#A35 INTEGERS 25 (2025)

ON SPLIT FAMILIES OF THUE EQUATIONS WITH LINEAR RECURRENCE SEQUENCES AS FACTORS

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Received: 6/3/24, Accepted: 3/24/25, Published: 4/25/25

Abstract

We consider a parameterized family of Thue equations,

$$(x - G_1(n)\, y) \cdots (x - G_d(n)\, y) - y^d = \pm 1,$$

which was first considered by Thomas and shown to have an explicit set of solutions for n greater than some computable constant. When the parameter functions are polynomials belonging to an explicitly described family, this is known to be true. We consider other parameter functions, namely linear recurrence sequences, for which it is not clear whether a similar result holds, and confirm that it does hold for an explicitly described family of linear recurrence sequences.

1. Introduction

Thue equations, i.e., integer equations of the form $f(x, y) = m$ for an irreducible homogeneous polynomial f of degree at least three, are an interesting object of study. Thue [17] used his improvement of Liouville's original result on the approximatability of algebraic numbers to prove that such equations can have at most finitely many integer solutions. As a result, a large class of Diophantine equations was proved to be decidable (in contrast to Hilbert's 10th problem, whose solution famously showed that the class of all Diophantine equations is undecidable). To the best of the author's knowledge, it is currently unknown whether even the class of all bivariate Diophantine equations is decidable.

However, for Thue equations, we can do even better. Baker [1] used his celebrated work on lower bounds for linear forms in logarithms to prove Thue's original result in an effective way—thus providing an algorithm that gives all solutions to any given Thue equation (by computing an upper bound on the absolute values of all solutions); let us call this property *effectively solvable*.

DOI: 10.5281/zenodo.15283686

Baker's work has inspired many authors to consider various generalizations of effectively solvable equations, such as dropping the homogeneity condition for certain families [4] or considering inequalities [14]. One of the first to successfully approach parameterized Thue equations (with positive discriminant)—where the coefficients of the Thue equation are themselves polynomials in one or more variables—was Thomas [16]. Since then, various authors have considered many parameterized Thue equations (for a survey, see [11]).

For monic polynomials $p_2(t), \ldots, p_d(t) \in \mathbb{Z}[t]$, Thomas considered the parameterized family

$$x(x - p_2(n)\, y) \cdots (x - p_d(n)\, y) + u\, y^d = 1, \qquad u = \pm 1, \tag{1}$$

which he called a *split family* of Thue equations with factors $p_2, \ldots, p_d$. This family always has the trivial solutions

$$\epsilon \left\{(1,0), (0,u), (p_2(n)u, u), \ldots, (p_d(n)u, u)\right\},$$

where $\epsilon = 1$ for odd d and $\epsilon = \pm 1$ for even d. He conjectured that if

$$0 < \deg p_2 < \cdots < \deg p_d,$$

then the split family of Thue equations with factors $p_2, \ldots, p_d$ is effectively solvable and has only the solutions listed above. The condition is necessary because there are known examples of split families that do not satisfy this condition, which have additional solutions not covered by Thomas' set. In the case of cubic split families, Thomas proved the conjecture under some additional technical conditions, which restrict the polynomials to a subclass he called *regular*.

Halter-Koch et al. [7] considered a special case of Thomas' conjecture where $p_2, \ldots, p_{d-1}$ are distinct integers and p_d is an integral parameter. They proved that in this case, Thomas' conjecture (for p_d larger than some computable constant) follows from the Lang–Waldschmidt conjecture.

Heuberger and Tichy [9] considered a multivariate version of Equation (1), where now $p_i \in \mathbb{Z}[t_1, \ldots, t_r]$ and they allowed for a non-zero first polynomial p_1. For $\mathrm{LH}(p)$, which they called the homogeneous part of maximal degree in p, they gave the following conditions:

1. The degrees satisfy $\deg p_1 < \cdots < \deg p_{d-2} < \deg p_{d-1} = \deg p_d$.
2. The homogeneous parts of maximal degree of p_{d-1} and p_d are the same, i.e., $\mathrm{LH}(p_{d-1}) = \mathrm{LH}(p_d)$, while still $p_{d-1} \neq p_d$.
3. For any $p \in \{p_1, \ldots, p_d, p_d - p_{d-1}\}$ there exist constants t_p, c_p such that whenever $t_1, \ldots, t_r \geq t_p$ it holds that

$$|\mathrm{LH}(p(t_1, \ldots, t_r))| \geq c_p \cdot \left(\min_{k \in \{1,\ldots,r\}} \{t_k\} \right)^{\deg p}.$$

They proved the effective solvability for all parameters $t_1, \ldots, t_r$ that satisfy the following condition, where t_0 and τ are computable constants:

$$t_0 \leq \min_{k \in \{1,\ldots,r\}} \{t_k\}, \qquad \max_{k \in \{1,\ldots,d\}} \{t_k\} \leq \left(\min_{k \in \{1,\ldots,r\}} \{t_k\} \right)^{\tau}.$$

Heuberger [10] later improved the result through highly technical but explicit conditions on terms involving the degrees of the polynomials. In the cubic case, his conditions are weaker than in Thomas' original result.

The "polynomial case" has been solved to the extent described above. One way to extend these investigations is to consider classes of parameter functions other than polynomials. An explicit cubic split family, parameterized by the Fibonacci and Lucas sequences, was considered in [12]. The family was proved to be effectively solvable, and an analog of the conjecture was confirmed. Using a combination of reduction methods, the bounds on the size of the parameters were reduced sufficiently for the remaining equations to be checked, fully solving the equation

$$x(x - F_n\, y)(x - L_n\, y) - y^3 = \pm 1.$$

With the exception of $n = 1$ and $n = 3$, only the trivial solutions were found.

In [13], the ideas developed for the polynomial case and adapted to tackle the exponential case were used to solve the cubic case in general, in a matter analogous to Thomas' original work. The conditions on the linear recurrence sequences $A(n)$ and $B(n)$ (which both have a dominant root) are very mild and become restrictive only if the dominant roots have the same absolute value. This is possible because the growth of the solutions can be described by 2×2 matrices and their determinants, which allow for very few term cancellations. If we consider families of higher degree, there is much greater potential for term cancellations, necessitating stricter conditions on the class of parameter functions.

In light of Heuberger and Tichy's result [9], our main result is as follows.

Theorem 1. *Let $(G_1(n))_{n \in \mathbb{N}}, \ldots, (G_d(n))_{n \in \mathbb{N}}$ be d simple linearly recurrent integer sequences satisfying the following conditions:*

1. *The sequences $G_1, \ldots, G_d$ satisfy a dominant root condition, with dominant roots $\gamma_1, \ldots, \gamma_d$ and $0 \leq \gamma_1 < \cdots < \gamma_{d-2} < \gamma_{d-1} = \gamma_d$.*

2. *The constant terms g_{d-1} and g_d corresponding to the dominant roots γ_{d-1} and γ_d in the closed formula for G_{d-1} and G_d satisfy $g_{d-1} = g_d$.*

3. *Both G_{d-1} and G_d have second dominant roots δ_{d-1} and δ_d with corresponding constant terms h_{d-1} and h_d satisfying $|\delta_{d-1}| < |\delta_d| < \gamma_{d-2}$ and $\gamma_{d-2}^2 < \gamma\, |\delta_d|$.*

For each $n \in \mathbb{N}$, define the homogeneous polynomial

$$f_n(x, y) = (x - G_1(n)\, y) \cdots (x - G_d(n)\, y) - y^d,$$

and let x, y, n be integers satisfying $|y| \geq 2$ and $f_n(x,y) = \pm 1$. Then there exists a computable constant κ, depending on the coefficients of $G_1, \ldots, G_d$, such that

$$\max\{\log|x|, \log|y|, n\} \leq \kappa.$$

If we consider the properties of the solutions to the equation $f_n(x,y) = \pm 1$ where $|y| \leq 1$ (see the beginning of Section 3), then the above theorem immediately implies the following corollary.

