#A40 INTEGERS 25 (2025)

ON THE k-ADDITIVE UNIQUENESS OF SUMS OF CONSECUTIVE SQUARES FOR MULTIPLICATIVE FUNCTIONS

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Received: 1/9/25, Accepted: 4/7/25, Published: 4/25/25

Abstract

Let $k \geq 2$ be an integer and $\mathcal{T}$ be the set consisting of all sums of consecutive squares. We prove that $\mathcal{T}$ is a k-additive uniqueness set for the set of multiplicative functions. That is, if a multiplicative function f satisfies a multivariate Cauchy's functional equation $f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$ for arbitrary $x_1, ..., x_k \in \mathcal{T}$, then f is the identity function $f(n) = n$ for all $n \in \mathbb{N}$.

1. Introduction

An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is called *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$. Let $\mathcal{M}$ denote the set of complex valued multiplicative functions.

A set $E \subseteq \mathbb{N}$ is called an *additive uniqueness set* of a set of arithmetic functions $\mathcal{F}$ if $f \in \mathcal{F}$ is uniquely determined under the condition

$$f(m + n) = f(m) + f(n) \text{ for all } m, n \in E. \tag{1}$$

For example, $\mathbb{N}$ and $\{1\} \cup 2\mathbb{N}$ are trivially additive uniqueness sets of $\mathcal{M}$.

This concept was first introduced by C. A. Spiro [9] in 1992. She proved that the set of primes is an additive uniqueness set of $\mathcal{M}_0 = \{f \in \mathcal{M} \mid \ f(p_0) \neq 0 \text{ for some prime } p_0\}$. Later on, Spiro's work has been extended in many directions.

Let $k \geq 2$ be a fixed integer. If there is only one function $f \in \mathcal{F}$ satisfying $f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$ for arbitrary $x_i \in E$, $i \in \{1, 2, \ldots, k\}$, then E is called a *k-additive uniqueness set* of $\mathcal{F}$.

In 2010, Fang [4] proved that the set of primes is a 3-additive uniqueness set of $\mathcal{M}_0$. In 2013, Dubickas and Šarka [3] generalized Fang's result to sums of arbitrary primes.

DOI: 10.5281/zenodo.15283720
[1]This research was supported by ADA University Faculty Research and Development Funds.

In 1996, Chung [1] proved the following result.

Theorem 1. *Let $f \neq 0$ be a multiplicative function. If f fulfills the condition*

$$f(m^2+n^2) = f(m^2) + f(n^2)$$

for all positive integers m and n, then

- $f(4) = 0$ *or* $f(4) = 4$,
- $f(q^{2k}) = (f(q^2))^k$ *for all positive integers q and k,*
- *if $f(4) = 4$, then $f(m^2) = m^2$ for all positive integers m.*

In particular, the set $\mathcal{S}$ of positive squares is not an additive uniqueness set for multiplicative functions.

In 1999, Chung and Phong [2] showed that the set of positive triangular numbers $T_n = \frac{n(n+1)}{2}$, $n \in \mathbb{N}$ and the set of positive tetrahedral numbers $Te_n = \frac{n(n+1)(n+2)}{6}$, $n \in \mathbb{N}$ are new additive uniqueness sets for $\mathcal{M}$ and Park [6] extended their work to sums of k triangular numbers, $k \geq 3$.

Park [5] proved that $\mathcal{S}$ is a k-additive uniqueness set of $\mathcal{M}$ for every $k \geq 3$. In 2022, he [7] proved that the set

$$W = \{a^2+b^2 : \ (a,b) \neq (0,0)\} = \{1,2,4,5,8,9,10,13,16,17,\ldots\}$$

of numbers which are representable as sums of two squares is an additive uniqueness set for multiplicative functions.

Set $s_n = \sum\limits_{i=0}^{n} i^2$ and let

$$\mathcal{T} = \{s_m - s_n |\ m > n \geq 0,\ m,n \in \mathbb{Z}\} = \{1,4,5,9,13,14,16,25,29,\ldots\}.$$

be the set of all finite sums of consecutive squares.

Note that if $W(x)$ is the counting function for the set W, then by Landau's theorem, $W(x) \sim \frac{Bx}{\sqrt{\log x}}$ as $x \to \infty$, where B is an explicitly defined positive constant. On the other hand, for the counting function of $\mathcal{T}$, we have

$$x^{2/3}(\log x)^{-2-\log 2-\epsilon} \ll T(x) \ll x^{2/3},$$

(see [10]). Thus, the set $\mathcal{T}$ is sparser than W.

