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WHICH INTEGER DIVIDES THE REVERSE OF ANY OF ITS MULTIPLES?

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Abstract

At the end of their paper titled *On the trail of reverse divisors: 1089 and all that follow*, Webster and Williams propose some problems including the following. Are there positive integers other than $1, 3, 9, 11, 33$ and 99 having the property that if they divide a positive integer, they divide its reverse? This paper serves to answer that question. We prove that no positive integer other than the six divisors of 99 satisfies this property.

1. Preliminary Results about the Reverse Divisors

Definition 1. For every integer $n \geq 1$ such that $n = a_k \cdots a_0$ in decimal writing, define the *reverse of* n to be the integer n^* such that, in decimal writing, $n^* = a_0 \cdots a_k$.
If $n = n^*$, the integer n is said to be *a palindrome*.

Definition 2. We say that an integer $n \geq 1$ is a *reverse divisor* if n is not a palindrome and $n \mid n^*$. The integer n^*/n is then called *the quotient of the reverse divisor* n.

Some authors, like Sloane, call this reverse divisor a *reverse multiple* [1]. We refer to Sloane's paper for a large bibliography on this subject of reverse divisor or reverse multiple.

Remark 1. A reverse divisor n includes, in decimal writing, at least two digits and cannot be a multiple of 10. So, it can be written in the form:

$$n = a_k \cdots a_0, \text{ where } k \geq 1, a_k \neq 0 \text{ and } a_0 \neq 0.$$

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Theorems 1 and 2 below are due to Webster and Williams [2].

Theorem 1. *The quotient of a reverse divisor is either* 4 *or* 9*. The two n-digit* ($n \geq 4$) *numbers*

$$11 \times (10^{n-2} - 1) = 1099 \cdots 9989 \text{ and } 22 \times (10^{n-2} - 1) = 2199 \cdots 9978,$$

each with $n - 4$ *central digits* 9*, are reverse divisors with respective quotients* 9 *and* 4*. No reverse divisor has fewer than four digits.*

For every $n \geq 4$, the n-digit reverse divisors displayed in this theorem are called *basic reverse divisors.* For $n = 4, 5, 6, 7$, they are the only reverse divisors.

The paper [1] deals with reverse divisors in an arbitrary numerical base B. In particular, Table 1 of [1] gives the values of the quotients for $3 \leq B \leq 20$.

Notation 1. For every integer $r \geq 0$ and any digit I, denote by I_r the string of r consecutive I's and by S_r the basic reverse divisor 109_r89 with quotient 9.

Theorem 2 (Structure theorem for reverse divisors with quotient 9)**.** *Non-basic reverse divisors with quotient* 9 *are precisely those natural numbers of the form*

$$S_{a_1}0_{b_1}S_{a_2}0_{b_2} \cdots S_{a_n}0_{b_n}VS_{a_n} \cdots 0_{b_2}S_{a_2}0_{b_1}S_{a_1},$$

where either $V = S_{a_0}$ *or* 0_{b_0}*, for some non-negative integers* $a_0, a_1, ..., a_n$ *and* $b_0, b_1, ..., b_n$*.*

The case of reverse divisors of quotient 4 was the subject of another paper [3].

2. The Property P^*

Definition 3. An integer $n \geq 1$ is said to have the property P^* if it divides the reverses of all of its multiples. That is, $n \mid (k \times n)^*$ for every integer $k \geq 1$.

Remark 2. If an integer $n \geq 1$ has the property P^*, then n is a palindrome or a reverse divisor.

Proposition 1. *The positive divisors of* 99 *have the property* P^**.*

Proof. It is obvious for $1, 3, 9$ and 11. For 33 and 99, notice that if n and n' are two coprime integers and both have the property P^*, then their product $n \times n'$ also does. □

Proposition 2. *If an integer* $n \geq 1$ *is not coprime with* 10*, then it does not have the property* P^**.*

Proof. It is obvious for $n \leq 9$ because 2 does not divide $(6 \times 2)^*$, 4 does not divide $(3 \times 4)^*$, 5 does not divide $(3 \times 5)^*$, 6 does not divide $(2 \times 6)^*$, and 8 does not divide $(2 \times 8)^*$. Suppose now that n is an $(k+1)$-digit $(k \geq 1)$ and $n = a_k \cdots a_0$ with $a_k \neq 0$ and $a_0 \neq 0$. Since n is not coprime with 10, then a_0 is even or equal to 5.

