicularities of arithmetical functions in the Abelian group $(U_1, \star)$. To that end, we first record a proposition whose proof can be found in [4].

Proposition 1. *We have*

$$U_M \leq U_1, \quad U_A \leq U_1, \quad U_1 \leq U,$$

where $\leq$ means the subgroup relation. In addition,

$$U_M \cap U_A = \{\delta\}.$$

Proposition 2. *The Abelian group $(U_1, \star)$ can be written as a direct product (or, equivalently, a direct sum)*

$$U_1 = U_M \oplus U_A.$$

Proof. It is known [4] that each $f \in U_1$ can be written uniquely as $f = g \star h$, where $g \in U_M$ and $h \in U_A$. □

Now we are ready to record two examples of perpendicularities. We leave it to the readers to check that these relations satisfy the axioms of perpendicularity in an Abelian group.

Example 1. The *trivial perpendicularity* is

$$g \perp_{triv} h \iff (g = \delta) \vee (h = \delta).$$

This is minimal in the poset of all perpendicularities.

Example 2. The *natural perpendicularity* is

$$g \perp_0 h \iff (g \in U_M \wedge h \in U_A) \vee (g \in U_A \wedge h \in U_M) \vee (g = \delta) \vee (h = \delta),$$

that is,

$$\perp_0 = \perp_{triv} \cup \{(g, h), (h, g) : \ g \in U_M, h \in U_A\}.$$

On the basis of Proposition 2, we can write $f \in U_1$ in the form $f = g \star h$, where $g \in U_M$ and $h \in U_A$. We say that $g \in U_M$ is the *multiplicative component* of f and $h \in U_A$ is the *antimultiplicative component* of f. As defined above, $g \perp_0 h$. We next present some concrete examples.

Example 3. If $f \in U_1$ is multiplicative, then its multiplicative component is f itself and antimultiplicative component is δ. Similarly, if $f \in U_1$ is antimultiplicative, then its multiplicative component is δ and antimultiplicative component is f itself. In both cases, $f \perp_0 \delta$.

Remark 1. Let f be an arithmetical function with $f(1) = 1$. Its multiplicative component g and antimultiplicative component h can be constructed as follows. Begin with the equations $h(p^k) = 0$ for all primes p and integers $k > 0$ and $f(p^k) = (g \star h)(p^k) = \sum_{i=0}^{k} g(p^i) h(p^{k-i})$ for all primes p and integers $k \geq 0$. These equations imply that $g(p^k) = f(p^k)$ for all primes p and integers $k \geq 0$, and this determines the multiplicative function g completely. Further, the antimultiplicative function h is given as $h = f \star g^{(-1)}$.

We illustrate this procedure in Examples 4, 5, 6, and 7.

Example 4. Let

$$f(n) = \begin{cases} 1 & \text{if } n = 1; \\ \mu(n) + \omega(n) - 1 & \text{if } n > 1. \end{cases}$$

Then $f(1) = 1$ and thus $f \in U_1$. Now,

$$g(p^k) = f(p^k) = \begin{cases} 1 & \text{if } k = 0; \\ -1 + 1 - 1 = -1 & \text{if } k = 1; \\ 0 + 1 - 1 = 0 & \text{if } k > 1, \end{cases}$$

and therefore the multiplicative component g of f is the Möbius function μ and the antimultiplicative component h of f is the summation function of f, $h(n) = (f \star g^{(-1)})(n) = \sum_{d|n} f(d)$. Now, $g \perp_0 h$.

Example 5. Consider the arithmetical function $f = N^{\omega}$ defined as $f(n) = n^{\omega(n)}$ for all positive integers n. Then $f(1) = 1$ and thus $f \in U_1$. Now,

$$g(p^k) = f(p^k) = (p^k)^1 = p^k,$$

and thus the multiplicative component of f is $g(n) = n = N(n)$. The antimultiplicative component h of f is

$$h = f \star g^{(-1)} = N^{\omega} \star N^{(-1)} = N^{\omega} \star (\mu N).$$

Thus, $g \perp_0 h$.

