#A38 INTEGERS 25 (2025)

THE r-FIBONACCI POLYNOMIAL AND ITS COMPANION SEQUENCES LINKED WITH SOME CLASSICAL SEQUENCES

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Received: 9/15/22, Revised: 9/22/24, Accepted: 3/29/25, Published: 4/25/25

Abstract

The aim of the present work is to provide some properties of the r-Fibonacci polynomial and its companion sequences. We present the Binet forms for both cases: distinct and multiple zeros of the characteristic polynomial. Then, we define their (p, q)-analogues and present the alternating sums of their terms using a matrix method. In addition, we extend the definition of the r-Fibonacci polynomial to include negative r values, we focus on the behavior of the resulting sequence, which is a solution of a recurrence relation, and establish its relationship with the generalized Padovan and Perrin numbers.

1. Introduction

The present paper is a continuation of our recent study [1] of the r-Fibonacci polynomial and its companion sequences. In [12], Raab introduced the r-Fibonacci bivariate polynomial sequence $(U_n^{(r)}(x, y))_n$ by the following recursion:

$$\begin{cases} U_0^{(r)} = 0,\ U_k^{(r)} = x^{k-1}\ (1 \leq k \leq r), \\ U_{n+1}^{(r)} = xU_n^{(r)} + yU_{n-r}^{(r)}\ (n \geq r), \end{cases}$$

where r is a positive integer, and x and y are two variables. Abbad et al. [1] defined a family of companion sequences $(V_n^{(r,s)})_{n\geq 0}$ indexed by s $(1 \leq s \leq r)$ as follows:

DOI: 10.5281/zenodo.15283706

$$\begin{cases} V_0^{(r,s)} = s+1,\ V_k^{(r)} = x^k\ (1 \le k \le r), \\ V_{n+1}^{(r,s)} = xV_n^{(r,s)} + yV_{n-r}^{(r,s)}\ (n \ge r). \end{cases}$$

The sequence $(V_n^{(r,s)})$ is called the r-Lucas polynomial of type s. The sequences $(V_n^{(r,s)})_n$ and $(U_n^{(r)})_n$ are linked, for $n \ge r$, by the following relation:

$$V_n^{(r,s)} = U_{n+1}^{(r)} + syU_{n-r}^{(r)}. \tag{1}$$

For more details about the r-Fibonacci polynomial and its companion sequences, see [1]. This paper is structured as follows. Section 2 presents the Binet formula for the r-Fibonacci polynomial and its companion sequences. Section 3 introduces (p, q)-analogues of these sequences. Section 4 employs a matrix method to evaluate the alternating sum of the r-Fibonacci polynomial and its companion sequences. Finally, the definition of the r-Fibonacci polynomial is extended to negative r values.

2. Binet Type Formulas

In this section, we present the Binet type formulas corresponding to the r-Fibonacci sequence and to the companion r-Lucas sequences.

Let $P(t) = t^{r+1} - xt^r - y = \prod_{j=1}^{h} (t - \alpha_j)^{r_j}$ be the characteristic polynomial of the sequences $(U_n^{(r)})_{n\ge 0}$ and $(V_n^{(r,s)})_{n\ge 0}$, where $\alpha_1, \ldots, \alpha_h$ are the zeros of P and r_j is the multiplicity of α_j for $1 \le j \le h$ such that $\sum_{j=1}^{h} r_j = r + 1$. Raab [12] showed that $\alpha = rx/(r + 1)$ is a real multiple zero of P and that the greatest multiplicity of any real zero is two. DeGua's rule for identifying imaginary roots states that if $2m$ consecutive terms of an equation are missing, the equation has $2m$ imaginary roots. Similarly, when $2m-1$ consecutive terms are missing, the equation has either $2m-2$ or $2m$ imaginary roots, depending on whether the two terms between which the $2m-1$ terms are missing have like or unlike signs. Consequently, we can deduce that the polynomial P has at most three real zeros. This is due to the presence of $(r-1)$ consecutive missing terms in the expression of the linear recurrence relation, which implies that at least $(r - 2)$ of its zeros are imaginary. Additionally, α is a multiple zero of P if and only if the discriminant of P equals zero, leading to $y = (-1/r)\,(rx/(r + 1))^{r+1}$. As a result, the characteristic polynomial P can be split in the following way:

$$P(t) = (t - \alpha_1)(t - \alpha_2)\cdots(t - \alpha_{r+1}) \text{ if } y \ne (-1/r)\,(rx/(r + 1))^{r+1},$$

or

$$P(t) = (t - \alpha)^2 \widetilde{P}(t) \text{ if } y = (-1/r)\,(rx/(r + 1))^{r+1}, \text{ where } \widetilde{P}(\alpha) \ne 0.$$

Theorem 1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ be the zeros of the characteristic polynomial $P(t) = t^{r+1} - xt^r - y$ associated with $(U_n^{(r)})_{n\geq 0}$ and $(V_n^{(r,s)})_{n\geq 0}$. Then for $1 \leq s \leq r$, we have the following*

(i) If $y \neq (-1/r)\,(rx/r+1)^{r+1}$,

$$U_{n+1}^{(r)} = \sum_{k=1}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx} \quad \text{and} \quad V_n^{(r,s)} = \sum_{k=1}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}.$$

(ii) If $y = (-1/r)\,(rx/r+1)^{r+1}$,

$$U_{n+1}^{(r)} = \frac{\alpha^{n+r-1}}{\widetilde{P}(\alpha)} \left((n+r) - \alpha \frac{\widetilde{P}'(\alpha)}{\widetilde{P}(\alpha)} \right) + \sum_{k=1, \alpha_k \neq \alpha}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx}$$

and

$$V_n^{(r,s)} = \frac{\alpha^{n+r-1}}{\widetilde{P}(\alpha)} \left(n(\frac{r-s}{r}) + (r + \frac{s}{r}) - \alpha(\frac{r-s}{r}) \frac{\widetilde{P}'(\alpha)}{\widetilde{P}(\alpha)} \right) + \sum_{k=1, \alpha_k \neq \alpha}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}.$$

Proof. (i) As noted in [8], the general term of the sequence $(U_n^{(r)})_{n\geq 0}$ can be expressed as $U_{n+1}^{(r)} = \sum_{k=1}^{r+1} b_k \alpha_k^n$, where the b_k are rational numbers and the α_k are the zeros of the characteristic polynomial. This system of equations can be solved using Cramer's rule and the Vandermonde determinant. To find the coefficients b_k, we utilize the first $(r+1)$ terms of the sequence $(U_n^{(r)})$ and the symmetric functions of the zeros of the characteristic polynomial. The result is:

$$\begin{aligned} b_k &= \frac{\alpha_k^r}{(\alpha_k - \alpha_1)(\alpha_k - \alpha_2)\cdots(\alpha_k - \alpha_{k-1})(\alpha_k - \alpha_{k+1})\cdots(\alpha_k - \alpha_{r+1})} \\ &= \frac{\alpha_k^r}{\prod_{j\neq k} (\alpha_k - \alpha_j)}. \end{aligned}$$

On the other hand, we notice that

$$P(t) = t^{r+1} - xt^r - y = (t - \alpha_1)(t - \alpha_2)\cdots(t - \alpha_{r+1}).$$