`Case 1`: a_0 is even. If a_k is odd, then n does not divide n^* and then n does not have the property P^*. If a_k is even, then it is clear that n, which is even, does not divide the odd integer $(k \times n)^*$ for $k = 6$ if $a_k = 2$ (resp. for $k = 3$ if $a_k = 4$, resp. for $k = 2$ if $a_k = 6$ or 8).

`Case 2`: $a_0 = 5$. If $a_k \neq 5$, then n does not divide n^*. If $a_k = 5$, then n does not divide $(3 \times n)^*$. □

3. The Main Result

The main result is the following.

Theorem 3. *An integer $n \geq 1$ has the property P^* if and only if n is one of the (positive) divisors of* 99*.*

For the proof of this theorem, taking into account Remark 2, it suffices to prove the two following results:

1) No palindrome other than divisors of 99 has the property P^*.

2) No reverse divisor has the property P^*.

3.1. The Case of Palindromes

Proposition 3. *No palindrome other than* $1, 3, 9, 11, 33$ *and* 99 *has the property* P^**.*

Proof. Let n be a palindrome other than $1, 3, 9, 11, 33$ and 99. We already know, by Proposition 2, that it suffices to prove that n does not satisfy the property P^* when n is coprime with 10. We reason according to the number of digits of the integer n.

`Case 1`: n has one or two digits. Then $n = 7$ or 77. But 7 does not divide $(2 \times 7)^* = 41$ and 77 does not divide $(2 \times 77)^* = 451$.

`Case 2`: n has three digits. In this case, $n = bab$ where a is any non-negative integer less than or equal to 9 and $b = 1, 3, 7$ or 9. We examine the following four subcases.

`Subcase a`: $n = 1a1$. If $a \geq 4$, then n does not divide $(3 \times n)^*$. Indeed, in this case, $3 \times n = bc3$ where $b = 4$ or 5 and if n divides $(3 \times n)^* = 3cb$, then the quotient would necessarily equal 2 (because $a \geq 4$). But the units digit of $2 \times n = 2 \times 1a1$ is

2 while that of $(3 \times n)^* = 3cb$ is $b \geq 4$. For $a \leq 3$, 131 does not divide $(9 \times 131)^*$, 121 does not divide $(5 \times 121)^*$, 111 does not divide $(879 \times 111)^*$and 101 does not divide $(879 \times 101)^*$.

`Subcase b`: $n = 3a3$. If $a \not\equiv 0 \pmod 3$, then $n = 3a3$ does not divide $(4 \times n)^*$. If $a = 3, 6$ or 9, then $n = 3a3$ does not divide $(13 \times 3a3)^*$. Finally, 303 does not divide $(103 \times 303)^*$.

`Subcase c`: $n = 7a7$. It is easy to verify that for every a, the integer $7a7$ does not divide $(2 \times 7a7)^*$.

`Subcase d`: $n = 9a9$. If $a \neq 0, 9$, then $9a9$ does not divide $(2 \times 9a9)^*$. For the two remaining values of a, 909 does not divide $(21 \times 909)^*$, and 999 does not divide $(8905 \times 999)^*$.