Example 6. Consider the function $f = N^{E_m}$, that is, $f(n) = n^{E_m(n)}$, where $E_m(n) = m^{\omega(n)}$, see [17]. Then $f(1) = 1$ and thus $f \in U_1$. Now,

$$g(p^k) = f(p^k) = (p^k)^{m^1},$$

and thus the multiplicative component of f is $g(n) = n^m = N_m(n)$. The antimultiplicative component h of f is

$$h = f \star g^{(-1)} = N^{E_m} \star N_m^{(-1)} = N^{E_m} \star (\mu N_m).$$

Thus, $g \perp_0 h$.

4. Perpendicularity in the Vector Space U_1

In Section 3, we investigated perpendicularity in the Abelian group $(U_1, \star)$. We here insert a (real) scalar multiplication in this Abelian group to obtain a vector space structure. We thus interpret the Dirichlet convolution as the addition of the vector space U_1. Therefore, it is natural that the power under the Dirichlet convolution serves as the scalar multiplication.

The *integer power* $f^{(n)}$ of $f \in U$ with respect to the Dirichlet convolution is defined in the natural manner: $f^{(n)} = f \star f \star \cdots \star f$ (n times) for $n > 0$, $f^{(n)} = \delta$ for $n = 0$, $f^{(n)} = f^{(-1)} \star f^{(-1)} \star \cdots \star f^{(-1)}$ ($-n$ times) for $n < 0$.

Define scalar multiplication from $\mathbb{Z} \times U_1$ to U_1 as

$$(n, f) \to f^{(n)}.$$

Then U_1 becomes a $\mathbb{Z}$-module. To obtain a vector space, we need rational or real powers of arithmetical functions under the Dirichlet convolution. Rational powers could be defined utilizing a classic algebraic approach, see [4, 13]. However, we go directly to real powers by adopting a discrete mathematics analog of a classic tool in mathematical analysis. This approach is based on observations made by Rearick [11, 12]. He defines real powers of arithmetical functions f with $f(1) \in \mathbb{R}_+$. Let P denote the set of arithmetical functions f with $f(1) \in \mathbb{R}_+$. Following Rearick [11], we define an operator L from P to A such that Lf is the arithmetical function with

$$(Lf)(1) = \log f(1), \tag{2}$$

and for $n > 1$,

$$(Lf)(n) = \sum_{ab=n} f(a)(\log a) f^{(-1)}(b), \tag{3}$$

where log denotes the usual real logarithm function.

Rearick [11] showed that the operator L is an isomorphism from P to A. Let E denote the inverse of L. For $f \in P$ and $\alpha \in \mathbb{R}$, Rearick [11] defined the *real power* $f^{(\alpha)}$ by

$$f^{(\alpha)} = E(\alpha L f). \tag{4}$$

It is known [11] that for any integer n, $f^{(n)}$ coincides with $f^{(\alpha)}$ defined by (4) when $\alpha = n$.

Rearick presented also slight modifications of L and E in [12] by defining an operator Log from P to A such that $\mathrm{Log} f$ is the arithmetical function with

$$(\mathrm{Log} f)(1) = \log f(1), \tag{5}$$

and for $n > 1$,

$$(\mathrm{Log} f)(n) = \frac{1}{\log n} \sum_{ab=n} f(a)(\log a) f^{(-1)}(b). \tag{6}$$

From (3) it is clear that $Lf(n) = (\log n)\mathrm{Log} f(n)$ if $n > 1$. Thus Log is also an isomorphism. Let Exp denote the inverse of the isomorphism Log. Then the real power $f^{(\alpha)}$ can also be defined by

$$f^{(\alpha)} = \mathrm{Exp}(\alpha \mathrm{Log} f), \tag{7}$$

where $\alpha \in \mathbb{R}$.

It is known [12, Theorem 6] that $\mathrm{Exp}(\alpha \mathrm{Log} f)$ and $E(\alpha L f)$ are equal. Therefore $f^{(\alpha)}$ can either be defined by (4) or (7). The advantage of using (7) lies in the fact that the Exp operator can be written inductively: If $f \in A$,

$$\mathrm{Exp} f(1) = \exp f(1), \tag{8}$$

and for $n > 1$,

$$\mathrm{Exp}f(n) = \frac{1}{\log n} \sum_{\substack{ab=n \\ a<n}} \mathrm{Exp}f(a)(\log b)f(b). \tag{9}$$

In (8), exp denotes the usual real exponential function.