Corollary 1. *Let $G_1, \ldots, G_d$ satisfy the conditions in Theorem 1. Then there exists a computable constant n_0 such that, for $n \geq n_0$, the parameterized Thue equation*

$$(x - G_1(n)\,y) \cdots (x - G_d(n)\,y) - y^d = \pm 1$$

has only the solutions

$$\{(\pm 1, 0), \pm(G_1(n), 1), \ldots, \pm(G_d(n), 1)\}.$$

If we compare the conditions in Theorem 1 or those in the result of Heuberger and Tichy with Thomas' original conjecture, we notice that the condition requiring sufficiently different growth of the parameter functions (whether by strictly increasing degrees or dominant roots) has been modified. Even the cubic case of linear recurrence sequences suggests that, in some sense, it should be easier if all the dominant roots are different. However, while Heuberger [10] was able to impose fewer restrictions and come closer to the original conjecture in the polynomial case, a key part of his proof is that the variables controlling the growth of the parameter functions—namely, the degrees of the polynomials—are integers. This contrasts with the situation in Theorem 1, where the controlling terms are the dominant roots (or their logarithms). Forcing them to be integers in order to apply similar ideas from [10] would impose even stricter conditions than those currently in place.

For simplicity, we assume that all dominant roots are positive real numbers. This is not a restriction—we can apply the theorem to alternating sequences by considering the positive and negative subsequences separately.

We use the standard O notation to describe asymptotic behavior in terms of $n \to \infty$ and write $f(n) = O(g(n))$ if, for some positive constants c and n_0, we have $|f(n)| \leq c\,g(n)$ for all $n \geq n_0$. Similarly, we write $f(n) = \Omega(g(n))$ for the other inequality, and $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold. If $f(n)$ does not influence the asymptotic behavior of $g(n)$ and we do not need to quantify the specifics, we succinctly write $f(n) = o(g(n))$. For example, we write $x + x^{-1} = x + o(x)$ if we do not care that the second addend is precisely x^{-1}. We use this to prevent some error terms from becoming unnecessarily complicated without adding conceptual or technical relevance.

Using this notation, we immediately obtain the following asymptotic bounds based on the requirements in Theorem 1:

$$G_i(n) = \Omega\left(\gamma_i^n\right), \qquad |G_i(n) - G_j(n)| = \begin{cases} \Omega\left(|\delta_d|^n\right) & \text{if } \{i,j\} = \{d-1,d\} \\ \Omega\left(\max\{\gamma_i,\gamma_j\}^n\right) & \text{otherwise.} \end{cases} \tag{2}$$

Other, more involved statements about the behavior of the recurrence sequences and the roots of the polynomial f_n will be discussed in a separate section before proceeding to the proof of Theorem 1 in the next section.

2. Auxiliary Results

We will refer to the dominant roots γ_{d-1} and γ_d as γ, and the corresponding constant terms g_{d-1} and g_d as g, since they are equal by the conditions in the theorem. Let $\alpha^{(1)} = \alpha^{(1)}(n), \ldots, \alpha^{(d)} = \alpha^{(d)}(n)$ be the roots of the polynomial f_n. Since $f_n(x) = f_n(x,1) = (x - G_1(n)) \cdots (x - G_d(n)) - 1$, we expect $\alpha^{(1)}$ to be close to $G_1(n)$, $\alpha^{(2)}$ to be close to $G_2(n)$, and so on. We quantify this in the next lemma, ensuring that we also have an explicit term for the expressions $\alpha^{(i)} - G_i(n)$; the subsequent errors are of little interest and are relevant only to the proof of the lemma.

Lemma 1. *The roots $\alpha^{(1)}, \ldots, \alpha^{(d)}$ are all real, and for $i = 1, \ldots, d$ and*

$$\gamma_\epsilon = \gamma_\epsilon(i) = \begin{cases} \gamma_i^{i-1} \cdot \prod_{k=i+1}^{d-2} \gamma_k \cdot \gamma^2 & \text{if } i \notin \{d-1,d\} \\ \gamma^{d-2}\delta_d & \text{otherwise,} \end{cases}$$

we have

$$\alpha^{(i)} = G_i(n) + \frac{1 + O\left(\gamma_\epsilon^{-n}\right)}{\prod_{\substack{k=1\\k\neq i}}^{d} (G_i(n) - G_k(n))}.$$

Proof. For $u = \pm 1$, we plug our approximation for the root $\alpha^{(i)}$ into f_n, replacing the O-term with $u\,\gamma_\epsilon^{-n}$. The sign of u determines the sign of the expression, and the statement then follows from the Intermediate Value Theorem.

Let $i \in \{1, \ldots, d\}$. We then consider the expression $f(\xi^{(i)}) + 1$ for

$$\xi^{(i)} = G_i(n) + \frac{1 + u\,\gamma_\epsilon^{-n}}{\prod_{\substack{k=1\\k\neq i}}^{d} (G_i(n) - G_k(n))}.$$

The form of our function is $f(\xi^{(i)}) + 1 = (\xi^{(i)} - G_1(n)) \cdots (\xi^{(i)} - G_d(n))$, and we split the product into the factor $(\xi^{(i)} - G_i(n))$, which is

$$\frac{1 + u\gamma_\epsilon^{-n}}{\prod_{\substack{k=1\\k\neq i}}^{d} (G_i(n) - G_k(n))},$$

and everything else. Note that, by the definition of $\gamma_\epsilon(i)$ and Equation (2), we have

$$\prod_{\substack{k=1\\k\neq i}}^{d}(G_i(n)-G_k(n))=\Omega\left(\gamma_\epsilon^n\right).$$

The remaining product is

$$\prod_{\substack{k=1\\k\neq i}}^{d}(\xi^{(i)}-G_k(n))=\prod_{\substack{k=1\\k\neq i}}^{d}\left(G_i(n)-G_k(n)+\frac{1+u\,\gamma_\epsilon^{-n}}{\prod_{\substack{k=1\\k\neq i}}^{d}(G_i(n)-G_k(n))}\right),$$

and we view each factor as made up of two addends. The first is $(G_i(n)-G_k(n))$, while the second is $O\left(\gamma_\epsilon^{-n}\right)$. If we expand the product, the highest-order term is the product of all $d-1$ factors $(G_i(n)-G_k(n))$, followed by the sum of terms where $(G_i(n)-G_k(n))$ appears $d-2$ times and $O\left(\gamma_\epsilon^{-n}\right)$ appears once, and so on.