Then for $1 \leq k \leq r+1$, we have

$$P'(\alpha_k) = (r+1)\alpha_k^r - rx\alpha_k^{r-1} = \prod_{j\neq k} (\alpha_k - \alpha_j),$$

which gives

$$b_k = \frac{\alpha_k^r}{(r+1)\alpha_k^r - rx\alpha_k^{r-1}} = \frac{\alpha_k^r}{\alpha_k^{r-1}((r+1)\alpha_k - rx)} = \frac{\alpha_k}{(r+1)\alpha_k - rx}.$$

Thus

$$U_{n+1}^{(r)} = \sum_{k=1}^{r+1} b_k \alpha_k^n = \sum_{k=1}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx}.$$

Now, we determine the Binet type formula for the sequence of polynomials $(V_n^{(r,s)})_{n\geq 0}$. Using Equation (1), we have

$$\begin{aligned} V_n^{(r,s)} &= U_{n+1}^{(r)} + syU_{n-r}^{(r)} \\ &= \sum_{k=1}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx} + sy\sum_{k=1}^{r+1} \frac{\alpha_k^{n-r}}{(r+1)\alpha_k - rx} \\ &= \sum_{k=1}^{r+1} \frac{\alpha_k^{n-r}(\alpha_k^{r+1} + sy)}{(r+1)\alpha_k - rx}. \end{aligned}$$

Now, since $y = \alpha_k^{r+1} - x\alpha_k^r$, it follows that

$$V_n^{(r,s)} = \sum_{k=1}^{r+1} \frac{\alpha_k^{n-r}((s+1)\alpha_k^{r+1} - sx\alpha_k^r)}{(r+1)\alpha_k - rx} = \sum_{k=1}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}.$$

(ii) Assuming that P has a multiple zero (let $\alpha_1 = \alpha_2 = \alpha$) and using the same technique as in (i), we have

$$\begin{aligned} U_{n+1}^{(r)} &= \sum_{k=1}^{r+1} b_k\alpha_k^n = b_1\alpha_1^n + b_2\alpha_2^n + \sum_{k=3}^{r+1} b_k\alpha_k^n \\ &= \frac{\alpha_1^{n+r}}{(\alpha_1 - \alpha_2)\widetilde{P}(\alpha_1)} + \frac{\alpha_2^{n+r}}{(\alpha_2 - \alpha_1)\widetilde{P}(\alpha_2)} \\ &\quad + \sum_{k=3}^{r+1} \frac{\alpha_k^{n+r}}{(\alpha_k - \alpha_1)(\alpha_k - \alpha_2)\cdots(\alpha_k - \alpha_{k-1})(\alpha_k - \alpha_{k+1})\cdots(\alpha_k - \alpha_{r+1})} \\ &= \frac{\alpha_1^{n+r}}{(\alpha_1 - \alpha_2)\widetilde{P}(\alpha_1)} - \frac{\alpha_2^{n+r}}{(\alpha_1 - \alpha_2)\widetilde{P}(\alpha_2)} + \sum_{\alpha_k \neq \alpha}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx} \\ &= \frac{1}{\widetilde{P}(\alpha_1)}\left[\frac{\alpha_1^{n+r} - \alpha_2^{n+r}}{\alpha_1 - \alpha_2} - \left(\frac{\widetilde{P}(\alpha_1)}{\widetilde{P}(\alpha_2)} - 1\right)\frac{\alpha_2^{n+r}}{\alpha_1 - \alpha_2}\right] + \sum_{\alpha_k \neq \alpha}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx} \\ &= \frac{1}{\widetilde{P}(\alpha_1)}\left[\frac{\alpha_1^{n+r} - \alpha_2^{n+r}}{\alpha_1 - \alpha_2} - \left(\frac{\widetilde{P}(\alpha_1) - \widetilde{P}(\alpha_2)}{\alpha_1 - \alpha_2}\right)\frac{\alpha_2^{n+r}}{\widetilde{P}(\alpha_2)}\right] + \sum_{\alpha_k \neq \alpha}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx}. \end{aligned}$$

To end the proof, let α_2 tend to $\alpha_1 = \alpha$. For the companion sequences $(V_n^{(r,s)})_{n\geq 0}$, we apply Equation (1) with $y = -\dfrac{\alpha^{r+1}}{r}$. □

3. The (p, q)-Analogue of the r-Fibonacci Polynomial and Its Companion Sequences

For $p, q \in \mathbb{R}$, the (p, q)-numbers are defined as:

$$[n]_{p,q} := p^{n-1} + p^{n-2}q + p^{n-2}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q},$$

$$[n]_{p,q}! := [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q}.$$

Also, we have

$$[n]_{p,q} = p^{n-k}[k]_{p,q} + q^k[n-k]_{p,q},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}, \tag{2}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}. \tag{3}$$

The theory of (p, q)-calculus has been studied by many mathematicians. Corcino [11] studied the (p, q)-extension of the binomial coefficients and derived some properties similar to those of ordinary and q-binomial coefficients. In [3], Bazeniar et al. gave an interpretation of generalized binomial coefficients and their (p, q)-analogue using a new type of symmetric function. According to Ahmia and Belbachir [2], the log-convexity is preserved under the (p, q)-binomial transformation. In this section, we propose the (p, q)-analogue of the r-Fibonacci polynomial and its companion sequences associated with the unified approach of Cigler and Carlitz [7, 10]. Belbachir et al. [5] introduced the generalized q-analogue of r-Fibonacci polynomials $\mathbf{U}_{n+1}^{(r)}(z, m)$, which is a unified approach of those of Carlitz and Cigler [7, 10]. They define

$$\mathbf{U}_{n+1}^{(r)}(z, m) := \sum_{k=0}^{\lfloor n/(r+1) \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n - rk \\ k \end{bmatrix}_q z^k,$$

with $\mathbf{U}_0^{(r)}(z,m)=0$. These polynomials satisfy the following recurrence formulas:

$$\mathbf{U}_{n+1}^{(r)}(z,m)=\mathbf{U}_n^{(r)}(qz,m)+qz\mathbf{U}_{n-r}^{(r)}(zq^{m+1},m)$$

and

$$\mathbf{U}_{n+1}^{(r)}(z,m)=\mathbf{U}_n^{(r)}(z,m)+q^{n-r}z\mathbf{U}_{n-r}^{(r)}(zq^{m-r},m).$$

Also, Benmezai proposed the (p,q)-analogue of the r-Fibonacci polynomial associated with the Cigler approach [6]. Inspired by the (p,q)-binomial definition given in [11], we propose the following definition for the (p,q)-analogue of r-Fibonacci polynomials.

Definition 1. For all $n\geq 0$, the (p,q)-analogue of the r-Fibonacci polynomial is defined as

$$\mathbf{U}_{n+1}^{(r)}(x,y,p,q,m):=\sum_{k=0}^{\lfloor n/(r+1)\rfloor}p^{\binom{n+1-(r+1)k}{2}}q^{\binom{k+1}{2}+m\binom{k}{2}}\begin{bmatrix}n-rk\\k\end{bmatrix}_{p,q}x^{n-(r+1)k}y^k,$$

with $\mathbf{U}_0^{(r)}=0$ and $\mathbf{U}_j^{(r)}=p^{\binom{j+1}{2}}x^{j-1}$ for $1\leq j\leq r$.

