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ABUNDANCE OF ARITHMETIC PROGRESSIONS IN CR-SETS

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Abstract

Furstenberg and Glasner proved that for an arbitrary $k \in \mathbb{N}$, any piecewise syndetic set of integers contains a k-term arithmetic progression and the collection of such progressions is itself piecewise syndetic in $\mathbb{Z}$. The above result was extended for arbitrary semigroups by Bergelson and Hindman, using the algebra of the Stone-Čech compactification of discrete semigroups. In addition, they provided an abundance of arithmetic progressions for various types of large sets. The first author, Hindman, and Strauss introduced two notions of large sets, namely, a J-set and a C-set. Bergelson and Glasscock introduced another notion of largeness, which is analogous to the notion of J-set, namely a CR-set. All these sets contain arithmetic progressions of arbitrary length. The second author and Goswami proved that for any J-set, $A \subseteq \mathbb{N}$, the collection $\{(a, b) : \{a, a + b, a + 2b, \dots, a + lb\} \subset A\}$, is a J-set in $(\mathbb{N} \times \mathbb{N}, +)$. In this article, we prove the same for CR-sets.

1. Introduction

For a general commutative semigroup $(S, +)$, a set $A \subseteq S$ is said to be *syndetic* in $(S, +)$ if there exists a finite set $F \subset S$ such that $\bigcup_{t \in F} -t + A = S$. A set $A \subseteq S$ is said to be *thick* if, for every finite set $E \subset S$, there exists an element $x \in S$ such that $E + x \subset A$. A set $A \subseteq S$ is said to be a *piecewise syndetic set* if there exists a finite set $F \subset S$ such that $\bigcup_{t \in F} -t + A$ is thick in S [12, Definition 4.38, page 101]. It can be proved that a piecewise syndetic set is the intersection of a thick set and a syndetic set [12, Theorem 4.49].

One of the famous Ramsey theoretic results is the so-called van der Waerden's

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theorem [13], which states that at least one cell of any partition $\{C_1, C_2, \ldots, C_r\}$ of $\mathbb{N}$ contains an arithmetic progression of arbitrary length. Since arithmetic progressions are invariant under shifts, it follows that every piecewise syndetic set contains arbitrarily long arithmetic progressions. The following theorem was proved algebraically by Furstenberg and Glasner in [9] and combinatorially by Beigelböck in [1].

Theorem 1. *Let $k \in \mathbb{N}$ and assume that $S \subseteq \mathbb{Z}$ is piecewise syndetic. Then $\{(a,d) : \{a, a+d, \ldots, a+kd\} \subset S\}$ is piecewise syndetic in $\mathbb{Z}^2$.*

To state the next theorem we need the following definition.

Definition 1. Let $(S,+)$ be a commutative semigroup and let $A \subseteq S$. The subset A is a *J-set* if and only if for every $F \in \mathcal{P}_f\left(S^{\mathbb{N}}\right)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$ we have $a + \sum_{n \in H} f(n) \in A$.

In [7], the authors proved the following theorem.

Theorem 2. *Let $k \in \mathbb{N}$ and assume that $S \subseteq \mathbb{N}$ is a J-set. Then*

$$\{(a,d) : \{a, a+d, \ldots, a+kd\} \subset S\}$$

is also a J-set in $\mathbb{N} \times \mathbb{N}$.

To express the main result of this article, we have to first define a combinatorially rich set (or CR-set) introduced in [2]. For $n, r \in \mathbb{N}$, let $S^{r \times n}$ denote the set of $r \times n$ matrices with elements in S. For $M = (M_{ij}) \in S^{r \times n}$ and a non-empty $\alpha \subseteq \{1, 2, \cdots, n\}$, $M_{\alpha j}$ denotes the sum $\sum_{i \in \alpha} M_{ij}$.

Definition 2. Let $(S,+)$ be a commutative semigroup. A subset $A \subseteq S$ is a *CR-set* if for all $n \in \mathbb{N}$, there exists an $r \in \mathbb{N}$ such that for all $M \in S^{r \times n}$, there exists a non-empty set $\alpha \subseteq \{1, 2, \ldots, r\}$ and $s \in S$ such that for all $j \in \{1, 2, \ldots, n\}$,

$$s + M_{\alpha,j} \in A.$$

We denote by $\mathcal{CR}(S,+)$ the class of combinatorially rich subsets of $(S,+)$.

