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SHIFTING LUCAS SEQUENCES AWAY FROM PRIMES

Dan Ismailescu
Mathematics Department, Hofstra University, Hempstead, New York
dan.p.ismailescu@hofstra.edu

Daniel Hyunwoo Jeon
Phillips Exeter Academy, Exeter, New Hampshire
dhjeon@exeter.edu

Aaron Kim
The Bronx High School of Science, Bronx, New York
kima13@bxscience.edu

Joonbeom Kwon
St. Andrew's School, Middletown, Delaware
dkwon@standrews-de.org

Austin Lee
Lawrenceville School, Lawrenceville, New Jersey
alee25@lawrenceville.org

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Abstract

We strengthen a result of Jones by showing that for any positive integer P, the Lucas sequence $(U_n)_n$ defined by $U_0 = 0$, $U_1 = 1$, $U_n = P \cdot U_{n-1} + U_{n-2}$ can be translated by a positive integer $K(P)$ such that the shifted sequence with general term $U_n + K(P)$ contains no primes, nor terms one unit away from a prime.

1. Introduction

Given integers P and Q, let $U_n(P,Q)$, the *Lucas sequence of the first kind with parameters P and Q*, be defined by

$$\begin{aligned} &U_0(P,Q) = 0,\ U_1(P,Q) = 1,\ \text{and} \\ &U_n(P,Q) = P \cdot U_{n-1}(P,Q) - Q \cdot U_{n-2}(P,Q),\ \text{for all } n \geq 2. \end{aligned} \tag{1}$$

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Lucas sequences are named after the French mathematician Édouard Lucas (1842-1891), who used them for testing the primality of numbers. Note that if $P = 1$ and $Q = -1$, then $U_n(1, -1) = F_n$, the Fibonacci sequence. Other well-known sequences are obtained for different choices of P and Q; for example, $U_n(2, -1) = P_n$ is the Pell sequence, $U_n(3, 2) = 2^n - 1 = M_n$ is the Mersenne sequence, and $U_n(2, 1) = n$ is the sequence of nonnegative integers.

It is a natural problem to inquire whether a given Lucas sequence contains infinitely many prime terms. To make the question meaningful we must impose the restriction that P and Q are coprime, as it is easy to prove that $\gcd(P, Q)$ divides $U_n(P, Q)$ for every $n \geq 2$. Heuristic arguments indicate that there are infinitely many Fibonacci primes as well as infinitely many Mersenne primes, and it is widely believed that for most choices of coprime integers P and Q there are infinitely many primes of the form $U_n(P, Q)$. However, besides the trivial exceptions $U_n(2, 1) = n$ and $U_n(-2, 1) = (-1)^{n+1}n$, there is no known example of a Lucas sequence which one can rigorously prove contains infinitely many prime terms.

Very recently, Broderius and Greene [2] considered the dual problem: *are there Lucas sequences which contain only finitely many primes?* They found two such classes of Lucas sequences and conjectured that there are no others. We found a simple example which is not covered by either one of their constructions.

Theorem 1. *Let $U_n(3, -4) : 0, 1, 3, 13, 51, 205, 819, \ldots$ be the Lucas sequence of the first kind with $P = 3$, $Q = -4$. Then $U_n(3, -4)$ is composite for all $n \geq 4$.*

While the question regarding the infinitude of Fibonacci primes is presently out of reach, it is quite surprising that one can easily show that neither $F_n + 1$ nor $F_n - 1$ can be prime if $n \geq 7$. The proof is part of mathematical folklore; we refer the reader to Honsberger's book [7]. From this result, it is clear that shifting the Fibonacci sequence by $+1$ or by -1 leads to sequences which contain only finitely many primes. Inspired by this fact, we found a class of Lucas sequences whose terms are, with finitely many exceptions, neither prime nor one unit away from a prime.