By setting $p=1$, we derive some particular cases of the (p,q)-analogue of the r-Fibonacci polynomial. This includes the q-analogue presented in [5] and the q-analogue introduced by Cigler in [9], where $r=1$ and $m=0$.

Theorem 2. *The (p,q)-analogue of the r-Fibonacci polynomials satisfy the following recurrence formulas:*

$$\boldsymbol{U}_{n+1}^{(r)}(x,y,p,q,m)=px\,\boldsymbol{U}_n^{(r)}(px,qy,p,q,m)+qy\,\boldsymbol{U}_{n-r}^{(r)}(px,q^{m+1}y,p,q,m)\tag{4}$$

and

$$\boldsymbol{U}_{n+1}^{(r)}(x,y,p,q,m)=px\,\boldsymbol{U}_n^{(r)}(px,py,p,q,m)+qy\,\boldsymbol{U}_{n-r}^{(r)}(qx,q^{m+1}y,p,q,m).\tag{5}$$

Proof. We use Equation (2) to prove the first identity. We have

$$\begin{aligned}&\mathbf{U}_{n+1}^{(r)}(x,y,p,q,m)\\&=\sum_{k=0}^{\lfloor n/(r+1)\rfloor}p^{\binom{n+1-(r+1)k}{2}}q^{\binom{k+1}{2}+m\binom{k}{2}}\\&\qquad\times\left(q^k\begin{bmatrix}n-rk-1\\k\end{bmatrix}_{p,q}+p^{n-(r+1)k}\begin{bmatrix}n-rk-1\\k-1\end{bmatrix}_{p,q}\right)x^{n-(r+1)k}y^k\\&=\quad px\sum_{k=0}^{\lfloor n/(r+1)\rfloor}p^{\binom{n-(r+1)k}{2}}q^{\binom{k+1}{2}+m\binom{k}{2}}\begin{bmatrix}n-rk-1\\k\end{bmatrix}_{p,q}(px)^{n-(r+1)k-1}(qy)^k\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n+1-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} p^{n-(r+1)k} \begin{bmatrix} n-rk-1 \\ k-1 \end{bmatrix}_{p,q} x^{n-(r+1)k} y^k \\
&= px\mathbf{U}_n^{(r)}(px, qy, p, q, m) + \sum_{k=1}^{\lfloor n/(r+1)\rfloor} p^{\binom{n+1-(r+1)(k+1)}{2}} q^{\binom{k+2}{2}+m\binom{k+1}{2}} p^{n-(r+1)(k+1)} \\
&\quad \times \begin{bmatrix} n-r(k+1)-1 \\ k \end{bmatrix}_{p,q} x^{n-(r+1)(k+1)} y^{k+1} \\
&= px\mathbf{U}_n^{(r)}(px, qy, p, q, m) + qy \sum_{k=1}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-r-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} \\
&\quad \times \begin{bmatrix} n-r-1-rk \\ k \end{bmatrix}_{p,q} (px)^{n-r-1-(r+1)k} (q^{m+1}y)^k \\
&= px\mathbf{U}_n^{(r)}(px, qy, p, q, m) + qy\mathbf{U}_{n-r}^{(r)}(px, q^{m+1}y, p, q, m).
\end{aligned}
$$

Now we use Equation (3) to prove the second identity. We have

$$
\begin{aligned}
&\mathbf{U}_{n+1}^{(r)}(x, y, p, q, m) \\
&= \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n+1-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} \\
&\quad \times \left(p^k \begin{bmatrix} n-rk-1 \\ k \end{bmatrix}_{p,q} + q^{n-(r+1)k} \begin{bmatrix} n-rk-1 \\ k-1 \end{bmatrix}_{p,q} \right) x^{n-(r+1)k} y^k \\
&= px \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-rk-1 \\ k \end{bmatrix}_{p,q} (px)^{n-(r+1)k-1} (py)^k \\
&\quad + \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n+1-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} q^{n-(r+1)k} \begin{bmatrix} n-rk-1 \\ k-1 \end{bmatrix}_{p,q} x^{n-(r+1)k} y^k, \\
&= px\mathbf{U}_n^{(r)}(px, py, p, q, m) + \sum_{k=1}^{\lfloor n/(r+1)\rfloor} p^{\binom{n+1-(r+1)(k+1)}{2}} q^{\binom{k+2}{2}+m\binom{k+1}{2}} q^{n-(r+1)(k+1)} \\
&\quad \times \begin{bmatrix} n-r(k+1)-1 \\ k \end{bmatrix}_{p,q} x^{n-(r+1)(k+1)} y^{k+1} \\
&= px\mathbf{U}_n^{(r)}(px, py, p, q, m) + qy \sum_{k=1}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-r-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} \\
&\quad \times \begin{bmatrix} n-r-1-rk \\ k \end{bmatrix}_{p,q} (qx)^{n-r-1-(r+1)k} (q^{m+1}y)^k \\
&= px\mathbf{U}_n^{(r)}(px, py, p, q, m) + qy\mathbf{U}_{n-r}^{(r)}(qx, q^{m+1}y, p, q, m).
\end{aligned}
$$

□

Definition 2. Let r and s be positive integers such that $1 \leq s \leq r$. For $n \geq 0$, we define the (p,q)-analogue of the r-Lucas polynomials of type s of the *first kind* and the *second kind*, respectively, as follows:

$$\mathbf{V}_n^{(r,s)}(x,y,p,q,m) := \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-(r+1)k}{2}} q^{(m+1)\binom{k}{2}} \begin{bmatrix} n-rk \\ k \end{bmatrix}_{p,q} \times \left(1 + \frac{sp^{n-(r+1)k}[k]_{p,q}}{[n-rk]_{p,q}}\right) x^{n-(r+1)k} y^k$$

and

$$\mathbb{V}_n^{(r,s)}(x,y,p,q,m) := \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-rk \\ k \end{bmatrix}_{p,q} \times p^{-k} \left(1 + \frac{sq^{(n-(r+1)k)}[k]_{p,q}}{[n-rk]_{p,q}}\right) x^{n-(r+1)k} y^k,$$

with $\mathbf{V}_0^{(r,s)}(x,y,p,q,m) = \mathbb{V}_0^{(r,s)}(x,y,p,q,m) = s+1$.

Remark 1. Note that for $p = 1$, we obtain the q-analogue of the r-Lucas polynomial of type s; see [1].

Let us now establish some links with the initial r-Fibonacci polynomial.