Although $\mathcal{T}$ is a 0-density subset of $\mathbb{N}$, it has a nice additive structure. Platiel and Rung [8] proved that $\mathcal{T}$ forms an additive basis of the exact order 3, i.e. each nonnegative integer can be written as the sum of at most three numbers from $\mathcal{T}$.

The main result of this short note is the following theorem.

Theorem 2. *Fix $k \geq 2$. The set $\mathcal{T}$ consisting of all finite sums of consecutive squares is a k-additive uniqueness set of $\mathcal{M}$, namely, if a multiplicative function f satisfies*

$$f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$$

for arbitrary $x_1, \ldots, x_k \in \mathcal{T}$, then f is the identity function.

Since $\mathcal{S} \subset \mathcal{T}$, the k-additivity of $\mathcal{T}$ for $k \geq 3$ immediately follows from the k-additivity of $\mathcal{S}$. For $k \geq 4$, we will present an alternative, much simpler proof.

2. Proof

Proof of Theorem 2. The proof consists of four cases.

`Case I:` $k = 2$. Trivially, $f(1) = 1$ and $f(2) = 2$. If $f(4) = 0$, then $f(5) = 1$, $f(9) = 2f(4) + 1 = 1$, and $f(14) = f(9) + f(5) = 2$. But this would lead to a contradiction:

$$\begin{aligned} f(28) &= f(7)f(4) = 0 \\ &= f(14) + f(14) = 4 \end{aligned}$$

Thus, $f(4) = 4$ and $f(n) = n$ for $n \leq 10$. Next, we use induction on n. Suppose that $f(n) = n$ for $n < N$. If N is not a prime power, then $N = ab$ with $\gcd(a, b) = 1$ and $f(N) = N$ by the multiplicativity of f and the induction hypothesis. Thus, we may assume that N is a prime power.

Let $N = 2^r$ with $r \geq 4$. Assume that 2^r is fixed by f for all $r \leq 2m$. Then $f(2^{2m+1}) = 2^{2m+1}$ since

$$f(2^{2m+1}) = f(2^{2m} + 2^{2m}) = f(2^{2m}) + f(2^{2m}).$$

We obtain $f(2^{2(m+1)}) = 2^{2(m+1)}$ by calculating $f(5 \cdot 2^{2m})$ in two ways:

$$f(5 \cdot 2^{2m}) = f(5) \cdot f(2^{2m})$$

and

$$f(5 \cdot 2^{2m}) = f(2^{2(m+1)} + 2^{2m}) = f(2^{2(m+1)}) + f(2^{2m}).$$

Now let $N = p^r$ with $p \equiv 1 \pmod 4$ and assume that $f(n) = n$ for all $n < N$. Then there exist positive integers x and y such that

$$p^r = x^2 + y^2.$$

So, from Theorem 1, we get

$$f(p^r) = f(x^2 + y^2) = f(x^2) + f(y^2) = x^2 + y^2 = p^r.$$

Let $P_n = 1^2 + 2^2 + \ldots + n^2$ be the nth square pyramidal number. Note that $P_n \in \mathcal{T}$ and $P_n = \frac{n(n+1)(2n+1)}{6}$. By an inductive argument, we have $f(P_n) = \sum_{i=1}^{n} f(i^2) = P_n$.

Next, consider the case $N = p^r$ with $p \equiv 3 \pmod 4$ and assume that $f(n) = n$ for all $n < N$. If r is even, then $p^r = 4m+1$ and it cannot be a sum of two positive squares. Note that

$$P_{2m} + P_{4m+1} = (2m+1)(3m+1)(4m+1).$$

In this case, $\gcd(2m+1, 4m+1) = \gcd(3m+1, 4m+1) = \gcd(2m+1, 3m+1) = 1$ and $2m+1, 3m+1 < 4m+1$. Hence, $f(4m+1) = 4m+1$.

If r is odd, then $N = p^r = 4m+3$ and we have the identity

$$P_{2m+1} + P_{4m+2} = (4m+3)(2m+1)(3m+2).$$

Note that, $\gcd(2m+1, 4m+3) = \gcd(3m+2, 4m+3) = \gcd(2m+1, 3m+2) = 1$ and $2m+1, 3m+2 < 4m+3$. Thus, $f(4m+3) = 4m+3$.

`Case II:` $k = 3$. Since $\mathcal{S} \subset \mathcal{T}$, this immediately follows from the 3-additivity of $\mathcal{S}$.