Real power also possesses an inductive formula:

$$f^{(\alpha)}(1) = (f(1))^{\alpha}, \tag{10}$$

and for $n > 1$,

$$f^{(\alpha)}(n) = \frac{\alpha}{\log n} \sum_{\substack{ab=n \\ a<n}} f^{(\alpha)}(a)(\log b)\mathrm{Log}\, f(b). \tag{11}$$

Note that (11) is obtained by replacing f with $\alpha \mathrm{Log}\, f$ in (9).

Now we are in a position to define scalar multiplication from $\mathbb{R} \times U_1$ to U_1 in order to obtain a vector space structure on U_1.

Theorem 1. *The class U_1 of arithmetical functions f with $f(1) = 1$ becomes a real vector space under the Dirichlet convolution and the scalar multiplication from $\mathbb{R} \times U_1$ to U_1 defined as*

$$(\alpha, f) \to f^{(\alpha)}. \tag{12}$$

Proof. As noted in Section 3, U_1 under the Dirichlet convolution is an Abelian group. It follows from (10) that (12) induces a function from $\mathbb{R} \times U_1$ to U_1. It is known [7, 11] that the real power $f^{(\alpha)}$ possesses the basic properties

$$(f^{(\alpha)})^{(\beta)} = f^{(\alpha\beta)}, f^{(\alpha+\beta)} = f^{(\alpha)} \star f^{(\beta)}, (f \star g)^{(\alpha)} = f^{(\alpha)} \star g^{(\alpha)}, f^{(1)} = f.$$

This means that (12) satisfies the vector space axioms. □

Theorem 2. *The classes U_M and U_A are subspaces of U_1 and $U_1 = U_M \oplus U_A$.*

Proof. On the basis of Propositions 1 and 2 it suffices to show that U_M and U_A are closed under the scalar multiplication (12). It is known [7, 11] that if f is multiplicative, then the real power $f^{(\alpha)}$ is multiplicative, that is, U_M is closed under the scalar multiplication. We prove that the same holds for antimultiplicative functions. In fact, assume that f is antimultiplicative, α is a real number and p is a prime number. Then $f(p^k) = 0$ for all $k > 0$. We show by induction on k that $f^{(\alpha)}(p^k) = 0$ for all $k > 0$. Let $k = 1$. Then, by (11),

$$f^{(\alpha)}(p) = \frac{\alpha}{\log p} \sum_{\substack{ab=p \\ a<p}} f^{(\alpha)}(a)(\log b)\mathrm{Log}\, f(b) = \frac{\alpha}{\log p} f^{(\alpha)}(1)(\log p)\mathrm{Log}\, f(p).$$

By (6),

$$\mathrm{Log}f(p) = \frac{1}{\log p} \sum_{ab=p} f(a)(\log a)f^{(-1)}(b).$$

Since f is antimultiplicative, $f^{(-1)}$ is also antimultiplicative. Therefore $f(p) = f^{(-1)}(p) = 0$, and thus $\mathrm{Log} f(p) = 0$. This implies that $f^{(\alpha)}(p) = 0$. Assume then that $f^{(\alpha)}(p^k) = 0$ for $1 \le k \le m$. Then, by (11),

$$\begin{aligned} f^{(\alpha)}(p^{m+1}) &= \frac{\alpha}{\log p^{m+1}} \sum_{\substack{ab=p^{m+1} \\ a<p^{m+1}}} f^{(\alpha)}(a)(\log b)\mathrm{Log}\, f(b) \\ &= \frac{\alpha}{\log p^{m+1}} \sum_{k=1}^{m} f^{(\alpha)}(p^k)(\log p^{m+1-k})\mathrm{Log}\, f(p^{m+1-k}) = 0. \end{aligned}$$

This shows that $f^{(\alpha)}(p^k) = 0$ for all $k > 0$. Thus $f^{(\alpha)}$ is antimultiplicative. □

Remark 2. The perpendicularities in Section 3 hold also in the vector space U_1. For example, Theorem 2 induces a perpendicularity on U_1 such that multiplicative functions are perpendicular to antimultiplicative functions. More precisely,

$$g \perp_0 h \iff (g \in U_M \wedge h \in U_A) \vee (g \in U_A \wedge h \in U_M) \vee (g = \delta) \vee (h = \delta).$$

We conclude this paper with an example of finding multiplicative and antimultiplicative component of a function in U_1.