We explicitly write the two highest-order terms (in the sense described above) and hide the rest in an error term $o\left(\gamma_\epsilon^{-n}\right)$. Doing this gives

$$\prod_{\substack{k=1\\k\neq i}}^{d}(G_i(n)-G_k(n))+\sum_{\substack{l=2\\l\neq i}}^{d}\frac{\prod_{\substack{k=1\\k\neq i,l}}^{d}(G_i(n)-G_k(n))}{\prod_{\substack{k=1\\k\neq i}}^{d}(G_i(n)-G_k(n))}\left(1+u\,\gamma_\epsilon^{-n}\right)+o\left(\gamma_\epsilon^{-n}\right),$$

and canceling the fraction gives

$$\prod_{\substack{k=1\\k\neq i}}^{d}(G_i(n)-G_k(n))+\sum_{\substack{k,l=1\\i\notin\{k,l\},k\neq l}}^{d}\frac{1+u\gamma_\epsilon^{-n}}{(G_i(n)-G_k(n))(G_i(n)-G_l(n))}+o\left(\gamma_\epsilon^{-n}\right).$$

We can also move the terms containing $u\gamma_\epsilon^{-n}$ into the error term $o\left(\gamma_\epsilon^{-n}\right)$, i.e.,

$$\prod_{\substack{k=1\\k\neq i}}^{d}(G_i(n)-G_k(n))+\sum_{\substack{k,l=1\\i\notin\{k,l\},k\neq l}}^{d}\frac{1}{(G_i(n)-G_k(n))(G_i(n)-G_l(n))}+o\left(\gamma_\epsilon^{-n}\right). \quad (3)$$

Multiplying by the factor $(\xi^{(i)}-G_i(n))$, the first product is then $1+u\gamma_\epsilon^{-n}$, while the product with the sum is again $o\left(\gamma_\epsilon^{-n}\right)$ with the same argument. Putting this together, we obtain

$$f(\xi^{(i)})+1=1+u\gamma_\epsilon^{-n}+o\left(\gamma_\epsilon^{-n}\right).$$

Setting $u=1$ and $u=-1$, we get $f(\xi^{(i)})>0$ and $f(\xi^{(i)})<0$ if n is sufficiently large, so that the error term can no longer compensate for $\pm\gamma_\epsilon^{-n}$. From this, the statement follows by the Intermediate Value Theorem. □

By combining the previous lemma with Equation (2), we immediately obtain the following result.

Lemma 2. *Let $\alpha^{(i)}$ and $\alpha^{(j)}$ be two roots of f_n. Then we have*

$$\left|\alpha^{(i)}\right| = \Omega\left(\gamma_i^n\right), \quad \left|\alpha^{(i)} - \alpha^{(j)}\right| = \begin{cases} \Omega\left(|\delta_d|^n\right) & \text{if } \{i,j\} = \{d-1,d\} \\ \Omega\left(\max\left\{\gamma_i, \gamma_j\right\}^n\right) & \text{otherwise.} \end{cases}$$

The same result holds if we replace $\alpha^{(j)}$ with $G_j(n)$.

Next, we examine the number field $\mathbb{K}_n = \mathbb{Q}\left(\alpha^{(1)}\right)$ generated by f_n. To do this, we define

$$\eta_j^{(i)} = \alpha^{(i)} - G_j(n) \quad \text{for } j = 1, \ldots, d,$$

for each $i = 1, \ldots, d$. This definition immediately gives

$$\eta_1^{(i)} \cdots \eta_d^{(i)} = f(\alpha^{(i)}) + 1 = 1 \tag{4}$$

for each $i = 1, \ldots, d$. If $i = j$, then by Lemma 1 and Equation (2) the asymptotic bound $\left|\eta_i^{(i)}\right| = O\left(\left(\gamma_2 \cdots \gamma_{d-2}\gamma^2\right)^{-n}\right)$ holds for any i, since the γ_ϵ defined in the lemma only swaps some of the factors of $\gamma_2 \cdots \gamma_{d-2}\gamma^2$ for larger ones. If instead $i \neq j$, then by Lemma 2 we have either $\left|\eta_j^{(i)}\right| = \Omega\left(\delta_d^n\right)$ or $\left|\eta_j^{(i)}\right| = \Omega\left(\max\left\{\gamma_i, \gamma_j\right\}^n\right)$, which we summarize in the following equation:

$$\left|\eta_j^{(i)}\right| = \begin{cases} O\left(\left(\gamma_2 \cdots \gamma_{d-2}\gamma^2\right)^{-n}\right) & \text{if } i = j \\ \Omega\left(\delta_d^n\right) & \text{if } \{i,j\} = \{d-1,d\} \\ \Omega\left(\max\left\{\gamma_i, \gamma_j\right\}^n\right) & \text{otherwise.} \end{cases} \tag{5}$$

We want to make statements about matrices containing logarithms of these $\eta_j^{(i)}$. To do this, we use the following theorem of Gershgorin [6], sometimes referred to as *Gershgorin's Circle Theorem*,

Theorem 2 ([6])**.** *Let $A = \left(a_{ij}\right)$ be an $n \times n$ matrix with complex entries and define, for each row $i = 1, \ldots, n$, the radius*

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|a_{ij}\right|.$$

Then every eigenvalue λ of A lies in at least one of the disks

$$\left\{z : |z - a_{ii}| \leq R_i\right\}, \qquad i = 1, \ldots, n.$$

Lemma 3. *Let $k \in \{1, \ldots, d-1\}$ and let*

$$B_k = \begin{pmatrix} \log\left|\eta_1^{(1)}\right| & \log\left|\eta_2^{(1)}\right| & \cdots & \log\left|\eta_k^{(1)}\right| \\ \vdots & \vdots & \ddots & \vdots \\ \log\left|\eta_1^{(k)}\right| & \log\left|\eta_2^{(k)}\right| & \cdots & \log\left|\eta_k^{(k)}\right| \end{pmatrix},$$

then we have $\det B_k = \Theta\left(n^k\right)$.

Proof. It follows immediately from Lemma 1 that $\log\left|\eta_j^{(i)}\right| = \Theta(n)$, which implies that $\det B_k = O\left(n^k\right)$. We need to prove the other direction: $\det B_k = \Omega\left(n^k\right)$.

Let λ be the smallest eigenvalue of B_k. According to Theorem 2, for at least one $i \in \{1, \ldots, k\}$ it must be that

$$\sum_{\substack{j=1\\ j\neq i}}^{k} \left|\log\left|\eta_j^{(i)}\right|\right| \geq \left|\lambda - \log\left|\eta_i^{(i)}\right|\right|.$$

From this inequality, it follows that

$$|\lambda| \geq \left|\log\left|\eta_i^{(i)}\right|\right| - \sum_{\substack{j=1\\ j\neq i}}^{k} \left|\log\left|\eta_j^{(i)}\right|\right|,$$

and we can ignore the outer absolute value in the sum since, for each $j \neq i$, we have $\log\left|\eta_j^{(i)}\right| > 0$. If we replace $\log\left|\eta_i^{(i)}\right|$ with $-\sum_{\substack{j=1\\ j\neq i}}^{d} \log\left|\eta_j^{(i)}\right|$ using Equation (4), then this implies

$$|\lambda| \geq \sum_{j=k+1}^{d} \log\left|\eta_j^{(i)}\right|.$$

We can bound the sum from below by $\log\left|\eta_d^{(i)}\right|$, which is $\Omega(n)$ by Lemma 2. If the smallest eigenvalue λ is $\Omega(n)$, then the determinant $\det B_k$ must be $\Omega\left(n^k\right)$, proving the lemma. □

Next, we examine the order $\mathcal{O} = \mathbb{Z}[\alpha^{(1)}]$ of $\mathbb{K}_n$ and its regulator $R_{\mathcal{O}}$. We use the following estimate by Pohst [15], whose proof, as noted by Heuberger [8], also works verbatim for non-maximal orders.