Theorem 3. *For positive integers r and s, the polynomials $\mathbf{V}_n^{(r,s)}(x,y,p,q,m)$ and $\mathbb{V}_n^{(r,s)}(x,y,p,q,m)$ satisfy the following recursions:*

1. *(Expression of $V_n^{(r,s)}$'s in terms of $\mathbf{U}_{n+1}^{(r)}$ and $\mathbf{U}_{n-r}^{(r)}$ without weight)*

$$\mathbf{V}_n^{(r,s)}(x,y,p,q,m) = \mathbf{U}_{n+1}^{(r)}(x/p,y/q,p,q,m) + sy\mathbf{U}_{n-r}^{(r)}(x,yq^m,p,q,m),$$

$$\mathbb{V}_n^{(r,s)}(x,y,p,q,m) = \mathbf{U}_{n+1}^{(r)}(x/p,y/p,p,q,m) + sy(q/p)^{n-r}\mathbf{U}_{n-r}^{(r)}(x,yp^rq^{m-r},p,q,m);$$

2. *(Expression of $V_n^{(r,s)}$'s in terms of $\mathbf{U}_{n+1}^{(r)}$ and $\mathbf{U}_n^{(r)}$ weighted by s)*

$$\mathbf{V}_n^{(r,s)}(x,y,p,q,m) = (s+1)\mathbf{U}_{n+1}^{(r)}(x/p,y/q,p,q,m) - sx\mathbf{U}_n^{(r)}(x,y,p,q,m),$$

$$\mathbb{V}_n^{(r,s)}(x,y,p,q,m) = (s+1)\mathbf{U}_{n+1}^{(r)}(x/p,y/p,p,q,m) - sx\mathbf{U}_n^{(r)}(x,y,p,q,m);$$

3. *(Expression of $V_n^{(r,s)}$'s in terms of $\mathbf{U}_n^{(r)}$ and $\mathbf{U}_{n-r}^{(r)}$)*

$$\mathbf{V}_n^{(r,s)}(x,y,p,q,m) = x\mathbf{U}_n(x,y,p,q,m) + (1+s)y\mathbf{U}_{n-r}(x,yq^m,p,q,m),$$

$$\mathbb{V}_n^{(r,s)}(x,y,p,q,m) = x\mathbf{U}_n^{(r)}(x,y,p,q,m) + (1+s)(q/p)^{n-r}y\mathbf{U}_{n-r}^{(r)}(x,yp^rq^{m-r},m).$$

Proof. We prove the first two relations. The other identities are proved in the same way. We have

$$\begin{aligned}
&\mathbf{V}_n^{(r,s)}(x,y,p,q,m)\\
&= \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-(r+1)k}{2}} q^{\binom{k}{2}(m+1)} \begin{bmatrix} n-rk \\ k \end{bmatrix}_{p,q} x^{n-(r+1)k} y^k \\
&\quad + s \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-(r+1)k}{2}+(n-(r+1)k)} q^{\binom{k}{2}(m+1)} \begin{bmatrix} n-rk-1 \\ k-1 \end{bmatrix}_{p,q} x^{n-(r+1)k} y^k \\
&= \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n+1-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-rk \\ k \end{bmatrix}_{p,q} (x/p)^{n-(r+1)k} (y/q)^k \\
&\quad + s \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n+1-(r+1)(k+1)}{2}} q^{\binom{k+1}{2}(m+1)} \begin{bmatrix} n-r(k+1)-1 \\ k \end{bmatrix}_{p,q} x^{n-(r+1)(k+1)} y^{k+1} \\
&= \mathbf{U}_{n+1}^{(r)}(x/p,y/q,p,q,m) \\
&\quad + sy \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-r-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-r-1-rk \\ k \end{bmatrix}_{p,q} x^{n-r-1-rk} (q^m y)^k \\
&= \mathbf{U}_{n+1}^{(r)}(x/p,y/q,p,q,m) + sy\mathbf{U}_{n-r}^{(r)}(x,q^m y,p,q,m).
\end{aligned}$$

For the second identity, we have

$$\begin{aligned}
&\mathbb{V}_n^{(r,s)}(x,y,p,q,m)\\
&= \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} p^{-k} \begin{bmatrix} n-rk \\ k \end{bmatrix}_{p,q} x^{n-(r+1)k} y^k \\
&\quad + s \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} p^{-k} q^{(n-(r+1)k)} \begin{bmatrix} n-rk-1 \\ k-1 \end{bmatrix}_{p,q} x^{n-(r+)k} y^k \\
&= \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n+1-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-rk \\ k \end{bmatrix}_{p,q} (x/p)^{n-(r+1)k} (y/p)^k \\
&\quad + s \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} p^{-k} q^{(n-(r+1)k)} \begin{bmatrix} n-rk-1 \\ k-1 \end{bmatrix}_{p,q} x^{n-(r+1)k} y^k \\
&= \mathbf{U}_{n+1}^{(r)}(x/p,y/p,p,q,m) + sy \sum_{k=0}^{\lfloor n/(r+1)\rfloor} p^{\binom{n-r-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}} q^{(m-r)k+n-r} \\
&\quad \times p^{-n+r(k+1)} \begin{bmatrix} n-r(k+1)-1 \\ k \end{bmatrix}_{p,q} x^{n-r-1-(r+1)k} y^k
\end{aligned}$$

$$= \mathbf{U}_{n+1}^{(r)}(x/p, y/p, p, q, m) + sy(q/p)^{n-r} \sum_{k=0}^{\lfloor n/(r+1) \rfloor} p^{\binom{n-r-(r+1)k}{2}} q^{\binom{k+1}{2}+m\binom{k}{2}}$$

$$\times \begin{bmatrix} n-r-1-rk \\ k \end{bmatrix}_{p,q} x^{n-r-1-(r+1)k} (p^r q^{m-r} y)^k$$

$$= \mathbf{U}_{n+1}^{(r)}(x/p, y/p, p, q, m) + sy(q/p)^{n-r} \mathbf{U}_{n-r}^{(r)}(x, yp^r q^{m-r}, p, q, m).$$

□

Using the previous identities, we show that the (p, q)-analogue of the r-Lucas polynomials of type s of the first and second kind satisfy the same recurrence relations as the (p, q)-analogue of the r-Fibonacci polynomial given in Relations (4) and (5).

Corollary 1. *The (p, q)-analogue of the r-Lucas polynomial of type s of the first and second kind satisfy the following recurrences, respectively:*

$$\mathbf{V}_{n+1}^{(r,s)}(x, y, p, q, m) = px\mathbf{V}_n^{(r,s)}(px, qy, p, q, m) + qy\mathbf{V}_{n-r}^{(r,s)}(px, q^{m+1}y, p, q, m)$$

and

$$\mathbb{V}_{n+1}^{(r,s)}(x, y, p, q, m) = px\mathbb{V}_n^{(r,s)}(px, py, p, q, m) + qy\mathbb{V}_{n-r}^{(r,s)}(qx, q^{m+1}y, p, q, m),$$

with $\mathbf{V}_0^{(r,s)}(x, y, p, q, m) = \mathbb{V}_0^{(r,s)}(x, y, p, q, m) = s + 1.$

4. The Alternating Sum of Finite Terms of the r-Fibonacci Polynomial and the Related Companion Sequences

To derive an explicit formula for the alternating sum of the r-Fibonacci polynomial terms and its companion sequences, we introduce a new sequence $\xi_n^{(r)} = m^{n-1}U_n^{(r)}$, where m is a positive integer. This sequence satisfies the recurrence relation

$$\xi_{n+1}^{(r)} = mx\xi_n^{(r)} + m^{r+1}y\xi_{n-r}^{(r)},$$

with initial conditions $\xi_0^{(r)} = 0, \xi_k^{(r)} = (mx)^{k-1}$ $(1 \le k \le r)$. Let $A_r(x, y)$ be the companion matrix of order $(r+1)$ associated with $\xi_n^{(r)}$:

$$A_r(x, y) := \begin{pmatrix} 0 & 0 & \cdots & 0 & m^{r+1}y \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & mx \end{pmatrix},$$

and its n-power

$$A_r^n(x,y)=\begin{pmatrix} m^{r+1}y\xi_{n-r}^{(r)} & y\xi_{n-r+1}^{(r)} & \cdots & m^{r+1}y\xi_{n-1}^{(r)} & m^{r+1}y\xi_{n}^{(r)} \\ m^{r+1}y\xi_{n-r-1}^{(r)} & y\xi_{n-r}^{(r)} & \cdots & m^{r+1}y\xi_{n-2}^{(r)} & m^{r+1}y\xi_{n-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m^{r+1}y\xi_{n-2r+1}^{(r)} & y\xi_{n-2r+2}^{(r)} & \cdots & m^{r+1}y\xi_{n-r}^{(r)} & m^{r+1}y\xi_{n-r+1}^{(r)} \\ \xi_{n-r+1}^{(r)} & \xi_{n-r+2}^{(r)} & \cdots & \xi_{n}^{(r)} & \xi_{n+1}^{(r)} \end{pmatrix}.$$

Using a matrix approach, we compute $S_n^{(r)}(m)$, the sum of the terms of the sequence $(\xi_n^{(r)})_n$, given by

$$S_n^{(r)}(m)=\sum_{j=1}^{n}\xi_j^{(r)}.$$

The following result is a generalization of Theorem 4 in [1]. The proof follows a similar approach.