Example 7. *Arithmetic derivative* D is the arithmetic function satisfying $D(p) = 1$ for all primes p and the *Leibniz rule* $D(mn) = mD(n) + nD(m)$ for all positive integers m and n. See e.g. [2, 15, 16]. It is easy to see that $D(p^k) = kp^{k-1}$ for all primes p and positive integers k. Since $D(1) = 0$, $D \notin U_1$. Therefore we consider its multiplicative analogue $f(n) = e^{D(n)} = \exp(D(n))$. This function satisfies the property $f(mn) = f(m)^n f(n)^m$ for all positive integers m and n. This property may be referred to as a multiplicative analogue of Leibniz rule.

Now, we calculate the multiplicative component g and the antimultiplicative component h of f. Then

$$g(p^k) = f(p^k) = \exp(kp^{k-1}).$$

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ denote the canonical factorization of n with $k_1, k_2, \ldots, k_r > 0$. Since g is multiplicative, we obtain

$$\begin{aligned} g(n) &= \exp(k_1 p_1^{k_1-1}) \exp(k_2 p_2^{k_2-1}) \cdots \exp(k_r p_r^{k_r-1}) \\ &= \exp(k_1 p_1^{k_1-1} + k_2 p_2^{k_2-1} + \cdots + k_r p_r^{k_r-1}) = \exp(s(n)), \end{aligned}$$

where s is the arithmetical function such that $s(1) = 0$ and, for $n > 1$, $s(n) = k_1 p_1^{k_1-1} + k_2 p_2^{k_2-1} + \cdots + k_r p_r^{k_r-1}$. Now, the antimultiplicative component h of f is $h = f \star g^{(-1)}$, and thus, $g \perp_0 h$.

Acknowledgement. The authors thank the reviewer for careful reading of our manuscript and for the valuable comments and suggestions, which have helped us improve our examples.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, UTM, Springer-Verlag, New York, 1976.

[2] E. J. Barbeau, Remarks on an arithmetic derivative, *Canad. Math. Bull.* **4** (1961), 117-122.

[3] G. Davis, Orthogonality relations on abelian groups, *J. Austral. Math. Soc.* (Series A) **19** (1975), 173-179.

[4] J. E. Delany, Groups of arithmetical functions, *Math. Mag.* **78** (2) (2005), 83-97.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed., Addison-Wesley, 1994.

[6] P. Haukkanen, M. Mattila, J. K. Merikoski, and T. Tossavainen, Perpendicularity in an abelian group, *Int. J. Math. Math. Sci.* **2013** (2013), Article ID 983607.

[7] V. Laohakosol, N. Pabhapote, and N. Wechwiriyakul, Arithmetic functions and operators, *Sci. Asia.* **27** (2001), 199-202.

[8] P. J. McCarthy, *Introduction to Arithmetical Functions*, UTX, Springer-Verlag, New York, 1986.

[9] M. Mattila, J. K. Merikoski, P. Haukkanen, and T. Tossavainen, Maximal perpendicularity in certain abelian groups, *Acta Univ. Sapientiae, Math.* **9** (2017), 235-247.

[10] J. K. Merikoski, P. Haukkanen, and T. Tossavainen, A note on perpendicularities and inner products, *Appl. Math. E-Notes* **23** (2023), 459-464.

[11] D. Rearick, Operators on algebras of arithmetic functions, *Duke Math. J.* **35** (1968), 761-766.

[12] D. Rearick, The trigonometry of numbers, *Duke Math. J.* **35** (1968), 767-776.

[13] J. Singh, On an arithmetic convolution, *J. Integer Seq.* **17** (2014), Article 14.6.7.

[14] R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 126, Marcel Dekker, Inc., New York, 1989.

[15] T. Tossavainen, P. Haukkanen, J. K. Merikoski, and M. Mattila, We can differentiate numbers, too, *College Math. J.* **55** (2) (2024), 100-108.

[16] V. Ufnarovski and B. Åhlander, How to differentiate a number, *J. Integer Seq.* **6** (2003), Article 03.3.4.

[17] R. Vaidyanathaswamy, The theory of multiplicative arithmetic functions, *Trans. Amer. Math. Soc.* **33** (1931), 579-662.