Choosing $(S,+) = (\mathbb{N},+)$ and $M_{ij} = j$ from the above definition, there exist a nonempty $\alpha \subseteq \{1, 2, \ldots, r\}$ and $s \in \mathbb{N}$ such that

$$\{s + \mid \alpha \mid, s + 2 \mid \alpha \mid, \ldots, s + n \mid \alpha \mid\} \subset A.$$

Thus, combinatorially rich sets in $(\mathbb{N},+)$ are AP-rich, i.e., they contain arbitrarily long arithmetic progressions. It can be stated that piecewise syndetic subsets of S are CR-sets and that CR-subsets of S are J-sets. Of course, the first inclusion implies that the set of CR-subsets of S is non-empty, and the second inclusion immediately implies that CR-subsets of S contain arbitrarily long arithmetic progressions. We prove that for any CR-set $A \subseteq \mathbb{N}$, the collection

$$\{(a,b) : \{a, a+b, a+2b, \dots, a+lb\} \subset A\}$$

is a CR-set in $(\mathbb{N} \times \mathbb{N}, +)$ and the same result for essential CR-sets. The next section is devoted to essential CR-sets.

2. Essential CR-Set

A collection $\mathcal{F} \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$ is *upward hereditary* if, whenever $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$, it follows that $B \in \mathcal{F}$. A nonempty and upward hereditary collection $\mathcal{F} \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$ will be called a *family.* If $\mathcal{F}$ is a family, the *dual family* $\mathcal{F}^*$ is given by

$$\mathcal{F}^* = \{E \subseteq S : \text{ for all } A \in \mathcal{F}, E \cap A \neq \emptyset\}.$$

A family $\mathcal{F}$ possesses the *Ramsey property* if, whenever $A \in \mathcal{F}$ and $A = A_1 \cup A_2$, there is some $i \in \{1,2\}$ such that $A_i \in \mathcal{F}$.

We give a brief review of the algebraic structure of the Stone–Čech compactification of discrete semigroups.

Let S be a discrete semigroup. The elements of βS are regarded as ultrafilters on S. Let $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subset S\}$ is a basis for the closed sets of βS. The operation '$\cdot$' on S can be extended to the Stone–Čech compactification βS of S so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for each $p \in \beta S$, the function $\rho_p(q) : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). There is a famous theorem [12, Theorem 2.5], due to Ellis which states that if S is a compact right topological semigroup then the set of idempotents, $E(S)$, is not empty. A nonempty subset I of a semigroup T is called a *left ideal* of S if $TI \subset I$, a *right ideal* if $IT \subset I$, and a *two-sided ideal* (or simply an *ideal*) if it is both a left and right ideal. A *minimal* left ideal is a left ideal that does not contain any proper left ideal. Similarly, we can define a minimal right ideal and the smallest ideal.

Any compact Hausdorff right topological semigroup T has the smallest two-sided ideal

$$\begin{aligned} K(T) &= \bigcup \{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup \{R : R \text{ is a minimal right ideal of } T\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. If p and q are idempotents in T, we write $p \leq q$

if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal $K(T)$ of T. Given $p, q \in \beta S$ and $A \subseteq S$, we know that $A \in p \cdot q$ if and only if the set $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. See [12] for an elementary introduction to the algebra of βS and any unfamiliar details.

It is known that the family $\mathcal{F}$ has the Ramsey property if and only if the family $\mathcal{F}^*$ is a filter. For a family $\mathcal{F}$ with the Ramsey property, let $\beta(\mathcal{F}) = \{p \in \beta S : p \subseteq \mathcal{F}\}$. Then we get the following from [4, Theorem 5.1.1].

Theorem 3. *Let S be a discrete set. For every family $\mathcal{F} \subseteq \mathcal{P}(S)$ with the Ramsey property, $\beta(\mathcal{F}) \subseteq \beta S$ is closed. Furthermore, $\mathcal{F} = \cup \beta(\mathcal{F})$. Also if $K \subseteq \beta S$ is closed, then $\mathcal{F}_K = \left\{E \subseteq S : \overline{E} \cap K \neq \emptyset\right\}$ is a family with the Ramsey property and $\overline{K} = \beta(\mathcal{F}_K)$.*

Let S be a discrete semigroup. Then, for every family $\mathcal{F} \subseteq \mathcal{P}(S)$ with the Ramsey property, $\beta(\mathcal{F}) \subseteq \beta S$ is closed. If $\beta(\mathcal{F})$ is a subsemigroup of βS, then $E(\beta \mathcal{F}) \neq \emptyset$.