Theorem 4. *Let $S_n^{(r)}(m)$ be the sum of the first n terms of the sequence $(\xi_k^{(r)})$, and let $P(t)=t^{r+1}-mxt^r-m^{r+1}y$ be its characteristic polynomial such that $P(1)\neq 0$. Then*

$$S_n^{(r)}(m)=\frac{1}{1-mx-m^{r+1}y}(1-\xi_{n+1}^{(r)}-m^{r+1}y\sum_{j=1}^{r}\xi_{n-r+j}^{(r)}). \tag{6}$$

Corollary 2 ([1, Theorem 4])**.** *The sum of the terms of $(U_n^{(r)}(x,y))_n$ is given by*

$$S_n^{(r)}(1)=\frac{1}{1-x-y}(1-U_{n+1}^{(r)}-y\sum_{j=1}^{r}U_{n-r+j}^{(r)}).$$

Corollary 3. *The alternating sum of the terms of $(U_n^{(r)}(x,y))_n$ is given by*

$$S_n^{(r)}(-1)=\frac{1}{1+x+(-1)^r y}(1-(-1)^n U_{n+1}^{(r)}+(-1)^r y\sum_{j=1}^{r}(-1)^{n-r+j-1}U_{n-r+j}^{(r)}).$$

Example 1. For $r=1$ and $(x,y)=(1,1)$, we obtain the alternating sum of Fibonacci numbers $(F_n)_{n\geq 0}$ (A000045 in the OEIS),

$$\sum_{j=1}^{n}(-1)^{j-1}F_j=1-(-1)^n F_{n-1}.$$

Example 2. For $r=1$ and $(x,y)=(2,1)$, the sequence $(U_n^{(r)})$ reduces to the usual Pell sequence $(P_n)_{n\geq 0}$ (A000129 in the OEIS). We have

$$\sum_{j=1}^{n}(-1)^{j-1}P_j=\frac{1}{2}(1+(-1)^{n-1}(P_{n+1}-P_n)).$$

Example 3. For $r = 2$ and $(x, y) = (1, 1)$, we obtain the 2-Fibonacci numbers $(T_n)_{n\geq 0}$ (A000930 in the OEIS), which satisfy the recursion $T_{n+1} = T_n + T_{n-2}$ with $T_0 = 0, T_1 = T_2 = 1$. Then

$$\sum_{j=1}^{n} T_j = (T_{n+3} - 1) \quad \text{and} \quad \sum_{j=1}^{n} (-1)^{j-1} T_j = \frac{1}{3}(1 - (-1)^n (T_{n+1} + T_{n-3})).$$

Now, we derive the expression for the sum of the terms of the companion sequences $(\eta_n^{(r,s)})$ of $(\xi_n^{(r)})$ defined by

$$S_n^{(r,s)}(m) = \sum_{j=1}^{n} \eta_j^{(r,s)} = \sum_{j=1}^{n} m^j V_j^{(r,s)}.$$

Theorem 5. *Let $S_n^{(r,s)}(m)$ be the sum of the first n terms of $(\eta_k^{(r,s)})$, and let $P(t) = t^{r+1} - mxt^r - m^{r+1}y$ be the corresponding characteristic polynomial such that $P(1) \neq 0$. Then*

$$S_n^{(r,s)}(m) = \frac{1}{1 - mx - m^{r+1}y}(1 + sm^{r+1}y - \eta_{n+1}^{(r,s)} - m^{r+1}y \sum\nolimits_{j=1}^{r} \eta_{n-r+j}^{(r,s)}) - 1.$$

Proof. According to Equation (1), the companion sequences $(\eta_n^{(r,s)})$ satisfy the relation

$$\eta_n^{(r,s)} = \xi_{n+1}^{(r)} + sm^{r+1}y\xi_{n-r}^{(r)}.$$

It follows that

$$\begin{aligned}
\sum\nolimits_{j=r}^{n} \eta_j^{(r,s)} &= \sum\nolimits_{j=r}^{n} (\xi_{j+1}^{(r)} + sm^{r+1}y\xi_{j-r}^{(r)}) \\
&= \sum\nolimits_{j=r+1}^{n+1} \xi_j^{(r)} + sm^{r+1}y \sum\nolimits_{j=1}^{n-r} \xi_j^{(r)} \\
&= S_{n+1}^{(r)}(m) - \sum\nolimits_{j=1}^{r} \xi_j^{(r)} + sm^{r+1}y S_{n-r}^{(r)}(m).
\end{aligned}$$

Since $\sum_{j=1}^{r} \xi_j^{(r)} = \sum_{j=1}^{r} (mx)^{j-1} = \sum_{j=0}^{r-1} (mx)^j = 1 + \sum_{j=1}^{r-1} \eta_j^{(r,s)}$, using Equation (6), we obtain

$$\begin{aligned}
\sum\nolimits_{j=1}^{n} \eta_j^{(r,s)} + 1 &= \frac{1}{1 - mx - m^{r+1}y}(1 - \xi_{n+2}^{(r)} - m^{r+1}y \sum\nolimits_{j=1}^{r} \xi_{n-r+j+1}^{(r)}) \\
&+ sy\frac{1}{1 - mx - m^{r+1}y}(1 - \xi_{n-r+1}^{(r)} - m^{r+1}y \sum\nolimits_{j=1}^{r} \xi_{n-2r+j}^{(r)}) \\
&= \frac{1 + sm^{r+1}y - \eta_{n+1}^{(r,s)} - m^{r+1}y \sum_{j=1}^{r} \eta_{n-r+j}^{(r,s)}}{1 - mx - m^{r+1}y}.
\end{aligned}$$

Finally, we have

$$\sum\nolimits_{j=1}^{n} \eta_j^{(r,s)} = \frac{1 + sm^{r+1}y - \eta_{n+1}^{(r,s)} - m^{r+1}y \sum_{j=1}^{r} \eta_{n-r+j}^{(r,s)}}{1 - mx - m^{r+1}y} - 1.$$

□

Theorem 5 allows us to evaluate the alternating sum for the terms of the companion sequences of the r-Fibonacci polynomial.