Theorem 3 ([15])**.** *Let $\mathbb{K}$ be a totally real algebraic number field of degree at least 4, and let $\mathfrak{D}$ be an order of $\mathbb{K}$ with discriminant $d_{\mathfrak{D}}$. Let $R_{\mathfrak{D}}$ be the regulator of $\mathfrak{D}$. Then there exists an explicit constant c, depending only on the degree of $\mathbb{K}$, such that*

$$R_{\mathfrak{D}} \geq c \log\left(d_{\mathfrak{D}}\right).$$

For our order $\mathcal{O}$, Theorem 3 and Lemma 2 immediately give the following corollary.

Corollary 2. *We have that*

$$R_{\mathcal{O}} = \Omega(n).$$

If we form the subgroup of $\mathcal{O}^\times$ generated by -1 and $\eta_1^{(i)}, \ldots, \eta_{d-1}^{(i)}$, we obtain the following lemma.

Lemma 4. *Consider the subgroup $G = \langle -1, \eta_1^{(i)}, \ldots, \eta_{d-1}^{(i)} \rangle$ of $\mathcal{O}^\times = \mathbb{Z}[\alpha^{(1)}]^\times$ with regulator R_G and index I. Then we have*

$$R_G = \Theta\left(n^{d-1}\right), \qquad I = O\left(n^{d-2}\right).$$

Proof. The estimate for the regulator follows from Lemma 3 with $k = d-1$. The estimate for the index follows from the relation $I = R_G / R_{\mathcal{O}}$ and Corollary 2. □

3. Proof of Main Theorem

Proof of Theorem 1. Let x, y, and n be integers satisfying $f_n(x, y) = \pm 1$. Note that for $y = 0$ this implies $x^d = \pm 1$, which leads to the solution $(x, y, n) = (\pm 1, 0, n)$ for every $n \in \mathbb{N}$. If instead $y = \pm 1$, then $f_n(x, y) = \pm 1$ implies either

$$(x - G_1(n)\, y) \cdots (x - G_d(n)\, y) = 0,$$

from which we get the solutions $(\pm G_i(n), 1, n)$ for every $n \in \mathbb{N}$ and $i = 1, \ldots, d$, or

$$(x - G_1(n)\, y) \cdots (x - G_d(n)\, y) = \pm 2.$$

Since the d factors on the left are all distinct integers, there are no solutions if $d \geq 4$. These are all solutions for which $|y| \leq 1$, and from now on, we can (and must) assume that $|y| \geq 2$.

We will refer to the terms $x - \alpha^{(i)} y$ as $\beta^{(i)}$ and call (x, y) a solution of type j if

$$\left|\beta^{(j)}\right| = \min\left\{\left|\beta^{(1)}\right|, \ldots, \left|\beta^{(d)}\right|\right\},$$

which, by the triangle inequality for $i \neq j$, implies that

$$2\left|\beta^{(i)}\right| \geq \left|\beta^{(i)} - \beta^{(j)}\right| = \left|y\left(\alpha^{(j)} - \alpha^{(i)}\right)\right|. \tag{6}$$

Analogous to Lemma 1 and/or in view of Lemma 2, we define the "correct" error term as $\gamma_\epsilon(j)$, where

$$\gamma_\epsilon(j) = \begin{cases} \gamma_j^{j-1} \cdot \prod_{i=j+1}^{d-2} \gamma_i \cdot \gamma^2 & \text{if } j \notin \{d-1, d\} \\ \gamma^{d-2} \delta_d & \text{otherwise.} \end{cases}$$

Moreover, the factor γ appears at least twice in any case. Combining this with $\beta^{(1)} \cdots \beta^{(d)} = f_n(x, y) = \pm 1$ gives

$$\left|\beta^{(j)}\right| \leq \prod_{\substack{i=1 \\ i \neq j}}^{d} \frac{2}{|y| \left|\alpha^{(j)} - \alpha^{(i)}\right|} = O\left(\frac{1}{|y|\, \gamma_\epsilon(j)^n}\right). \tag{7}$$

Furthermore, if we add and then subtract $G_j(n)\,y$ from $\beta^{(i)}$, for $i \neq j$, we get

$$\begin{aligned}\log\left|\beta^{(i)}\right| &= \log\left|-\alpha^{(i)}y + \beta^{(j)} + \alpha^{(j)}y\right| \\ &= \log\left|-\eta_j^{(i)}y + \beta^{(j)} + \eta_j^{(j)}\right|.\end{aligned}$$

Together with Equations (7) and (5), this gives a representation for $\log\left|\beta^{(i)}\right|$:

$$\log\left|\beta^{(i)}\right| = \log|y| + \log\left|\eta_j^{(i)}\right| + O\left(\frac{1}{|y|\,\gamma_\epsilon(j)^n}\right) \qquad i \in \{1,\ldots,d\}\setminus\{j\}\,. \tag{8}$$

A second representation is obtained via the group $G = \langle -1, \eta_1^{(i)}, \ldots, \eta_{d-1}^{(i)}\rangle$. Since $\beta^{(i)} \in \mathbb{Z}[\alpha^{(1)}]^\times$, there exist integers $b_1, \ldots, b_{d-1}$, such that, for the index $I = [\mathbb{Z}[\alpha^{(1)}]^\times : G]$, the relation

$$\log\left|\beta^{(i)}\right| = \frac{b_1}{I}\log\left|\eta_1^{(i)}\right| + \cdots + \frac{b_{d-1}}{I}\log\left|\eta_{d-1}^{(i)}\right| \qquad i \in \{1,\ldots,d\}\setminus\{j\} \tag{9}$$

holds. By comparing both representations, we want to derive a lower bound for $\log|y|$.