Definition 3. Let $\mathcal{F}$ be a family with the Ramsey property such that $\beta(\mathcal{F})$ is a subsemigroup of βS, and let p be an idempotent in $\beta(\mathcal{F})$. Then each member of p is called an *essential $\mathcal{F}$-set*.

The family $\mathcal{F}$ is called *left (right) shift-invariant* if for all $s \in S$ and all $E \in \mathcal{F}$, one has $sE \in \mathcal{F}(Es \in \mathcal{F})$. The family $\mathcal{F}$ is called *left (right) inverse shift-invariant* if for all $s \in S$ and all $E \in \mathcal{F}$, one has $s^{-1}E \in \mathcal{F}(Es^{-1} \in \mathcal{F})$. We derive the following theorem from [4, Theorem 5.1.2].

Theorem 4. *If $\mathcal{F}$ is a family having the Ramsey property then $\beta \mathcal{F} \subseteq \beta S$ is a left ideal if and only if $\mathcal{F}$ is left shift-invariant. Similarly, $\beta \mathcal{F} \subseteq \beta S$ is a right ideal if and only if $\mathcal{F}$ is right shift-invariant.*

From [4, Theorem 5.1.10], we can identify those families $\mathcal{F}$ with Ramsey property for which $\beta(\mathcal{F})$ is a subsemigroup of βS. The condition is a rather technical weakening of left-shift invariance.

Theorem 5. *Let S be any discrete semigroup, and let $\mathcal{F}$ be a family of subsets of S having the Ramsey property. Then the following are equivalent.*

(1) $\beta(\mathcal{F})$ is a subsemigroup of βS.

(2) $\mathcal{F}$ has the following property:

If $E \subseteq S$ is any set, and if there is $A \in \mathcal{F}$ such that for all finite $H \subseteq A$, one has $\left(\cap_{q \in H} x^{-1}E\right) \in \mathcal{F}$, then $E \in \mathcal{F}$.

The elementary characterization of essential $\mathcal{F}$-sets is known from [5, Theorem 5].

Definition 4. Let ω be the first infinite ordinal and let each ordinal denote the set of all its predecessors. In particular, $0=\emptyset$, and for each $n\in\mathbb{N}$, $n=\{0,1,...,n-1\}$.

(a) If f is a function and $dom\,(f)=n\in\omega$, then for all x, $f^\frown x=f\cup\{(n,x)\}$.

(b) Let T be a set of functions whose domains are members of ω. For each $f\in T$, $B_f\,(T)=\{x:f^\frown x\in T\}$.

We get the following theorem from [5, Theorem 5], which plays a vital role in this article.

Theorem 6. *Let $(S,\cdot)$ be a semigroup, and assume that $\mathcal{F}$ is a family of subsets of S with the Ramsey property such that $\beta\,(\mathcal{F})$ is a subsemigroup of βS. Let $A\subseteq S$. Then the statements (a), (b), and (c) are equivalent and are implied by statement (d). If S is countable, then all five statements are equivalent.*

(a) A is an essential $\mathcal{F}$-set.

(b) There is a non empty set T of functions such that

(i) for all $f\in T$, $domain(f)\in\omega$ and $rang\,(f)\subseteq A$;

(ii) for all $f\in T$ and all $x\in B_f\,(T)$, $B_{f^\frown x}\subseteq x^{-1}B_f\,(T)$; and

(iii) for all $F\in\mathcal{P}_f\,(T)$, $\bigcap_{f\in F}B_f(T)$ is an $\mathcal{F}$-set.

(c) There is a downward directed family $\langle C_F\rangle_{F\in I}$ of subsets of A such that

(i) for each $F\in I$ and each $x\in C_F$ there exists $G\in I$ with $C_G\subseteq x^{-1}C_F$; and

(ii) for each $\mathcal{F}\in\mathcal{P}_f\,(I)$, $\bigcap_{F\in\mathcal{F}}C_F$ is an $\mathcal{F}$-set.