`Case 3`: n has at least 4 digits. We will examine the following four subcases:

$$n = 1a_1 \cdots a_1 1, \quad n = 3a_1 \cdots a_1 3, \quad n = 7a_1 \cdots a_1 7 \quad \text{and} \quad n = 9a_1 \cdots a_1 9.$$

`Subcase a`: $n = 7a_1 \cdots a_1 7$. Let r be the number of digits of n. Then $2 \times n = 1b \cdots c4$ is an integer with $(r+1)$ digits. If n divides $(2 \times n)^* = 4c \cdots b1$, then the quotient k is an integer such that $1 \leq k \leq 9$ and it verifies $k \times 7 \equiv 1 \pmod{10}$, so $k = 3$. But $3 \times n = 3 \times 7a_1 \cdots a_1 7 < 4c \cdots b1 = (2 \times n)^*$.

`Subcase b`: $n = 1a_1 \cdots a_1 1$. Let r be the number of digits of n. For this subcase, we have two possibilities: some digit of n is greater than or equal to 2 or all its digits are less than 2. If there exist digits greater than or equal to 2, let i be the smallest index such that $a_i \geq 2$. Then $5 \times n$ is an integer with r digits. If $i = 1$, then $5 \times n$ ends with 5 and begins with b, where $6 \leq b \leq 9$. Since $(5 \times n)^\star$ begins with b, n ends with 1 and $n \mid (5 \times n)^\star$, then $(5 \times n)^\star = b \times n$. Let t be the carry coming from the multiplication of b by a_1 in the penultimate stage of the multiplication process of b by n. The last stage consists in multpling b by 1 and adding t. Since $b \times n$ has r digits, then $b + t \leq 9$, so that $b \times n$ begins with $b + t$ and not with 5, thus a contradiction. If $i \geq 2$, then $5 \times n$ ends with $c_i b_{i-1} \cdots b_1 5$, where $b_1 = 5 \times a_1, \ldots, b_{i-1} = 5 \times a_{i-1}$ and c_i is the unit digit of $5 \times a_i$. Then $(5 \times n)^\star$ begins with $5b_1 \cdots b_{i-1}$. Since $n \mid (5 \times n)^\star$, then the quotient is necessarily equal to 5; that is, $(5 \times n)^\star = 5 \times n$. But $5 \times n$ begins with $5b_1 \cdots b'_{i-1}$, where $b'_{i-1} > b_{i-1}$ because of the carry arizing from the multiplication $5 \times a_i$, hence a contradiction. If for every i, $a_i \leq 1$, we shall prove that n does not divide $[(10^r - 1) \times n]^*$. Indeed, we have

$$\begin{array}{rcccccccccccc} (10^r - 1) \times n & = & 1 & a_1 & \cdots & a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ & - & & & & & & 1 & a_1 & \cdots & a_1 & 1 \\ & = & 1 & a_1 & \cdots & a_1 & 0 & 8 & (9 - a_1) & \cdots & (9 - a_1) & 9. \end{array}$$

Then,

$$\begin{aligned}[(10^r-1)\times n]^* &= 9(9-a_1)\cdots(9-a_1)80a_1\cdots a_1 1\\ &= 9(9-a_1)\cdots(9-a_1)79\times 10^{r-1} \quad + \quad n.\end{aligned}$$

It suffices to show that n does not divide $9(9-a_1)\cdots(9-a_1)79$. By way of contradiction, if $n = 1a_1\cdots a_1 1$ divides $9(9-a_1)\cdots(9-a_1)79$, which is an $(r+1)$-digit, then the quotient is a 2-digit number which ends with 9 and begins with 8 or 9. But, $89\times n$ is clearly lower and $99\times n$ does not end with 79.