Corollary 4 ([1, Theorem 5])**.** *For any integer $n \geq 1$, we have*

$$\sum_{j=1}^{n} V_j^{(r,s)} = S_n^{(r,s)}(1) = \frac{1 + sy - V_{n+1}^{(r,s)} - y \sum\limits_{j=1}^{r} V_{n-r+j}^{(r,s)}}{1 - x - y} - 1.$$

Corollary 5. *The alternating sum of the terms of $(V_n^{(r,s)}(x,y))_n$ is given by*

$$\sum_{j=1}^{n} (-1)^j V_j^{(r,s)} = \frac{1 - (-1)^r sy + (-1)^n V_{n+1}^{(r,s)} + (-1)^r y \sum\limits_{j=1}^{r} (-1)^{n-r+j} V_{n-r+j}^{(r,s)}}{1 + x + (-1)^r y} - 1.$$

Example 4. For $(r,s) = (1,1)$ and $(x,y) = (1,1)$, we obtain the finite alternating sum of the Lucas numbers (A000032 in the OEIS), and

$$\sum_{k=1}^{n} (-1)^k L_k = 1 + (-1)^n L_{n-1}.$$

Example 5. For $(r,s) = (2,1)$ and $(x,y) = (1,1)$, we obtain $(T_n^{(2,1)})_{n\geq 0}$, the 2-Fibonacci-Lucas numbers of type 1, and we have

$$\sum_{k=1}^{n} T_k^{(2,1)} = T_{n+3}^{(2,1)} - 3 \quad \text{and} \quad \sum_{k=1}^{n} (-1)^k T_k^{(2,1)} = \frac{1}{3}((-1)^n (T_{n+1}^{(2,1)} + T_{n-3}^{(2,1)}) - 1.$$

Example 6. For $(r,s) = (2,2)$ and $(x,y) = (1,1)$, we get $(T_n^{(2,2)})_{n\geq 0}$, the 2-Fibonacci-Lucas numbers of type 2, and we find

$$\sum_{k=1}^{n} T_k^{(2,2)} = T_{n+3}^{(2,2)} - 4 \quad \text{and} \quad \sum_{k=1}^{n} (-1)^k T_k^{(2,2)} = \frac{1}{3}(-1 + (-1)^n (T_{n+1}^{(2,1)} + T_{n-3}^{(2,1)}) - 1.$$

5. Extension of the r-Fibonacci Polynomial to Negative r Values

Our purpose in this section is to expand the definition of the r-Fibonacci polynomial to include negative values of r. We establish an explicit formula for the general term of this extended polynomial sequence. Subsequently, we determine its generating function and provide Binet-like formulae. We consider x and y to be invertible elements within a commutative unitary ring $\mathcal{A}$.

Definition 3. For any integer $r \geq 2$, we define the $(-r)$-Fibonacci bivariate polynomial sequence $(U_n^{(-r)}(x,y))_n$ by the following recursion:

$$\begin{cases} U_0^{(-r)} = 0,\ U_1^{(-r)} = 1,\ U_2^{(-r)} = \cdots = U_r^{(-r)} = 0, \\ U_{n+1}^{(-r)} = y^{-1}U_{n-r+1}^{(-r)} - y^{-1}xU_{n-r}^{(-r)}\ (n \geq r). \end{cases} \tag{7}$$

Theorem 6. *Let r be a nonnegative integer, and x, y two elements of a commutative unitary ring $\mathcal{A}$. We suppose that x and y are reversible in $\mathcal{A}$. Then for $n \geq 1$, we have*

$$U_{n+1}^{(-r)} = \sum_k \binom{k}{n-rk} y^{-k}(-x)^{n-rk},$$

or

$$U_{n+1}^{(-r)} = \sum_k \binom{(n-k)/r}{k} y^{-(n-k)/r}(-x)^k.$$

The first sum is confined to integer values of k ranging from $\lfloor n/(r+1)\rfloor$ to $\lfloor n/r\rfloor$. The second sum is limited to integer k values between 0 and $\lfloor n/r\rfloor$ such that r divides $(n-k)$.

Proof. Using Theorem 3 given in [4], we consider the sequence $(U_n^{(-r)})_n$ given by $U_n^{(-r)} = y^{-1}U_{n-r}^{(-r)} - y^{-1}xU_{n-r-1}^{(-r)}$ with $a_1 = a_2 = \cdots = a_{r-1} = 0$, $a_r = y^{-1}$ and $a_{r+1} = -xy^{-1}$. Then, for $n \geq -r$, we have

$$y_n^{(-r)} = \sum_{ri+(r+1)j=n} \binom{i+j}{j} (y^{-1})^i(-xy^{-1})^j = \sum_{r(i+j)+j=n} \binom{i+j}{j} (y^{-1})^{i+j}(-x)^j.$$

Letting $i+j=k$, we obtain

$$y_n^{(-r)} = \sum_k^{\lfloor n/r\rfloor} \binom{k}{n-rk} (-x)^{n-rk}(y^{-1})^k,$$

with initial conditions, $U_{-j}^{(-r)} = 0$ for $0 \leq j \leq r-1$, and $U_{-r}^{(-r)} = -x^{-1}y$. On the other hand, let $(\lambda_j)_{0\leq j\leq r}$ be the sequence defined by $\lambda_j = \sum_{k=0}^{r-j} a_k U_{k+j}^{(-r)}$, with

$a_0 = -1$. We have $\lambda_j = 0$ for $1 \leq j \leq r-1$, $\lambda_0 = x^{-1}$ and $\lambda_r = -x^{-1}y$. The sequence $(U_n^{(-r)})_n$ can be expressed as

$$U_n^{(-r)} = \sum_{j=0}^{r} \lambda_j y_{n+j}^{(-r)} = \lambda_0 y_n^{(-r)} + \lambda_r y_{n+r}^{(-r)} = \sum_{k}^{\lfloor n/r \rfloor} \binom{k}{n-1-rk} (-x)^{n-1-rk} (y^{-1})^k.$$

Setting $n - rk = k'$, the second sum is easily obtained. □

Remark 2. The companion matrix of the $(-r)$-Fibonacci sequence of order $(r+1)$ is given as follows:

$$B_r(x,y) := \begin{pmatrix} 0 & 0 & \cdots & y^{-1} & -xy^{-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

By an inductive argument and using Equation (7), we get its nth power

$$B_r^n(x,y) = \begin{pmatrix} U_{n+1}^{(-r)} & U_{n+2}^{(-r)} & \cdots & U_{n+r}^{(-r)} & -xy^{-1}U_n^{(-r)} \\ U_n^{(-r)} & U_{n+1}^{(-r)} & \cdots & U_{n+r-1}^{(-r)} & -xy^{-1}U_{n-1}^{(-r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{n-r+2}^{(-r)} & U_{n-r+3}^{(-r)} & \cdots & U_{n+1}^{(-r)} & -xy^{-1}U_{n-r+1}^{(-r)} \\ U_{n-r+1}^{(-r)} & U_{n-r+2}^{(-r)} & \cdots & U_n^{(-r)} & -xy^{-1}U_{n-r}^{(-r)} \end{pmatrix}.$$

The sequence $(U_n^{(-r)})_n$ possesses combinatorial properties. It immediately follows that

$$U_{n+m}^{(-r)} = \sum_{j=0}^{r-1} U_{n+j}^{(-r)} U_{m+1-j}^{(-r)} - xy^{-1} U_{n-1}^{(-r)} U_{m-r+1}^{(-r)}.$$

Setting $n = m$, we obtain

$$U_{2n}^{(-r)} = \sum_{j=0}^{r-1} U_{n+j}^{(-r)} U_{n+1-j}^{(-r)} - xy^{-1} U_{n-1}^{(-r)} U_{n-r+1}^{(-r)}.$$

Specifically, we find the following identity for Padovan numbers by entering $r = 2$ and $(x,y) = (-1,1)$,

$$P_{2n} = 2P_n P_{n+1} + (P_{n-1})^2.$$

Now, we define the companion sequence associated with the $(-r)$-Fibonacci polynomial.