Theorem 2. *Let $P \geq 3$ and $Q = 1$. Then for all but finitely many n, $U_n(P, 1) - 1$, $U_n(P, 1)$, and $U_n(P, 1) + 1$ are all composite.*

Finally, we study a problem first considered by Jones [8]: *Is it true that for a given Lucas sequence $U_n(P, Q)$, there exists a positive number K such that the sequence with general term $U_n(P, Q) + K$ contains no prime numbers?* For example, we will show that $F_n + 14475$ is never a prime - see Section 3. Jones proved that such a K exists for every Lucas sequence $U_n(P, -1)$. We further strengthen his result in the following:

Theorem 3. *Let $P \geq 1$ and $Q = -1$. Then there exists a positive integer K such that $U_n(P, -1) + K - 1$, $U_n(P, -1) + K$, and $U_n(P, -1) + K + 1$ are composite for all $n \geq 0$.*

The rest of the paper is organized as follows: in Section 2, we introduce the necessary background, then present proofs of Theorem 1 and Theorem 2. In Section 3, we put forward the technique required for Theorem 3, whose proof is then presented over Sections 4 and 5. Section 6 contains some final comments and several open questions, and the paper concludes with an appendix that contains several tables relevant to the proof of Theorem 3.

2. Preliminaries

In this section, we present several well known properties of the Lucas sequences $U_n(P,Q)$ which will be used for proving Theorem 2 and Theorem 3.

We denote by x and y the roots of the characteristic equation $z^2 - Pz + Q = 0$. Since $x + y = P$ and $xy = Q$, we will write $U_n(x + y, xy)$ when focusing on x and y. It can be easily shown via induction that the general term can be explicitly expressed in terms of x and y as follows:

$$U_n(P,Q) = \frac{x^n - y^n}{x - y} \text{ if } x \neq y, \text{ and } U_n(P,Q) = nx^{n-1} \text{ if } x = y. \tag{2}$$

In order to maintain the simplicity of the exposition, we will only consider the situation when x and y are distinct real numbers. This is equivalent to requiring $P^2 - 4Q > 0$. Moreover, we will assume that $P \geq 1$. These assumptions imply that $U_n(P,Q)$ is a strictly increasing sequence for all $n \geq 2$ - see for example Lemma 3 in [6]. In particular, we are only concerned with Lucas sequences whose terms are all nonnegative.

We will need the following identities, which are well known (see [10]) and can be easily proved using the Binet formula (2):

$$U_{m+n} = U_m U_{n+1} - Q U_{m-1} U_n, \tag{3}$$

$$U_n^2 = U_{n-1} U_{n+1} + Q^{n-1}. \tag{4}$$

As mentioned earlier, we assume that $\gcd(P,Q) = 1$. It is worth noting that in this case, $U_n(P,Q)$ is a *strong divisibility sequence*, meaning that $\gcd(U_m, U_n) = U_{\gcd(m,n)}$ for all $m, n \geq 1$.

Broderius and Greene [2] identified two classes of Lucas sequences which are eventually prime-free. For completeness, we state their result under the additional assumptions $P \geq 1$ and $P^2 - 4Q > 0$.

Theorem 4. *([2]) Let P and Q be integers such that $P \geq 1$ and $P^2 - 4Q > 0$.*

(a) If Q is a perfect square, then the sequence with general term $U_n(P,Q)$ contains only finitely many primes.

(b) If x and y are distinct real numbers such that $x + y$ and xy are both integers

and $r > 1$ is an integer, then the sequence with general term $U_n(x^r + y^r, x^r y^r)$ contains only finitely many primes.

Proof. We briefly sketch the proofs, as we find them instructive. For part (a), let $Q = R^2$ for some integer $R \geq 1$. Taking $m = n$ in Equation (3), we obtain

$$U_{2n} = U_n U_{n+1} - QU_{n-1}U_n = U_n(U_{n+1} - QU_{n-1}). \tag{5}$$

Next taking $m = n + 1$ in Equation (3), we obtain

$$U_{2n+1} = U_{n+1}^2 - QU_n^2 = (U_{n+1} - RU_n)(U_{n+1} + RU_n). \tag{6}$$

It follows that both U_{2n} and U_{2n+1} can be expressed as the product of two integers. One can easily prove that both factors are greater than 1 for all sufficiently large n.

For part (b), Broderius and Greene make use of the identity

$$U_n(x^r + y^r, x^r y^r) = \frac{U_n(x + y, xy)U_r(x^n + y^n, x^n y^n)}{U_r(x + y, xy)}, \tag{7}$$

which implies the desired conclusion as soon as one proves that both factors in the numerator are greater than the denominator for a fixed r and a sufficiently large n. For the sake of brevity, we omit this argument. □

Broderius and Greene conjectured that every Lucas sequence that does not belong to one of these two families will contain infinitely many prime terms. Theorem 1 provides a counterexample.