`Subcase c`: $n = 3a_1\cdots a_1 3$. For this subcase, we have three possibilities. If there exist digits greater than or equal to 5, let i be the smallest index such that $a_i \geq 5$. Then $2\times n$ ends with $c_i b_{i-1}\cdots b_1 6$ where $b_1 = 2\times a_1, ..., b_{i-1} = 2\times a_{i-1}$ and c_i is the units digit of $2\times a_i$. Then $(2\times n)^*$ begins with $6b_1\cdots b_{i-1}c_i$. If n satisfies P^*, then n divides $(2\times n)^*$ and, therefore, the quotient is necessarily 2. Then $2\times n = (2\times n)^*$, which is false because $2\times n$ does not begin with $6b_1\cdots b_{i-1}c_i$. If there exist no digits greater than or equal to 5 but there exist digits equal to 4, let i be the smallest index such that $a_i = 4$. Then $3\times n$ ends with $b_{i-1}\cdots b_1 9$, where $b_1 = 3\times a_1, ..., b_{i-1} = 3\times a_{i-1}$. Then $(3\times n)^*$ begins with $9b_1\cdots b_{i-1}$. If n satisfies P^*, then n divides $(3\times n)^*$ and the quotient is, therefore, necessarily 3. Then $3\times n = (3\times n)^*$, which is false because $3\times n$ does not begin with $9b_1\cdots b_{i-1}$. If $a_i \leq 3$ for every i, we will prove that n does not divide $B^* := (10_{r-2}3\times n)^* = \left[(10^{r-1}+3)\times n\right]^*$. Indeed,

$$\begin{array}{rccccccccccc} B & = & & 3 & a_1 & \cdots & a_1 & 3 & 0 & \cdots & 0 & 0\\ & & + & & & & & 9 & (3\times a_1) & \cdots & (3\times a_1) & 9\\ & = & & 3 & a_1 & \cdots & (a_1+1) & 2 & (3\times a_1) & \cdots & (3\times a_1) & 9. \end{array}$$

Then,

$$\begin{array}{rcccccccccc} B^* & = & 9 & (3\times a_1) & \cdots & (3\times a_1) & 2 & (a_1+1) & \cdots & a_1 & 3\\ & = & 9 & (3\times a_1) & \cdots & (3\times a_1) & (9-7) & (a_1+1) & \cdots & a_1 & 3\\ & = & 30_{r-1}\times n & - & A, & & & & & & \end{array}$$

where $A = 70_{r-1} - (a_1+1)\cdots a_1 3$. It suffices to verify that n does not divide A. But,

$$\begin{aligned} A &= 70_{r-1} - 3a_1\cdots a_1 3 + 30_{r-1} - 10_{r-2},\\ &= -n + 990_{r-2}, \end{aligned}$$

which is not divisible by n since n is coprime with 10 and it does not divide 99.

`Subcase d`: $n = 9a_1\cdots a_1 9$. Let r be the number of digits of n. We will distinguish the following two situations: when at least one of the a_i's is not 9 and when all the a_i are equal to 9. We suppose first that there exists an index i such that $a_i < 9$.

We shall prove that n does not divide $[10_{r-2}1 \times n]^* = \left[(10^{r-1}+1) \times n\right]^*$. Indeed,

$$\begin{array}{rcccccccccc} 10_{r-2}1 \times n & = & & 9 & a_1 & \cdots & a_1 & 9 & 0 & \cdots & 0 & 0 \\ & & + & & & & & 9 & a_1 & \cdots & a_1 & 9 \\ & = & & 9 & b_{r-2} & \cdots & b_1 & 8 & a_1 & \cdots & a_1 & 9, \end{array}$$

where the integer $9b_{r-2}\cdots b_1$ is obtained by adding 1 to $9a_1 \cdots a_1$. Then,

$$\begin{aligned} [10_{r-2}1 \times n]^* &= 9a_1 \cdots a_1 8 b_1 \cdots b_{r-2}9 \\ &= 9a_1 \cdots a_1 9 \times 10^{r-1} - (10^{r-1} - b_1 \cdots b_{r-2}9) \\ &= 10^{r-1} \times n - (10^{r-1} - b_1 \cdots b_{r-2}9), \end{aligned}$$

which is not divisible by n, since $10^{r-1} - b_1 \cdots b_{r-2}9$ is positive and smaller than n. We suppose now that $a_i = 9$ for all i; that is, $n = 9_r = 99\cdots 9$ (r times). We will prove then that n divides $m = 75_{r-2}464_{r-1}6$ but does not divide m^*. Indeed, for any integer $l = \cdots a_k a_{k-1} \cdots a_1 a_0 = \sum 10^i a_i$, since $9_r = 10^r - 1$, we have

$$\begin{aligned} l &\equiv a_{r-1}a_{r-2}\cdots a_0 + a_{2r-1}a_{2r-2}\cdots a_r + \cdots \pmod{9_r} \\ &\equiv \sum_{i \in r\mathbb{N}} a_{i+r-1}a_{i+r-2}\cdots a_i \pmod{9_r}. \end{aligned}$$