(d) There is a decreasing sequence $\langle C_n\rangle_{n=1}^{\infty}$ of subsets of A such that

(i) for each $n\in\mathbb{N}$ and each $x\in C_n$, there exists $m\in\mathbb{N}$ with $C_m\subseteq x^{-1}C_n$; and

(ii) for each $n\in\mathbb{N}$, C_n is an $\mathcal{F}$-set.

Let $(S,+)$ be a commutative semigroup. By [2, Lemma 2.14], the class $\mathcal{CR}$ is partition regular; also, it is trivial that the class $\mathcal{CR}$ is translation invariant. Hence $\beta\,(\mathcal{CR})$ is a closed subsemigroup of $\beta\,(S)$ by Theorem 4. Let A be a subset of $(S,+)$. We call A an *essential CR-set* if and only if $A\in p$ for some $p\in E\,(\beta\,(\mathcal{CR}))$. Then from the above, we get the following theorem.

Theorem 7. *Let $(S,+)$ be a countable commutative semigroup. Then the following are equivalent.*

(a) A is an essential CR-set.

(b) There is a decreasing sequence $\langle C_n\rangle_{n=1}^{\infty}$ of subsets of A such that

(i) for each $n \in \mathbb{N}$ and each $x \in C_n$, there exists $m \in \mathbb{N}$ with $C_m \subseteq x^{-1}C_n$; and

(ii) for each $n \in \mathbb{N}$, C_n is a CR-set.

3. Proof of the Main Theorem

Now we are going to prove the abundance of arithmetic progressions in CR-sets.

Definition 5. Let $A = (a_{ij})_{m\times n_1}$ and $B = (b_{ij})_{m\times n_2}$ be two matrices. The *concatenation* of A and B is $C = A^\frown B$, defined by $C = (c_{ij})_{m\times(n_1+n_2)}$, where

$$c_{ij} = \begin{cases} a_{ij} \text{ if } j \le n_1 \\ b_{ij} \text{ if } j > n_2. \end{cases}$$

Example 1. Let $A = \begin{pmatrix} 3 & 6 \\ 7 & 4 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 8 & 9 & 1 \\ 6 & 8 & 3 & 5 \\ 7 & 9 & 2 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 6 \\ 9 \\ 8 \end{pmatrix}$. Then

$$A^\frown B^\frown C = \begin{pmatrix} 3 & 6 & 5 & 8 & 9 & 1 & 6 \\ 7 & 4 & 6 & 8 & 3 & 5 & 9 \\ 1 & 3 & 7 & 9 & 2 & 1 & 8 \end{pmatrix}.$$

Theorem 8. *Let $(S, +)$ be a commutative semigroup. Let A be a CR-set in S and $l \in \mathbb{N}$. Then the set*

$$\{(a, b) : \{a, a+b, a+2b, \ldots, a+lb\} \subset A\}$$

is a CR-set in $(S \times S, +)$.

Proof. Let $C = \{(a, b) : \{a, a+b, a+2b, \ldots, a+(l-1)\,b\} \subset A\}$. Since A is CR-set, for any $n \in \mathbb{N}$, we can find r such that $M \in S^{r\times ln}$, there exist $a \in S$, and $\alpha \subset \{1, 2, \ldots, r\}$ such that $a + \sum_{i\in\alpha} M_{i,j} \in A$ for all $j \in \{1, 2, \ldots, ln\}$. If $M' \in (S \times S)^{r\times ln}$, then $M' = \left(\left(M^1_{ij}, M^2_{ij}\right)\right)_{r\times ln}$. Let $s \in S$ and

$$M^k = \begin{pmatrix} M^1_{11} + k(s + M^2_{11}) & M^1_{12} + k(s + M^2_{12}) & \cdots & M^1_{1n} + k(s + M^2_{1n}) \\ M^1_{21} + k(s + M^2_{21}) & M^1_{22} + k(s + M^2_{22}) & \ldots & M^1_{2n} + k(s + M^2_{2n}) \\ \vdots & \vdots & \ldots & \vdots \\ M^1_{r1} + k(s + M^2_{r1}) & M^1_{r2} + k(s + M^2_{r2}) & \ldots & M^1_{rn} + k(s + M^2_{rn}) \end{pmatrix}$$

for $k \in \{0, 1, 2, \ldots, l-1\}$. So, $M = M^{0\frown} M^{1\frown} \ldots^\frown M^{l-1}$ and M is an $r\times ln$ matrix. There exists $\alpha \in \{1, 2, \ldots, r\}$ such that $a + \sum_{i\in\alpha} M_{i,j} \in A$ for all $j \in \{1, 2, \ldots, ln\}$.