Proof of Theorem 1. The characteristic equation of the Lucas sequence $U_n(3, -4)$ is $z^2 - 3z - 4 = (z - 4)(z + 1) = 0$, which means that $U_n(3, -4) = (4^n - (-1)^n)/5$ for all $n \geq 0$. It immediately follows that

$$\begin{aligned} 5U_{2n}(3, -4) &= 4^{2n} - 1 = (4^n - 1)(4^n + 1), \quad \text{and} \\ 5U_{2n+1}(3, -4) &= 4^{2n+1} + 1 = (2 \cdot 4^n + 1)^2 - (2^{n+1})^2 \\ &= (2^{2n+1} - 2^{n+1} + 1)(2^{2n+1} + 2^{n+1} + 1). \end{aligned}$$

Hence, $5U_n(3, -4)$ can be written as the product of two integers which are both greater than 5 as soon as $n \geq 4$. The conclusion of Theorem 1 follows. □

Note that $U_n(3, -4)$ is not among the sequences covered by Theorem 4, even if we allow x and y be complex numbers. First, $Q = -4$, so it is not a perfect square. Second, suppose that there exist complex numbers x, y so that $x + y$ and xy are integers, and there exists an integer $r > 1$ with $x^r + y^r = 3$ and $x^r y^r = -4$. This immediately implies that $x^r = 4$ and $y^r = -1$, which further gives $x = 4^{1/r}(\cos(2k\pi/r) + i\sin(2k\pi/r))$ and $y = \cos((2l + 1)\pi/r) + \sin((2l + 1)\pi/r)$ for some integers k and l.

The product xy equals $4^{1/r}(\cos((2k+2l+1)\pi/r)+i\sin((2k+2l+1)\pi/r))$. Since xy is an integer, it must be that $xy = \pm 4^{1/r}$, and furthermore the only option is $r = 2$. But in this case, $x^2 = 4$ and $y^2 = -1$; this means that $x+y = \pm 2 \pm i$, which is not an integer.

We next present a proof of Theorem 2. We will use the abbreviated notation U_n to represent $U_n(P,1)$.

Proof of Theorem 2. Since $Q = 1$, the fact that U_n is prime for at most finitely many n is a consequence of Theorem 4. Moreover, identities (5) and (6) become

$$U_{2n} = U_nU_{n+1} - U_{n-1}U_n = U_n(U_{n+1} - U_{n-1}), \tag{8}$$

$$U_{2n+1} = U_{n+1}^2 - U_n^2 = (U_{n+1} - U_n)(U_{n+1} + U_n). \tag{9}$$

Note that in this case, identity (4) becomes

$$U_n^2 = U_{n-1}U_{n+1} + 1, \tag{10}$$

which is sometimes referred to as *Cassini's identity.*

Using the identities (8), (9), and (10), the quantities of the form $-1+U_n$ and $1+U_n$ can be expressed as follows:

$$\begin{aligned} -1 + U_{2n} &= (U_{n+1} - U_n)(U_n + U_{n-1}), \\ -1 + U_{2n+1} &= U_n(U_{n+2} - U_n), \\ 1 + U_{2n} &= (U_n - U_{n-1})(U_{n+1} + U_n), \\ 1 + U_{2n+1} &= U_{n+1}(U_{n+1} - U_{n-1}). \end{aligned}$$

It follows that for every sufficiently large n, any term of the form $-1+U_n$, U_n, or $1+U_n$ can be written as a product of two integers greater than 1, which completes the proof. □

3. Method

Our proof of Theorem 3 is based on two concepts.

On one hand, given any Lucas sequence $U_n = U_n(P,-1)$ and any positive integer m, the sequence $U_n(P,-1)$ is purely periodic modulo m. We include the standard argument.

Due to the nature of the recurrence relation $U_n \equiv P \cdot U_{n-1} + U_{n-2} \pmod{m}$, from any point of the sequence, the future terms of the sequence are completely determined by two consecutive terms. As there are m^2 possible values of the pairs $(U_{n-2} \pmod{m}, U_{n-1} \pmod{m})$, we deduce that the sequence must eventually be periodic modulo m. Since $U_{n-2} \equiv U_n - P \cdot U_{n-1} \pmod{m}$, it follows that this process is reversible; that is, the values of the earlier terms are uniquely determined by any two consecutive terms. As we must eventually reach the beginning of the sequence, we conclude that $U_n(P,-1)$ must be purely periodic.