In particular, if l is a $(2r+1)$-digit number; that is, $l = a_{2r}a_{2r-1}\cdots a_1 a_0$, then

$$l \equiv a_{2r} + a_{2r-1}a_{2r-2}\cdots a_r + a_{r-1}a_{r-2}\cdots a_0 \pmod{9_r}.$$

Then, for $l = m = 75_{r-2}464_{r-1}6$, we obtain:

$$m = 75_{r-2}464_{r-1}6 \equiv 7 + 5_{r-2}46 + 4_{r-1}6 \pmod{9_r}$$

and

$$m^* = 64_{r-1}645_{r-2}7 \equiv 6 + 4_{r-1}6 + 45_{r-2}7 \pmod{9_r}.$$

Since $7 + 5_{r-2}46 + 4_{r-1}6 = 9_r$ and $6 + 4_{r-1}6 + 45_{r-2}7 = 90_{r-2}9$, which is positive and smaller than 9_r, it follows that 9_r divides m but does not divide m^*. □

3.2. The Case of Reverse Divisors

For the following proposition (Proposition 4), we need the four following lemmas.

Lemma 1. *No reverse divisor with quotient 4 has the property P^*.*

Proof. Let n be a reverse divisor. Then, according to Theorem 1, it has at least 4 digits and by Proposition 2, we can assume that n is written as follows:

$$n = a_k \cdots a_0 \text{ where } k \geq 3, a_k \neq 0 \text{ and } a_0 \neq 0.$$

If n is a reverse divisor with quotient 4, then $n^* = a_0 \cdots a_k = 4 \times a_k \cdots a_0$. Then, we necessarily have $a_k = 1$ or 2 and $a_0 \geq 4$. But if a_k is the units digit of $4 \times a_0$, then $4 \times a_0 \equiv 1$ or 2 (mod 10). Since $4 \times a_0 \equiv 1 \pmod{10}$ is impossible, then $4 \times a_0 \equiv 2 \pmod{10}$ and finally, $a_0 = 8$. By Proposition 2, the integer n, which is even, does not have the property P^*. □

Lemma 2. *No basic reverse divisor with quotient* 9 *has the property* P^**.*

Proof. By Theorem 1, a basic reverse divisor with quotient 9 is of the form

$$S_r = 109_r89.$$

If $r = 0$, that is, $S_r = S_0 = 1089$, it is easy to verify that S_0 does not divide $(11 \times S_0)^*$. Assume now that $r \geq 1$. We will prove that S_r does not divide $(10_r1 \times S_r)^*$. Indeed, $10_r1 \times S_r = 10_r1 \times 109_r89 = 110_r989_{r-1}89$. To show that $S_r = 109_r89$ does not divide $(10_r1 \times S_r)^* = 989_{r-1}890_r11$, observe that $90_{r+1} \times S_r = (10_r1 \times S_r)^* + S_{r-1}$ and S_{r-1} is positive and smaller than S_r. □

Remark 3. By the proof of Lemma 2, $(10_r1 \times S_r)^* = 90_{r+1} \times S_r - S_{r-1}$. On the other hand, $S_r = 10 \times S_{r-1} + 99$, and then $0 < \dfrac{S_{r-1}}{S_r} < \dfrac{1}{10}$. Therefore, the quotient $(10_r1 \times S_r)^* \div S_r$, which is not an integer, verifies

$$89_{r+1}9 < (10_r1 \times S_r)^* \div S_r < 90_{r+1}.$$

For example, if $r = 2$, that is, $S_r = 109989$, the quotient $(1001 \times S_r)^* \div S_r$ is

$$8999.900090009000 \cdots .$$

Lemma 3. *Let* n *be any non-basic reverse divisor. Then* n *begins and ends with* S_r *for some* r*.*

a) If $r = 0$*, then* $(11 \times n)^*$ *begins with* 97 *and ends with* 11*.*

b) If $r \geq 1$*, then* $(10_r1 \times n)^*$ *begins with* $989_{r-1}8$ *and ends with* 0_r11*.*