3.1. Double-Exponential Lower Bound

We solve Equation (9) using Cramer's rule and get

$$R\frac{b_k}{I} = u_k \log|y| + v_k + O\left(\frac{n^{d-2}}{|y|\,\gamma_\epsilon(j)^n}\right) \tag{10}$$

for $1 \leq k \leq d-1$, where

$$\begin{aligned}u_k &= \det\left(\log\left|\eta_1^{(i)}\right|, \ldots, \log\left|\eta_{k-1}^{(i)}\right|, 1, \log\left|\eta_{k+1}^{(i)}\right|, \ldots, \log\left|\eta_{d-1}^{(i)}\right|\right)_{i\neq j}, \\ v_k &= \det\left(\log\left|\eta_1^{(i)}\right|, \ldots, \log\left|\eta_{k-1}^{(i)}\right|, \log\left|\eta_j^{(i)}\right|, \log\left|\eta_{k+1}^{(i)}\right|, \ldots, \log\left|\eta_{d-1}^{(i)}\right|\right)_{i\neq j}.\end{aligned}$$

If we consider, for some $\lambda_0, \lambda_1, \ldots, \lambda_{d-1}$, the linear combinations

$$\mathbf{b} = \lambda_0 I + \sum_{k=1}^{d-1}\lambda_k b_k, \quad \mathbf{u} = \sum_{k=1}^{d-1}\lambda_k u_k, \quad \mathbf{v} = \lambda_0 R + \sum_{k=1}^{d-1}\lambda_k v_k,$$

then this preserves Identity (10) in the sense that

$$R\frac{\mathbf{b}}{I} = \mathbf{u}\log|y| + \mathbf{v} + O\left(\frac{n^{d-2}}{|y|\,\gamma_\epsilon(j)^n}\right). \tag{11}$$

We now distinguish between different cases for the type j and show that, for a suitable $\mathbf{u}$, $\log|y|$ grows exponentially in n.

`Case 1:` $j \leq d-2$. If $j \leq d-2$, then the column $\left(\log \left|\eta_j^{(i)}\right|\right)_{i \neq j}$ appears twice in v_k, so $v_k = 0$ for all $k \neq j$. If we choose $\lambda_0 = \cdots = \lambda_{d-2} = 0$ and $\lambda_{d-1} = 1$, then Equation (11) is

$$R\frac{b_{d-1}}{I} = u_{d-1} \log |y| + v_{d-1} + O\left(\frac{n^{d-2}}{|y|\,\gamma_\epsilon(j)^n}\right), \tag{12}$$

and since $d-1 \neq j$, we have $v_{d-1} = 0$.

In u_{d-1}, we subtract the penultimate row, $i = d-1$, from the last row, $i = d$. Writing $l_{i'}^{(i)}$ for $\log\left|\eta_{i'}^{(i)}\right|$ and $\mathbf{l}_{i'}$ for the corresponding column vector (excluding by context the last two rows), we have

$$u_{d-1} = \begin{vmatrix} \mathbf{l}_1 & \cdots & \mathbf{l}_{d-2} & \mathbf{1} \\ l_1^{(d-1)} & \cdots & l_{d-2}^{(d-1)} & 1 \\ l_1^{(d)} - l_1^{(d-1)} & \cdots & l_{d-2}^{(d)} - l_{d-2}^{(d-1)} & 0 \end{vmatrix}.$$

The entries in the last row are very small for all $i = 2, \ldots, d-2$: We use Lemma 1, factor out the dominant term $g\gamma^n$, and write $\log|1+x| = x + O\left(x^2\right)$, since the remaining terms surely have absolute value less than 1 for sufficiently large n. This gives

$$\begin{aligned} l_i^{(d)} - l_i^{(d-1)} &= \log\left|\frac{\alpha^{(d)} - G_i(n)}{\alpha^{(d-1)} - G_i(n)}\right| \\ &= \log\left|\frac{g\gamma^n\left(1 + \frac{h_d}{g}\left(\frac{\delta_d}{\gamma}\right)^n - \frac{G_i(n)}{g\gamma^n} + o\left(\left|\frac{\delta_d}{\gamma}\right|^n\right)\right)}{g\gamma^n\left(1 + \frac{h_{d-1}}{g}\left(\frac{\delta_{d-1}}{\gamma}\right)^n - \frac{G_i(n)}{g\gamma^n} + o\left(\left|\frac{\delta_{d-1}}{\gamma}\right|^n\right)\right)}\right| \\ &= \frac{h_d}{g}\left(\frac{\delta_d}{\gamma}\right)^n - \frac{G_i(n)}{g\gamma^n} - \frac{h_{d-1}}{g}\left(\frac{\delta_{d-1}}{\gamma}\right)^n + \frac{G_i(n)}{g\gamma^n} \\ &\quad + o\left(\left|\frac{\delta_d}{\gamma}\right|^n\right) + O\left(\frac{\max\left\{|\delta_d|, \gamma_i\right\}^{2n}}{\gamma^{2n}}\right), \end{aligned}$$

and the same holds true for $i = 1$ if we set $\gamma_1 = 0$. By Condition (3) in Theorem 1, we have $\gamma_{d-2}^{2n} = o\left(\gamma^n \delta_d^n\right)$, i.e., the O-term is absorbed by the o-term. Succinctly put, we can say that

$$l_i^{(d)} - l_i^{(d-1)} = \frac{h_d}{g}\left(\frac{\delta_d}{\gamma}\right)^n + o\left(\left|\frac{\delta_d}{\gamma}\right|^n\right). \tag{13}$$

We expand u_{d-1} along the last row and separate the explicit term, which we can then factor, from $o\left(|\delta_d/\gamma|^n\right)$ in $l_i^{(d)} - l_i^{(d-1)}$, shifting the latter into the error term. The minors, other than the last one, which has coefficient 0 in the expansion, are all of order $O\left(n^{d-3}\right)$ by Equation (5), and thus

$$u_{d-1} = \frac{h_d}{g}\left(\frac{\delta_d}{\gamma}\right)^n \begin{vmatrix} \mathbf{l}_1 & \cdots & \mathbf{l}_{d-2} & \mathbf{1} \\ l_1^{(d-1)} & \cdots & l_{d-2}^{(d-1)} & 1 \\ 1 & \cdots & 1 & 0 \end{vmatrix} + o\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right).$$

We then subtract the last row from the penultimate row $l_1^{(d-1)}$ times. Using the same arguments as above, except that now the δ_{d-1} terms cancel, we have for $i = 2, \ldots, d-2$ that

$$l_i^{(d-1)} - l_1^{(d-1)} = \log\left|\frac{\alpha^{(d-1)} - G_i(n)}{\alpha^{(d-1)}}\right| = -\frac{g_i}{g}\left(\frac{\gamma_i}{\gamma}\right)^n + o\left(\frac{\max\left\{|\delta_{d-1}|, \gamma_i\right\}^n}{\gamma^n}\right), \tag{14}$$

and for $i = 1$ the entry is 0. We expand along the penultimate row. All minors in which the factor is $l_i^{(d-1)} - l_1^{(d-1)}$ can be shifted into an error term. Explicitly writing only the last minor (with factor 1) gives

$$u_{d-1} = \frac{h_d}{g}\left(\frac{\delta_d}{\gamma}\right)^n \left(-\begin{vmatrix} \mathbf{l}_1 & \cdots & \mathbf{l}_{d-2} \\ 1 & \cdots & 1 \end{vmatrix} + O\left(n^{d-3}\left(\frac{\gamma_{d-2}}{\gamma}\right)^n\right)\right) + o\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right).$$

We then multiply the last row by the constant $l_1^{(d)} + l_1^{(d-1)} = \log\left|\alpha^{(d)}\right| + \log\left|\alpha^{(d-1)}\right|$, which is of order $\Theta(n)$. For each $i = 2, \ldots, d-2$, going from $l_1^{(d)}$ to $l_i^{(d)}$ introduces an error of

$$\log\left|\frac{\alpha^{(d)}}{\alpha^{(d)} - G_i(n)}\right| = -\log\left|1 - \frac{G_i(n)}{\alpha^{(d)}}\right| = O\left(\left|\frac{G_i(n)}{\alpha^{(d)}}\right|\right) = O\left(\left(\frac{\gamma_i}{\gamma}\right)^n\right).$$