`Case III:` $k = 4$. Platiel and Rung's theorem guarantees that every positive integer can be written as a sum of three numbers from $\mathcal{T}$ some of which possibly vanish. But, since $9 \in \mathcal{T}$ and

$$n = (n-9) + 9 = (n-9) + 4 + 5 = (n-9) + 4 + 4 + 1,$$

every integer $n > 9$ can be written as a sum of four positive numbers from $\mathcal{T}$.

Note that $f(4) = 4$, $f(7) = f(4+1+1+1) = 7$ and $f(10) = f(4+4+1+1) = 10$. From

$$\begin{cases} f(20) = 4f(5) = f(14+4+1+1) = 7f(2) + 6 \\ f(2)f(5) = 10 \end{cases}$$

we obtain two solutions

$$f(2) = -\frac{20}{7}, f(5) = -\frac{7}{2} \quad \text{or} \quad f(2) = 2, f(5) = 5.$$

But the first case would lead to a contradiction:

$$\begin{aligned} f(14) = f(5+4+4+1) &= \frac{11}{2} \\ &= f(7 \cdot 2) = 7f(2) = -20. \end{aligned}$$

Thus, we can conclude that $f(2) = 2$ and $f(5) = 5$. From the equalities $f(12) = 4f(3) = f(5+5+1+1)$, $f(8) = f(5+1+1+1)$, $f(18) = 2f(9) = f(5+5+4+4)$, we obtain $f(3) = 3$, $f(8) = 8$, $f(9) = 9$ and f should be the identity function by induction.

`Case IV:` $k \geq 5$. It is clear that the sum of k numbers from $\mathcal{T}$ can represent k, but cannot represent any number from 1 through $k-1$. Since sums of four numbers from $\mathcal{T}$ represent all integers greater than or equal to 10, the sum

$$\underbrace{1+\cdots+1}_{k-4 \text{ times}}+x+y+z+w,$$

where $x, y, z, w \in \mathcal{T}$, can represent all integers greater than or equal to $k+6$.

Note that

$$\begin{aligned}
(k-2)+14 &= (k-2)\cdot 1+13+1 \\
&= (k-2)\cdot 1+9+5, \\
(k-3)+15 &= (k-3)\cdot 1+5+5+5 \\
&= (k-3)\cdot 1+13+1+1, \\
(k-4)+16 &= (k-4)\cdot 1+4+4+4+4 \\
&= (k-4)\cdot 1+9+5+1+1 \\
(k-4)+43 &= (k-4)\cdot 1+13+13+13+4 \\
&= (k-4)\cdot 1+36+5+1+1.
\end{aligned}$$

Let $x = f(4)$, $y = f(5)$, $z = f(9)$ and $w = f(13)$. The above equalities give rise to the system of equations

$$\begin{cases} 1+w = y+z \\ 3y = w+2 \\ 4x = y+z+2 \\ 3w+x = xz+y+2. \end{cases}$$

The solutions are

$$f(4) = f(5) = f(9) = f(13) = 1$$
$$f(4) = 4, f(5) = 5, f(9) = 9, f(13) = 13.$$

Consider the first solution set $f(4) = f(5) = f(9) = f(13) = 1$. Arrange positive numbers from $\mathcal{T}$ into an increasing sequence and let x_n denote the nth term. Then $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5) = 1$. Observe that, every x_n with $n \geq 5$ can be written as a sum of four numbers from $\mathcal{T}$. From the equality

$$\begin{aligned}
&(k-5)+1+4+4+4+x_e \\
&= (k-5)+13+x_a+x_b+x_c+x_d,
\end{aligned} \tag{2}$$

with $a, b, c, d < e$, we infer that $f(x_n) = 1$ for all $n \geq 5$ inductively.

But for sufficiently large n, x_n can be represented as a sum of k numbers from $\mathcal{T}$. So $f(x_n) = k$, which is a contradiction.

Hence, we conclude that $f(4) = 4$, $f(5) = 5$, $f(9) = 9$ and $f(13) = 13$. Moreover, (2) yields $f(x_n) = x_n$ for every $n \geq 1$.

If N is a sum of k positive numbers from $\mathcal{T}$ then $f(N) = N$. Otherwise, choose an integer $M \geq k+6$ such that $\gcd(M, N) = 1$. Then M and MN can be represented as sums of k positive numbers from $\mathcal{T}$. By the multiplicativity of f, we get $Mf(N) = f(M)f(N) = f(MN) = MN$. Therefore, $f(N) = N$ and this completes the proof. $\square$

Acknowledgment. The author would like to thank Professor Poo-Sung Park for fruitful discussions. He is also grateful to the anonymous referee for his/her helpful comments.

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