This implies that $a+\sum_{i\in\alpha}M^1_{i,j}+k(s+M^2_{ij})\in A$ for all $j\in\{1,2,\dots,n\}$ and $k\in\{0,1,2,\dots,l-1\}$, that is, $a+\sum_{i\in\alpha}M^1_{i,j}+k\sum_{i\in\alpha}(s+M^2_{ij})\in A$ for all $j\in\{1,2,\dots,n\}$ and $k\in\{0,1,2,\dots,l-1\}$. This implies that

$$a+\sum_{i\in\alpha}M^1_{i,j}+k(\mid\alpha\mid s+\sum_{i\in\alpha}M^2_{ij})\in A$$

for all $j\in\{1,2,\dots,n\}$ and $k\in\{0,1,2,\dots,l-1\}$. So,

$$a+\sum_{i\in\alpha}M^1_{i,j}+k(\mid\alpha\mid s+\sum_{i\in\alpha}M^2_{ij})\in A$$

for all $j\in\{1,2,\dots,n\}$ and $k\in\{0,1,2,\dots,l-1\}$, that is,

$$\left(a+\sum_{i\in\alpha}M^1_{i,j},\mid\alpha\mid s+\sum_{i\in\alpha}M^2_{ij}\right)\in C$$

for all $j\in\{1,2,\dots,n\}$. Finally, we get $(a,\mid\alpha\mid s)+\sum_{i\in\alpha}\left(M^1_{i,j}+M^2_{i,j}\right)\in C$ for all $j\in\{1,2,\dots,n\}$. Hence C is a CR-set. □

Now we can prove the abundance of arithmetic progressions in an essential CR-set, using elementary characterization and the above theorem.

Theorem 9. *Let A be an essential CR-set in $\mathbb{N}$, and $l\in\mathbb{N}$. Then the set*

$$\{(a,b):\ \{a,a+b,a+2b,\dots,a+lb\}\subset A\}$$

is an essential CR-set in $(\mathbb{N}\times\mathbb{N},+)$.

Proof. As A is an essential CR-set, there exists a decreasing sequence of CR-sets in $\mathbb{N}$, say $\{A_n : n\in\mathbb{N}\}$, satisfying the property (b)(i) of Theorem 7. So,

$$A\supseteq A_1\supseteq A_2\supseteq\cdots\supseteq A_n\supseteq\dots$$

and for $i\in\mathbb{N}$, $B_i=\{(a,b)\in\mathbb{N}\times\mathbb{N}:\ \{a,a+b,a+2b,\dots,a+(l-1)\,b\}\subset A_i\}$ are CR-sets in $\mathbb{N}\times\mathbb{N}$. Consider

$$B\supseteq B_1\supseteq B_2\supseteq\cdots\supseteq B_n\supseteq\dots.$$

Pick $n\in\mathbb{N}$ and $(a,b)\in B_n$. Then $\{a,a+b,a+2b,\dots,a+(l-1)\,b\}\subset A_n$. Now, by property (b)(i) of Theorem 7, there exists $N_i\in\mathbb{N}$ such that $a+ib\in A_i$ for $i=0,1,2,\dots,l-1$ and $A_{N_i}\subseteq-(a+ib)+A_n$. Taking $N=\max\{N_0,N_1,\dots,N_{l-1}\}$, we get

$$A_N\subseteq\bigcap_{i=0}^{l-1}\left(-(a+ib)+A_n\right).$$

Now, if (a_1, b_1) is any element of B_N, then

$$\{a_1, a_1 + b_1, a_1 + 2b_1, \ldots, a_1 + (l-1)\, b_1\} \subseteq A_N \subseteq \bigcap_{i=0}^{l-1} (-(a+ib) + A_n)\,.$$

So $(a_1+a)+i.(b_1+b) \in A_n$ for all $i \in \{0, 1, 2, \ldots, l-1\}$. Hence $(a_1, b_1) \in -(a, b)+B_n$. This implies $B_N \subseteq -(a, b) + B_n$. Therefore, for any $(a, b) \in B_n$, there exists $N \in \mathbb{N}$ such that $B_N \subseteq -(a, b) + B_n$, showing the property (b)(i) of Theorem 7. This proves theorem. □

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