Similarly, we can go from $l_1^{(d-1)}$ to $l_i^{(d-1)}$. Taken together, this means that

$$\begin{vmatrix} \mathbf{l}_1 & \cdots & \mathbf{l}_{d-2} \\ 1 & \cdots & 1 \end{vmatrix} = \Theta\left(\frac{1}{n}\right)\begin{vmatrix} \mathbf{l}_1 & \cdots & \mathbf{l}_{d-2} \\ l_1^{(d)} + l_1^{(d-1)} & \cdots & l_{d-2}^{(d)} + l_{d-2}^{(d-1)} \end{vmatrix} + O\left(n^{d-3}\left(\frac{\gamma_i}{\gamma}\right)^n\right).$$

If we now add all the other rows to the last one, the entries sum to $-l_i^{(j)}$ according to Equation (4). This means that

$$\begin{vmatrix} \mathbf{l}_1 & \cdots & \mathbf{l}_{d-2} \\ 1 & \cdots & 1 \end{vmatrix} = -\Theta\left(\frac{1}{n}\right)\begin{vmatrix} \mathbf{l}_1 & \cdots & \mathbf{l}_{d-2} \\ l_1^{(j)} & \cdots & l_1^{(j)} \end{vmatrix} + O\left(n^{d-3}\left(\frac{\gamma_i}{\gamma}\right)^n\right).$$

After suitably swapping rows, the determinant is exactly the one from Lemma 3 for $k = d-2$, and is thus of order $\Theta\left(n^{d-2}\right)$. We can absorb the error term and get that

$$\begin{vmatrix} \mathbf{l}_1 & \cdots & \mathbf{l}_{d-2} \\ 1 & \cdots & 1 \end{vmatrix} = \pm\,\Theta\left(n^{d-3}\right),$$

and thus

$$|u_{d-1}| = \Theta\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right) + o\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right) = \Theta\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right).$$

Returning to Equation (12) and plugging in our asymptotic expression for u_{d-1} gives

$$R\frac{|b_{d-1}|}{I} = \Theta\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right)\log|y| + O\left(\frac{n^{d-2}}{|y|\,\gamma_\epsilon(j)^n}\right).$$

Since $\gamma_\epsilon(j)$ contains the factor γ at least twice, the error term cannot asymptotically cancel the Θ-term, i.e.,

$$R\frac{|b_{d-1}|}{I} = \Theta\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right)\log|y|.$$

In particular, the left-hand side, and thus $|b_{d-1}|$, is nonzero. Since b_{d-1} is an integer, we have $|b_{d-1}| \geq 1$. Furthermore, we have $R/I = \Omega(n)$ by Lemma 4. Going back to the above equation, this gives

$$\log|y| = \Omega\left(n^{-(d-4)}\left|\frac{\gamma}{\delta_d}\right|^n\right). \tag{15}$$

`Case 2:` $j = d-1$ or $j = d$. For $j = d-1$ and $k < d-1$, we again have the column $\log\left|\eta_j^{(i)}\right|_{i\neq j} = \mathbf{l}_j$ twice in v_k, and thus $v_k = 0$. This is not the case for $j = d$, where instead we take $\mathbf{v} = v_{d-2} - v_{d-3}$. After swapping the antepenultimate and penultimate columns in v_{d-3}, we can join the determinants, and adding every other column to the antepenultimate one gives us, by Equation (4),

$$\begin{aligned}
\mathbf{v} &= \begin{vmatrix}\mathbf{l}_1 & \cdots & \mathbf{l}_{d-4} & \mathbf{l}_{d-3} & \mathbf{l}_d & \mathbf{l}_{d-1}\end{vmatrix} - \begin{vmatrix}\mathbf{l}_1 & \cdots & \mathbf{l}_{d-4} & \mathbf{l}_d & \mathbf{l}_{d-2} & \mathbf{l}_{d-1}\end{vmatrix}\\
&= \begin{vmatrix}\cdots & \mathbf{l}_{d-3} & \mathbf{l}_d & \mathbf{l}_{d-1}\end{vmatrix} + \begin{vmatrix}\cdots & \mathbf{l}_{d-2} & \mathbf{l}_d & \mathbf{l}_{d-1}\end{vmatrix}\\
&= \begin{vmatrix}\cdots & (\mathbf{l}_{d-3}+\mathbf{l}_{d-2}) & \mathbf{l}_d & \mathbf{l}_{d-1}\end{vmatrix} = \begin{vmatrix}\cdots & 0 & \mathbf{l}_d & \mathbf{l}_{d-1}\end{vmatrix} = 0.
\end{aligned}$$

In both cases, setting $\mathbf{v} = v_{d-2} - v_{d-3}$ gives us $\mathbf{v} = 0$. We calculate the corresponding $\mathbf{u} = u_{d-2} - u_{d-3}$, which, with the analogue of the above calculation, gives us

$$\begin{aligned}
\mathbf{u} &= \begin{vmatrix}\mathbf{l}_1 & \cdots & \mathbf{l}_{d-4} & (\mathbf{l}_{d-3}+\mathbf{l}_{d-2}) & \mathbf{1} & \mathbf{l}_{d-1}\end{vmatrix}\\
&= \begin{vmatrix}\cdots & -\mathbf{l}_d & \mathbf{1} & \mathbf{l}_{d-1}\end{vmatrix}\\
&= \begin{vmatrix}\cdots & (\mathbf{l}_{d-1}-\mathbf{l}_d) & \mathbf{1} & \mathbf{l}_{d-1}\end{vmatrix}.
\end{aligned}$$

The matrix has rows $i \in \{1,\ldots,d-2,d-1,d\}\setminus\{j\}$, so the penultimate row is $i = d-2$, while the last row is either $i = d$ or $i = d-1$, depending on whether $j = d-1$ or $j = d$. Computing the last entry in the antepenultimate column gives us by Lemma 2 either

$$\log\left|\frac{\alpha^{(d-1)} - G_{d-1}(n)}{\alpha^{(d-1)} - G_d(n)}\right| = -\Theta(n) \qquad \text{or} \qquad \log\left|\frac{\alpha^{(d)} - G_{d-1}(n)}{\alpha^{(d)} - G_d(n)}\right| = \Theta(n).$$

Thus, up to sign, the entry is $\Theta(n)$. The entries in all other rows $i \leq d-2$ are instead

$$\log\left|\frac{\alpha^{(i)} - G_{d-1}(n)}{\alpha^{(i)} - G_d(n)}\right| = -\frac{h_d}{g}\left(\frac{\delta_d}{\gamma}\right)^n + o\left(\left|\frac{\delta_d}{\gamma}\right|^n\right) = O\left(\left|\frac{\delta_d}{\gamma}\right|^n\right),$$

analogous to Equation (13). Expanding the determinant $\mathbf{u}$ along the antepenultimate column and shifting all but the last minor into the error term, we obtain

$$\pm\mathbf{u} = \Theta(n)\left|\mathbf{l}_1 \quad \cdots \quad \mathbf{l}_{d-4} \quad \mathbf{1} \quad \mathbf{l}_{d-1}\right| + O\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right).$$