Following Jones [8], the *period* of $U_n(P,-1)$ modulo m is the smallest positive integer $h = h(P,m)$ such that $U_h \equiv 0 \pmod{m}$ and $U_{h+1} \equiv 1 \pmod{m}$. We refer to the actual list of residues modulo m that occur from index 0 to index $h-1$ as the *cycle* of $U_n(P,-1)$ modulo m, and denote it by $\mathcal{C}(P,m)$.

For future purposes, we list the cycles corresponding to the Fibonacci sequence ($P=1$) and $m \in \{2,3,5,7,11,23,31\}$. These cycles are

$$\begin{aligned}
\mathcal{C}(1,2) &= [0,1,1],\\
\mathcal{C}(1,3) &= [0,1,1,2,0,2,2,1],\\
\mathcal{C}(1,5) &= [0,1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1],\\
\mathcal{C}(1,7) &= [0,1,1,2,3,5,1,6,0,6,6,5,4,2,6,1],\\
\mathcal{C}(1,11) &= [0,1,1,2,3,5,8,2,10,1],\\
\mathcal{C}(1,23) &= [0,1,1,2,3,5,8,13,21,11,9,20,6,3,9,12,21,10,8,18,3,21,1,22,0,22,\\
&\qquad 22,21,20,18,15,10,2,12,14,3,17,20,14,11,2,13,15,5,20,2,22,1],\\
\mathcal{C}(1,31) &= [0,1,1,2,3,5,8,13,21,3,24,27,20,16,5,21,26,16,11,27,7,3,10,13,23,\\
&\qquad 5,28,2,30,1].
\end{aligned}$$

Note that $h(1,2)=3$, $h(1,3)=8$, $h(1,5)=20$, $h(1,7)=16$, $h(1,11)=10$, $h(1,23)=48$, and $h(1,31)=30$.

The second useful concept is that of a finite covering system.

Definition 1. A *finite covering system*, or simply a *covering*, of the integers is a system of congruences $x \pmod{m_i} \in R_i$ such that every integer satisfies at least one of the congruences. Here, the m_i-s are all distinct and greater than 1, and each R_i is a nonempty subset of $\{0,1,2,\ldots,m_i-1\}$.

We write a covering system as a set of ordered pairs $\{(R_i,m_i)\}$. Typically, each set R_i consists of a single element, but occasionally (as will be the case in the sequel) multiple entries are allowed.

Finite covering systems were introduced by Erdős [3], who used the covering

$$\{(\{0\},2),(\{0\},3),(\{1\},4),(\{3\},8),(\{7\},12),(\{23\},24)\}$$

to prove that numbers of the form $|2^n - 7629217|$ are composite for all $n \geq 0$.

In order to illustrate how these two ideas can be combined and better motivate our next steps, we first present a proof of the following result.

Theorem 5. *Let F_n be the n-th Fibonacci number. Then, $F_n + 14475$ is composite for all $n \geq 0$.*

Proof. Let $K = 14475$. Then the cycles $\mathcal{C}(1, m)$, $m \in \{2, 3, 5, 7, 11, 23, 31\}$ listed above allow us the following observations:

- Since $K \equiv 1 \pmod 2$, then $F_n + K \equiv 0$ if and only if $n \equiv 1, 2 \pmod 3$.
- Since $K \equiv 0 \pmod 3$, then $F_n + K \equiv 0$ if and only if $n \equiv 0, 4 \pmod 8$.
- Since $K \equiv 0 \pmod 5$, then $F_n + K \equiv 0$ if and only if $n \equiv 0, 5, 10, 15 \pmod{20}$.
- Since $K \equiv 6 \pmod 7$, then $F_n + K \equiv 0$ if and only if $n \equiv 1, 2, 6, 15 \pmod{16}$.
- Since $K \equiv 10 \pmod{11}$, then $F_n + K \equiv 0$ if and only if $n \equiv 1, 2, 9 \pmod{10}$.
- Since $K \equiv 8 \pmod{23}$, then $F_n + K \equiv 0$ if and only if $n \equiv 30, 42 \pmod{48}$.
- Since $K \equiv 29 \pmod{31}$, then $F_n + K \equiv 0$ if and only if $n \equiv 3, 37 \pmod{30}$.