Lemma 4. $9_r89 \times 9_{r+2}$ *ends with* 0_r11*.*

Proof. Indeed, $9_r89 \times 9_{r+2} = 9_r89 \times 10^{r+2} - 9_r89$, which ends with 0_r11. □

Proposition 4. *No reverse divisor has the property* P^**.*

Proof. Lemmas 1 and 2 show that there is no reverse divisor with quotient 4 or basic reverse divisor with quotient 9 which satisfies the property P^*. It remains to examine the case of non-basic reverse divisors with quotient 9. Let n be a non-basic reverse divisor. Then, by Theorem 2, n begins and ends with S_r for some r and we can apply Lemma 3. Suppose that n has the property P^* and let s be the number

of its digits.
`Case 1`: $r = 0$. Then $(11 \times n)^*$ ends with 11. Since $11 \times n$ and $(11 \times n)^*$ are $(s+1)$-digit numbers, $(11 \times n)^* = k \times n$ where k is a 2-digit number. Let $k = a_2a_1$. Since $(11 \times n)^*$ ends with 11, we obtain $a_1 = a_2 = 9$. But $99 \times n > (11 \times n)^*$ since $(11 \times n)^*$ has $s + 1$ digits and $99 \times n = 10^2 \times n - n$ has $s + 2$ digits.
`Case 2`: $r \geq 1$. Then $(10_r1 \times n)^*$ ends with 0_r11. Since $10_r1 \times n$ (and then $(10_r1 \times n)^*$) is an $(s + r + 2)$-digit number, then $(10_r1 \times n)^* = k \times n$ where k is an $(r+2)$-digit number. Let $k = a_{r+2}a_{r+1} \cdots a_1$. Since, by Lemma 3, $(10_r1 \times n)^*$ ends with 0_r11, the $(r + 2)$-tuple $(a_1, ..., a_{r+2})$ satisfies a system of $(r + 2)$ congruences modulo 10 which has a unique solution. On the other hand, by Lemma 4, $9_r89 \times 9_{r+2}$ ends with 0_r11. Then $(9, ..., 9)$ is this solution; that is, $k = 9_{r+2}$. But clearly, $9_{r+2} \times n$ is not equal to $(10_r1 \times n)^*$, since $(10_r1 \times n)^*$ is an $(s + r + 1)$-digit number while $9_{r+2} \times n = 10^{r+2}n - n$ is an $(s + r + 2)$-digit number. □

4. On Two Questions

In Theorem 3, we proved that a positive integer n satisfies the property P^* if and only if n divides 99. In this section, we first discuss the similar property when the reverse of a multiple of n is taken in a given numerical base B. Secondly, in the decimal expansion, we replace the condition "n divides the reverse of any of its multiples" by "n divides the reverses of infinitely many of its multiples ".

4.1. The Problem in a Given Base

Let $B \geq 2$ be a number base. Extend the notion of reverse and the property P^* as follows.

- For every positive integer $m = a_kB^k + \cdots + a_1B + a_0$, we call the integer

$$m^*_B = a_0B^k + a_1B^{k-1} + \cdots + a_k.$$

  the *reverse of* m *in base* B

- A positive integer n is said to have the property P^*_B if n divides m^*_B for any positive multiple m of n, that is, if for every positive integer m, if n divides m, then n divides m^*_B.

Notice that the property P^*_{10} is the property P^* and that, by Theorem 3, a positive integer n has the property P^*_{10} if and only if n divides $10^2 - 1$.