The matrix now consists only of the rows $i \in \{1, \ldots, d-2\}$. We add the penultimate column to the last column $-l_{d-1}^{(1)}$ times. For $i = 1$, the entry is 0, and for $i = 2, \ldots, d-2$, this yields

$$\log\left|\eta_{d-1}^{(i)}\right| - \log\left|\eta_{d-1}^{(1)}\right| = -\Theta\left(\left(\frac{\gamma_i}{\gamma}\right)^n\right),$$

analogous to Equation (14). Expanding along this column, the last entry $i = d-2$ dominates and we get

$$\begin{aligned}\left|\mathbf{l}_1 \quad \cdots \quad \mathbf{l}_{d-4} \quad \mathbf{1} \quad \mathbf{l}_{d-1}\right| = {} & \Theta\left(\left(\frac{\gamma_{d-2}}{\gamma}\right)^n\right)\left|\mathbf{l}_1 \quad \cdots \quad \mathbf{l}_{d-4} \quad \mathbf{1}\right| \\ & + o\left(n^{d-4}\left(\frac{\gamma_{d-2}}{\gamma}\right)^n\right).\end{aligned}$$

We multiply the last column by $-l_{d-2}^{(d-3)} - l_{d-1}^{(d-3)} - l_d^{(d-3)} = \Theta(-n)$. Using the same argument as in Equation (13), we can go from $l_{d-2}^{(d-3)}$ [$l_{d-1}^{(d-3)}$, resp. $l_d^{(d-3)}$] to $l_{d-2}^{(i)}$ [$l_{d-1}^{(i)}$, resp. $l_d^{(i)}$] while making an error of order $|\gamma_{d-3}/\gamma_{d-2}|^n$ [$|\gamma_{d-3}/\gamma|^n$]. Taken together, this means that

$$\begin{aligned}\left|\mathbf{l}_1 \quad \cdots \quad \mathbf{l}_{d-4} \quad \mathbf{1}\right| = {} & -\Theta\left(\frac{1}{n}\right)\left|\mathbf{l}_1 \quad \cdots \quad \mathbf{l}_{d-4} \quad (-\mathbf{l}_{d-2} - \mathbf{l}_{d-1} - \mathbf{l}_d)\right| \\ & + O\left(n^{d-4}\left|\frac{\gamma_{d-3}}{\gamma_{d-2}}\right|^n\right).\end{aligned}$$

If we subtract all other columns from the last one, the result sums to $\mathbf{l}_{d-3}$ according to Equation (4). Thus, by Lemma 3, the determinant is $\Theta\left(n^{d-3}\right)$. Put together, this means that

$$\left|\mathbf{l}_1 \quad \cdots \quad \mathbf{l}_{d-4} \quad \mathbf{1}\right| = -\Theta\left(n^{d-4}\right) + O\left(n^{d-4}\left|\frac{\gamma_{d-3}}{\gamma_{d-2}}\right|^n\right) = -\Theta\left(n^{d-4}\right).$$

If we go back one equation further and plug this in, we have

$$\begin{aligned}\left|\mathbf{l}_1 \quad \cdots \quad \mathbf{l}_{d-4} \quad \mathbf{1} \quad \mathbf{l}_{d-1}\right| &= -\,\Theta\left(n^{d-4}\left(\frac{\gamma_{d-2}}{\gamma}\right)^n\right) + o\left(n^{d-4}\left(\frac{\gamma_{d-2}}{\gamma}\right)^n\right)\\ &= -\,\Theta\left(n^{d-4}\left(\frac{\gamma_{d-2}}{\gamma}\right)^n\right).\end{aligned}$$

So for $\mathbf{u}$, in conjunction with $|\delta_d| < \gamma_{d-2}$ from Condition (3), we get that

$$|\mathbf{u}| = \Theta\left(n^{d-3}\left(\frac{\gamma_{d-2}}{\gamma}\right)^n\right) + O\left(n^{d-3}\left|\frac{\delta_d}{\gamma}\right|^n\right) = \Theta\left(n^{d-3}\left|\frac{\gamma_{d-2}}{\gamma_d}\right|^n\right).$$

We plug this along with $\mathbf{v} = 0$ into Equation (11) and get

$$R\frac{|\mathbf{b}|}{I} = \Theta\left(n^{d-3}\left|\frac{\gamma_{d-2}}{\gamma}\right|^n\right)\log|y| + O\left(\frac{n^{d-2}}{|y|\,\gamma_\epsilon(j)^n}\right).$$

Again, $\gamma_\epsilon(j)$ contains the factor γ at least twice, making the error term asymptotically negligible. So the right-hand side, and hence $\mathbf{b}$, is nonzero. We derive $|\mathbf{b}| \geq 1$, which, together with $R/I = \Omega(n)$, implies

$$\log|y| = \Omega\left(n^{-(d-4)}\left(\frac{\gamma}{\gamma_{d-2}}\right)^n\right). \tag{16}$$

If we compare the two bounds from Equations (15) and (16), the worse one, Equation (16), holds regardless of the type j of the solution.

3.2. Exponential-Polynomial Upper Bound

Returning to our notation $x - \alpha^{(i)}y = \beta^{(i)}$ for $i \in \{1, \ldots, d\}$, we can take any three indices i_1, i_2, i_3 and eliminate both x and y from these equations, giving the relation

$$\left(\alpha^{(i_3)} - \alpha^{(i_2)}\right)\beta^{(i_1)} + \left(\alpha^{(i_1)} - \alpha^{(i_3)}\right)\beta^{(i_2)} + \left(\alpha^{(i_2)} - \alpha^{(i_1)}\right)\beta^{(i_3)},$$

often called Siegel's equation. A standard application of Baker's method (e.g., [5]) is to rewrite Siegel's identity as an S-unit equation and apply lower bounds to the associated linear form in logarithms. The same approach can also be used for parameterized Thue equations (e.g., [16], for a survey, also see [11]) to attempt to derive asymptotic bounds on the size of the solutions. Alternatively, one can use Bugeaud and Győry's explicit bounds [3] directly to obtain the same asymptotic bounds, often with worse numerics due to their greater generality.

Since it is not too much work, we derive an asymptotic polynomial upper bound for $\log|y|$ ourselves, and set $(i_1, i_2, i_3) = (j, k, l)$ in Siegel's identity for the type j and some k, l. If $j \in \{d-1, d\}$, it is advantageous to choose $k, l \notin \{d-1, d\}$, e.g., $k = 1, l = 2$, whereas if $j \leq d-2$, it is advantageous to choose $k = d, l = d-1$.