Definition 4. For any integer $r \geq 2$, we define the companion sequence related to $(U_n^{(-r)})_n$ by the following recurrence relation:

$$\begin{cases} V_0^{(-r)} = r+1,\ V_1^{(-r)} = \cdots = V_{r-1}^{(-r)} = 0,\ V_r^{(-r)} = ry^{-1}, \\ V_{n+1}^{(-r)} = y^{-1}V_{n-r+1}^{(-r)} - y^{-1}xV_{n-r}^{(-r)},\ (n \geq r). \end{cases}$$

The $(-r)$-Fibonacci polynomial and its companion sequence satisfy a similar identity to (1). The following proposition gives an explicit form for $V_n^{(-r)}$ in terms of r and $U_n^{(-r)}$.

Proposition 1. *Let $r \geq 2$ be an integer and x, y be reversible elements of a commutative unitary ring $\mathcal{A}$. For $n \geq r$, we have*

$$V_n^{(-r)} = rU_{n+1}^{(-r)} - xy^{-1}U_{n-r}^{(-r)}. \tag{8}$$

Proof. We consider the sequence $(V_n^{(-r)})$ given by the recurrence relation

$$V_n^{(-r)} = y^{-1}V_{n-r}^{(-r)} - y^{-1}xV_{n-r-1}^{(-r)}.$$

Applying Theorem 3 in [4], for $a_1 = a_2 = \cdots = a_{r-1} = 0$, $a_r = y^{-1}$ and $a_{r+1} = -xy^{-1}$, we get $V_{-j}^{(-r)} = (x^{-j})$ for $1 \leq j \leq r$ and $V_0^{(-r)} = r+1$. Thus, the sequence $(\lambda_j)_{0\leq j\leq r}$ becomes

$$\lambda_j = -\sum_{k=0}^{r-j} a_k V_{k+j}^{(-r)},$$

with $a_0 = -1$. So, $\lambda_0 = r+1-y^{-1}(x^{-1})^r$ and $\lambda_j = x^{-j}$ for $1 \leq j \leq r$. Finally, we get

$$\begin{aligned} V_n^{(-r)} &= \lambda_0 U_{n+1}^{(-r)} + \lambda_1 U_{n+2}^{(-r)} + \cdots + \lambda_r U_{n+r+1}^{(-r)} \\ &= rU_{n+1}^{(-r)} + U_{n+1}^{(-r)} - y^{-1}U_{n-r+1}^{(-r)} \\ &= rU_{n+1}^{(-r)} - y^{-1}xU_{n-r}^{(-r)}. \end{aligned}$$

□

Theorem 7. *For $n \geq 1$, the sequence $(V_n^{(-r)})_{n\geq 1}$ satisfies the following two equivalent identities:*

$$V_n^{(-r)} = \sum_k \frac{n}{n-rk}\binom{k-1}{n-1-rk} y^{-k}(-x)^{n-rk} + ry^{-n/r}[r \mid n]$$

or

$$V_n^{(-r)} = \sum_k \frac{n}{k+1}\binom{(n-1-r-k)/r}{k} y^{-(n-1-k)/r}(-x)^{k+1} + ry^{-n/r}[r \mid n],$$

with $V_0^{(-r)} = r+1$ *and* $[r \mid n] = 1$ *for* r *dividing* n *and* $[r \mid n] = 0$ *otherwise. The first summuation is restricted to integers* $k \geq 0$ *such that* $\lfloor n/(r+1) \rfloor \leq k \leq \lfloor n/r \rfloor$*; the second summation is limited to integers* $0 \leq k \leq \lfloor n/(r+1) \rfloor - 1$ *for which* r *divides* $(n-k-1)$.

Proof. The proof is done using Equation (8) and Theorem 6. □

We mention that for any integer $r \geq 2$, the $(-r)$-Fibonacci polynomial $(U_n^{(-r)})_n$ and its companion sequence $(V_n^{(-r)})_n$ are linked with some classical sequences. There are many studies in the literature that concern the particular case $(x, y) = (-1, 1)$ that include Padovan numbers $(P_n)_{n\geq 0}$ (A000931 in the OEIS) and Perrin (Padovan-Lucas) numbers $(E_n)_{n\geq 0}$ (A001608 in the OEIS) for $r = 2$ (see for instance, [13] and references therein). We also have the sequences (A127838 and A050443 in the OEIS) for $r = 3$. In Table 1, we present a chart of these sequences for the first values of r.

Name	Sloane's code	First terms
$U_n^{(-2)}$	A000931	0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16,...
$U_n^{(-3)}$	A127838	0, 1, 0, 0, 1, 1, 0, 1, 2, 1, 1, 3, 3, 2, 4,...
$U_n^{(-4)}$	A127839	0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 2, 1, 0, 1, 3,...
$U_n^{(-5)}$	A127840	0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 2, 1, 0,...
$V_n^{(-2)}$	A001608	3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39,...
$V_n^{(-3)}$	A050443	4, 0, 0, 3, 4, 0, 3, 7, 4, 3, 10, 11, 7, 13, 21,...
$V_n^{(-4)}$	A087935	5, 0, 0, 0, 4, 5, 0, 0, 4, 9, 5, 0, 4, 13, 14, 5, 4,...
$V_n^{(-5)}$	A087936	6, 0, 0, 0, 0, 5, 6, 0, 0, 0, 5, 11, 6, 0, 0, 5,16,...

Table 1: The first terms of $(U_n^{(-r)})_n$ and $(V_n^{(-r)})_n$.

The sequences $(U_n^{(-r)}(x,y))_n$ and $(V_n^{(-r)}(x,y))_n$ are called the *r-generalized Padovan* and *r-generalized Perrin* numbers, respectively.

The generating functions of the $(-r)$-Fibonacci sequence and its companion sequence are given by the following theorem.

Theorem 8. *For* $z \in \mathbb{C}$ *the generating functions of* $(U_n^{(-r)})_{n\geq 0}$ *and* $(V_n^{(-r)})_{n\geq 0}$ *are given by*

$$U(z) = \sum_{n\geq 0} U_{n+1}^{(-r)} z^n = \frac{1}{1 - y^{-1}z^r + xy^{-1}z^{r+1}}$$

and

$$V(z) = \sum_{n\geq 0} V_n^{(-r)} z^n = \frac{r+1-y^{-1}z^r}{1 - y^{-1}z^r + xy^{-1}z^{r+1}},$$

respectively.

Proof. Let $U(z) = U_1^{(-r)} + U_2^{(-r)}z + U_3^{(-r)}z^2 \dots$; then using Equation (7), we have

$$(1 - y^{-1}z^r + xy^{-1}z^{r+1})U(z) = U_1^{(-r)} \text{ and } U(z) = \frac{1}{1 - y^{-1}z^r + xy^{-1}z^{r+1}}.$$

Finally, the expression of $V(z)$ is obtained by Equation (8). □

5.1. Binet Formulae

In this subsection, we give the Binet formula associated with the $(-r)$-Fibonacci sequence and its companion sequence. We start with the following result.