Proposition 5. *Let* $B \geq 2$ *be a number base and* n *be a positive divisor of* $B^2 - 1$. *Then* n *has the property* P^*_B.

Proof. Let $m = a_0 + a_1B + \cdots + a_kB^k$ be any positive integer. Then $m_B^* = a_k + a_{k-1}B + \cdots + a_0B^k$. Since $B^2 \equiv 1 \pmod{n}$, we have:

$$m = a_0 + a_1B + \cdots + a_kB^k \equiv \sum_{0 \le i \le \frac{k}{2}} a_{2i} + B \sum_{0 \le i < \frac{k}{2}} a_{2i+1} \pmod{n}. \tag{1}$$

We then obtain:

$$m_B^* \equiv \begin{cases} \sum_{0 \le i \le \frac{k}{2}} a_{2i} + B \sum_{0 \le i < \frac{k}{2}} a_{2i+1} \pmod{n} & \text{if } k \text{ even,} \\ B \sum_{0 \le i \le \frac{k}{2}} a_{2i} + \sum_{0 \le i < \frac{k}{2}} a_{2i+1} \pmod{n} & \text{if } k \text{ odd.} \end{cases} \tag{2}$$

Suppose that n divides m. We will prove that n divides m_B^*.

`Case 1`: k is even. Then by (1) and (2) we have $m \equiv m_B^* \pmod{n}$, and since n divides m, it divides m_B^*.

`Case 2`: k is odd. Since n divides m, then by (1), $\sum a_{2i} \equiv -B \sum a_{2i+1} \pmod{n}$. Then, by (2), we obtain:

$$\begin{aligned} m_B^* &\equiv B(-B \sum_{0 \le i < \frac{k}{2}} a_{2i+1}) + \sum_{0 \le i < \frac{k}{2}} a_{2i+1} \pmod{n} \\ &\equiv -(B^2 - 1) \sum_{0 \le i < \frac{k}{2}} a_{2i+1} \pmod{n}, \end{aligned}$$

and since n divides $B^2 - 1$, it divides m_B^*. □

Problem 1. Let $B \ge 2$ be any number base. Are the divisors of $B^2 - 1$ the only positive integers satisfying the property P_B^*?

4.2. Weakening the Property in the Decimal Expansion

We have seen that only the divisors of 99 satisfy the property P^*. Instead of asking for integers n which divide the reverses of their multiples, we weaken the conditions of P^* and look for integers n which divide the reverses of infinitely many of their multiples. We are unable to completely answer this question. However, we state the following.

Proposition 6. *Let $n \ge 1$ be an integer, $n \notin 10\mathbb{N}$, such that there exists an integer $k_0 \ge 1$ with $n \mid (k_0 \times n)^*$. Then there exist infinitely many integers $k \ge 1$ such that $n \mid (k \times n)^*$. Moreover, the stated conditions are satisfied if* $\gcd(n, 10) = 1$ *or n is a palindrome or a reverse divisor.*

Proof. Set $k_0 \times n = a_r \cdots a_1$ and let $K = a_r \cdots a_1 a_r \cdots a_1 \cdots a_r \cdots a_1$ be its concatenation as many times as desired. Then obviously $n \mid K$, so that $K = k \times n$ for

some integer $k \geq 1$ and $n \mid K^*$, thus $n \mid (n \times k)^*$.If $\gcd(n, 10) = 1$, let e be the order of $\overline{10}$ in $(\mathbb{Z}/n\mathbb{Z})^*$, then $10^e - 1 = nk_0$ where k_0 is a positive integer. This shows that $k_0 \times n = 9_e$ and thus n divides $(k_0 \times n)^* = k_0 \times n = 9_e$.
If n is a palindrome or a reverse divisor, one may take $k_0 = 1$. □

Remark 4. If $\gcd(n, 10) = 1$, one may give an alternative proof not using k_0. With the same meaning of e and for any positive integer d, we have $10^{ed} - 1 = nk(d)$ where $k(d)$ is a positive integer and this proves the result.

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