Dividing by the second addend and subtracting the third (flipping the sign of $\alpha^{(k)} - \alpha^{(j)}$) gives the S-unit equation

$$\frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \frac{\beta^{(j)}}{\beta^{(k)}} + 1 = \frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \frac{\beta^{(l)}}{\beta^{(k)}}. \tag{17}$$

From Equations (7) and (6), we obtain

$$\frac{\beta^{(j)}}{\beta^{(k)}} = O\left(\frac{1}{|y|^2 \gamma_\epsilon(j)^n \left|\alpha^{(j)} - \alpha^{(k)}\right|}\right).$$

Our choice of k, l ensures optimal conditions in Corollary (2) in the sense that

$$\frac{\alpha^{(l)} - \alpha^{(k)}}{\left(\alpha^{(j)} - \alpha^{(l)}\right)\left(\alpha^{(j)} - \alpha^{(k)}\right)} = O\left(\frac{\max\left\{|\delta_d|, \gamma_2\right\}^n}{\gamma^{2n}}\right),$$

and thus, from Equation (17), we obtain

$$\frac{\alpha^{(l)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}} \frac{\beta^{(j)}}{\beta^{(k)}} + 1 = 1 + O\left(\frac{\max\left\{|\delta_d|, \gamma_2\right\}^n}{|y|^2 \gamma_\epsilon(j)^n \gamma^{2n}}\right),$$

where the factor γ^n appears at least four times in the denominator, and the factor in the numerator more than cancels out with another factor in $\gamma_\epsilon(j)^n$. Returning to Equation (17), we now have for the right-hand side that its logarithm is

$$\log\left|\frac{\beta^{(l)}}{\beta^{(k)}}\right| + \log\left|\frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}}\right| = O\left(\frac{\max\left\{|\delta_d|, \gamma_2\right\}^n}{|y|^2 \gamma_\epsilon(j)^n \gamma^{2n}}\right). \tag{18}$$

We want to apply the following lower bound for linear forms in logarithms by Baker and Wüstholz [2].

Theorem 4 ([2])**.** *Let $\alpha_1, \ldots, \alpha_n$ be real algebraic numbers greater than 1, and let $b_1, \ldots, b_n$ be integers. Let K be the number field generated by $\alpha_1, \ldots, \alpha_n$ over the rationals $\mathbb{Q}$, and let d be its degree. Put, for the standard logarithmic Weil height h,*

$$h'(\alpha) = \frac{1}{d} \max\left\{h(\alpha), |\log \alpha|, 1\right\}.$$

If $\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \neq 0$, then

$$\log|\Lambda| > -c(n,d) h'(\alpha_1) \cdots h'(\alpha_n) h'(\mathbf{b}),$$

where $\mathbf{b} = (b_1 : \cdots : b_n)$.

The coefficients of our linear form are both 1, so we can disregard their contributions to the lower bound. To bound the second term in Equation (18), we use the properties

$$h(\alpha \pm \beta) \leq h(\alpha) + h(\beta) + \log 2, \quad h(\alpha \cdot \beta) \leq h(\alpha) + h(\beta)$$

of the logarithmic Weil height (for an overview of this notion of height and its various properties, see, for example, Chapter 3 of [18]).

Since $\alpha^{(j)}, \alpha^{(k)}, \alpha^{(l)}$ are conjugates and therefore of equal height, this gives

$$h\left(\frac{\alpha^{(j)} - \alpha^{(k)}}{\alpha^{(j)} - \alpha^{(l)}}\right) = O\left(h(\alpha^{(j)})\right).$$

Using the connection to the Mahler measure of the minimal polynomial f, combined with Lemma 1, we can easily conclude that

$$h(\alpha^{(j)}) = O(n).$$

For the height of $\beta^{(k)}$ and $\beta^{(l)}$, we use Equation (9) and get

$$h(\beta^{(k)}) \leq h(b_1)h(\eta_1^{(k)}) + \cdots + h(b_{d-1})h(\eta_{d-1}^{(k)}).$$

By plugging in the definition of $\eta_i^{(k)}$, we get

$$h(\eta_i^{(k)}) \leq h(\alpha^{(k)}) + h(G_i(n)) + \log 2,$$

and since the height of $G_i(n)$ is also $O(n)$, we get

$$h(\eta_i^{(k)}) = O(n).$$

For the heights $h(b_i) = \max\{\log|b_i|, 0\}$, we look again at the system of equations (9), denote it by $\beta = \eta \cdot \frac{1}{I}\mathbf{b}$, and consider its inverse problem $\eta^{-\mathbf{1}}\beta = \frac{1}{I}\mathbf{b}$. We can do this: for $j = d$, the matrix η is the matrix from Lemma 3 with $k = d-1$ and therefore has a nonzero determinant. For $j \neq d$, we add all the other rows $i \neq j$ to the row $i = d$ and obtain the negative of the row $i = j$ by Equation (4). This changes at most the sign of the determinant and not the invertibility of η. Considering the system of equations $\eta^{-\mathbf{1}}\beta = \frac{1}{I}\mathbf{b}$, we then take the (column-wise) maximum norm $\|\cdot\|_\infty$, which gives

$$\frac{1}{I}\|\mathbf{b}\|_\infty = \left\|\eta^{-\mathbf{1}}\beta\right\|_\infty \leq \left\|\eta^{-\mathbf{1}}\right\|_\infty \cdot \|\beta\|_\infty.$$

We have $I = O\left(n^{d-2}\right)$ by Lemma 4, $\det\eta = \Theta\left(n^{d-1}\right)$ by Lemma 3, $\log\left|\beta^{(i)}\right| = O\left(\log|y| + \log\left|\eta_j^{(i)}\right|\right)$ by Equation (8) and $\log\left|\eta_j^{(i)}\right| = O(n)$ by definition and Lemma 1. Taken together, this gives us

$$h(b_i) = \max\{\log|b_i|, 0\} = O\left(\max\{\log\log|y|, \log n, 0\}\right),$$

and we can assume $h(b_i) = O(\log\log|y|)$, as otherwise we would already have $\log|y| = O(n)$. Combining the bounds for $h(b_i)$ and $h(\eta_i^{(k)})$ gives us

$$h(\beta^{(k)}) = O(n\log\log|y|),$$

and the same bound holds for $h(\beta^{(l)})$.

If we plug everything into Theorem 4 (the asymptotic bounds do not change for the modified height h'), we get

$$\log\left|\log\left|\frac{\beta^{(l)}}{\beta^{(k)}}\right|+\log\left|\frac{\alpha^{(j)}-\alpha^{(k)}}{\alpha^{(j)}-\alpha^{(l)}}\right|\right| = -\,\Omega\left(n^2\log\log|y|\right).$$

Comparing this with the upper bound of Equation (18) gives

$$n\log|y| = O\left(n^2\log\log|y|\right),$$

which implies $\log|y| = O(n\log n)$: If the implied constant is c, i.e.,

$$\log|y| \le c\,n\log\log|y|,$$

then the assertion is true if, for example, the relation $\log|y| \le 2c\,n\log n$ holds. If instead $\log|y| > 2c\,n\log n$, then we have

$$2c\,n\log n < \log|y| \le c\,n\log\log|y|,$$

which gives $\log n < \frac{1}{2}\log\log|y|$ and therefore $n < \sqrt{\log|y|}$. But going back to the original inequality, we have

$$\log|y| < c\sqrt{\log|y|}\log\log|y|,$$

and therefore $\log|y| = O(1)$, which is even stronger than the assertion.

In summary, we have the asymptotically almost linear upper bound

$$\log|y| = O(n\log n),$$

and by comparing this with the asymptotically exponential lower bound from Equation (16) we get $n = O(1)$, which in turn implies $\log|y| = O(1)$. This concludes our proof of Theorem 1. □

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