Lemma 1. *Let $P(t) = t^{r+1} - y^{-1}t + xy^{-1}$ be the characteristic polynomial of the sequence $(U_n^{(-r)})_n$. We suppose that $x \neq (\frac{r}{r+1})((r+1)y)^{-1/r}$. Then, the polynomial P has no multiple zeros.*

Proof. Suppose that $P(t) = 0$ has a multiple root β with $\beta \neq 0$. Then, we have $P(\beta) = \beta^{r+1} - y^{-1}\beta + xy^{-1} = 0$ and $P'(\beta) = (r+1)\beta^r - y^{-1} = 0$, so $\beta = (\frac{y^{-1}}{r+1})^{\frac{1}{r}}$. Hence $P(\beta) = (\frac{y^{-1}}{r+1})^{\frac{r+1}{r}} - y^{-1}(\frac{y^{-1}}{r+1})^{\frac{1}{r}} + xy^{-1} = 0$, which gives

$$x = (\frac{r}{r+1})((r+1)y)^{\frac{-1}{r}}.$$

□

Theorem 9. *Let $\beta_1, \beta_2, \dots, \beta_{r+1}$ be the zeros of the characteristic polynomial associated with $(U_n^{(-r)})_{n\geq 0}$ such that $x \neq (\frac{r}{r+1})((r+1)y)^{-1/r}$. Then*

$$U_{n+1}^{(-r)} = \sum_{k=1}^{r+1} \frac{\beta_k^{n+r+1}}{r\beta_k^{r+1} - xy^{-1}} \qquad \textit{and} \qquad V_n^{(-r)} = \sum_{k=1}^{r+1} \beta_k^n.$$

Proof. Let $\beta_1, \beta_2, \dots, \beta_{r+1}$ be the eigenvalues of the matrix $B_r(x,y)$ and $M_r(x,y)$ be the Vandermonde matrix given as follows:

$$M_r(x,y) = \begin{pmatrix} \beta_1^r & \beta_2^r & \cdots & \beta_{r+1}^r \\ \beta_1^{r-1} & \beta_2^{r-1} & \cdots & \beta_{r+1}^{r-1} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_1 & \beta_2 & \vdots & \beta_{r+1} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

The eigenvalues of the matrix $B_r(x,y)$ are all distinct if we suppose the discriminant of the characteristic polynomial associated with $(U_n^{(-r)})_{n\geq 0}$ is different from zero, so the matrix $B_r(x,y)$ is diagonalizable and the relation $B_r(x,y)\times M_r(x,y) = M_r(x,y)\times D$ is satisfied, where $D = Diag(\beta_1, \beta_2, \ldots, \beta_{r+1})$. Then,

$$M_r^{-1}(x,y)\times B_r^n(x,y)\times M_r(x,y) = D^n.$$

Letting $B_r(x,y) = (b_{ij})$, we have the following linear system of equations:

$$\begin{cases} b_{i1}\beta_1^r + b_{i2}\beta_1^{r-1} + \cdots + b_{i(r+1)} = \beta_1^{n+r+1-i}, \\ b_{i1}\beta_2^r + b_{12}\beta_2^{r-1} + \cdots + b_{i(r+1)} = \beta_2^{n+r+1-i}, \\ \vdots \\ b_{i1}\beta_{r+1}^r + b_{i2}\beta_{r+1}^{r-1} + \cdots + b_{i(r+1)} = \beta_{r+1}^{n+r+1-i}. \end{cases}$$

Thus, for each $1 \le i,j \le r$ we have

$$b_{ij} = \frac{det(M_r^{(j)}(x,y))}{det(M_r(x,y))},$$

where $(M_r^j(x,y))$ is the matrix obtained from $(M_r(x,y))$ by replacing the j^{th} column with the vector

$$M_r^i(x,y) = \begin{pmatrix} \beta_1^{n+r+1-i} \\ \beta_2^{n+r+1-i} \\ \vdots \\ \beta_r^{n+r+1-i} \end{pmatrix}.$$

Setting $i = j = 1$, we get $b_{11} = U_{n+1}^{(-r)} = \dfrac{det(M_r^1(x,y))}{det(M_r(x,y))}$.

Finally, using Equation (8), we have

$$\begin{aligned} V_n^{(-r)} &= rU_{n+1}^{(-r)} - xy^{-1}U_{n-r}^{(-r)} \\ &= \sum_{k=1}^{r+1} \frac{r\beta_k^{n+r+1} - xy^{-1}\beta_k^{n-r+r}}{r\beta_k^{r+1} - xy^{-1}} \\ &= \sum_{k=1}^{r+1} \beta_k^n \frac{r\beta_k^{r+1} - xy^{-1}}{r\beta_k^{r+1} - xy^{-1}} \\ &= \sum_{k=1}^{r+1} \beta_k^n. \end{aligned}$$

□

Acknowledgments. The authors would like to thank the anonymous referee for reviewing the work thoroughly. Also, we would like to thank Professor Atmane Benmezai for their numerous helpful suggestions.

References

[1] S. Abbad, H. Belbachir, and B. Benzaghou, Companion sequences associated to the r-Fibonacci sequence: algebraic and combinatorial properties, *Turkish J. Math* **43** (2019), 1095-1114.

[2] M. Ahmia and H. Belbachir, p, q-Analogue of a linear transformation preserving log-convexity, *Indian J. Pure Appl. Math* **49** (2018), 549-557.

[3] A. Bazeniar, M. Ahmia, and H. Belbachir, Connection between bi^{s}nomial coefficients and their analogs and symmetric functions, *Turkish J. Math* **42** (2018), 807-818.

[4] H. Belbachir and F. Bencherif, Linear recurrent sequences and powers of a square matrix, *Integers* **6** (2006), #A12, 17 pp.

[5] H. Belbachir, A. Benmezai, and A. Bouyakoub, Generalized Carlitz's approach for q-Fibonacci and q-Lucas polynomials, submitted.

[6] A. Benmezai, *Le Q-analogue des suites de Fibonacci et de Lucas*, Ph.D. thesis, Oran 1 University, (2016).

[7] L. Carlitz, Fibonacci notes, 4: q-Fibonacci polynomials, *The Fibonacci Quart* **13** (1975), 97-102.

[8] L. Cerlienco, M. Mignotte, and F. Piras, Suites récurrentes linéaires, propriétés algébriques et arithmétiques, *Enseign. Math* **33** (1987), 67-108.

[9] J. Cigler, A new class of q-Fibonacci polynomials, *Electron. J. Combin* **10** (2003), #R19, 15pp.

[10] J. Cigler, Some beautiful q-analogues of Fibonacci and Lucas polynomials, preprint, `arXiv:1104.2699`.

[11] R. B. Corcino. On (p, q)-binomial coefficients, *Integers* **8** (2008), #A29, 16 pp.

[12] J. A. Raab, A generalization of the connection between the Fibonacci sequence and Pascal's triangle, *The Fibonacci Quart* **1** (1963), 21-31.

[13] Y. Soykan, On Generalized Padovan Numbers, preprint, `Preprints:2021100101`.