It is straightforward to verify that the pairs $(\{1, 2\}, 3)$, $(\{0, 4\}, 8)$, $(\{0, 5, 10, 15\}, 20)$, $(\{1, 2, 6, 15\}, 16)$, $(\{1, 2, 9\}, 10)$, $(\{30, 42\}, 48)$, $(\{3, 27\}, 30)$ form a covering system. It follows that for every $n \geq 0$, $F_n + K$ has at least one factor in $\{2, 3, 5, 7, 11, 23, 31\}$, so $F_n + 14475$ is always composite. □

The success of our construction relied on the existence of several coprime moduli

$$q_1 = 2, q_2 = 3, q_3 = 5, q_4 = 7, q_5 = 11, q_6 = 23, q_7 = 31,$$

and a set of residues

$$s_1 = 1, s_2 = 0, s_3 = 0, s_4 = 6, s_5 = 10, s_6 = 8, s_7 = 29,$$

such that the system

$$\{(R_i, h(1, q_i)) \mid F_n + s_i \equiv 0 \pmod{q_i} \text{ if and only if } n \pmod{h(1, q_i)} \in R_i\} \quad (11)$$

is a covering. Note that it is not essential that $q_1, q_2, \ldots$ are prime numbers, but coprimality is required to ensure that the system $K \equiv s_i \pmod{q_i}$ has solutions.

We plan to extend Theorem 5 in two different ways. First, we will prove that for every $P \geq 1$, there exists a positive integer $K = K(P)$ such that $U_n(P, -1) + K$ is composite for all $n \geq 0$. This has already been proved by Jones [8] using a different approach.

As in the Fibonacci case, we will identify a finite set of coprime numbers $q_1, q_2, \ldots$ and a corresponding set of residues $s_1, s_2, \ldots$, such that the system

$$\{(R_i, h(P, q_i)) \mid U_n(P, -1) + s_i \equiv 0 \pmod{q_i} \text{ if and only if } n \pmod{h(P, q_i)} \in R_i\} \quad (12)$$

is a covering. This would imply that if we let $K(P)$ be a sufficiently large solution of the system $K \equiv s_1 \pmod{q_1}$, $K \equiv s_2 \pmod{q_2}$, $\ldots$, then $U_n(P, -1) + K(P)$ will be divisible by at least one of the numbers $q_1, q_2, \ldots$, and therefore composite.

Next, we will prove that $K(P)$ can be chosen with the additional property that for every $n \geq 0$, both $U_n(P,-1) + K(P) - 1$ and $U_n(P,-1) + K(P) + 1$ are also divisible by at least one of the numbers $q_1, q_2, \ldots$. This may sound challenging at first, given the somewhat awkward formulation given in (12). However, it can be done, and with relatively little difficulty. The key lies in the way we will be choosing the coprime numbers $q_1, q_2, \ldots$. From now on we use the shortened notation U_n for $U_n(P,-1)$.
We will always take $q_1 = 2$ and $q_2 = 3$. The generic cycle U_n modulo 2 is

$$\mathcal{C}(P,2) = [0,1,P,P+1,P,1] \pmod{2}, \tag{13}$$

which becomes $[0,1]$ if P is even and $[0,1,1]$ if P is odd.
The generic cycle U_n modulo 3 is

$$\mathcal{C}(P,3) = [0,1,P,P^2+1,0,P^2+1,-P,1] \pmod{3}, \tag{14}$$

which is $[0,1]$, $[0,1,1,2,0,2,2,1]$, or $[0,1,2,2,0,2,1,1]$ depending on whether $P \equiv 0 \pmod{3}$, $P \equiv 1 \pmod{3}$, or $P \equiv 2 \pmod{3}$.
Consider the quantity

$$\gcd(U_{24}, U_{25} - 1) = P(P^2+1)(P^2+2)(P^2+3)(P^4+4P^2+1). \tag{15}$$

Computing the cycles of U_n modulo each of the factors on the right-hand side of equality (15), we obtain

$$\begin{aligned}
\mathcal{C}(P,P) &= [0,1],\\
\mathcal{C}(P,P^2+1) &= [0,1,P,0,P,-1,0,-1,-P,0,-P,1],\\
\mathcal{C}(P,P^2+2) &= [0,1,P,-1,0,-1,-P,1],\\
\mathcal{C}(P,P^2+3) &= [0,1,P,-2,-P,1],\\
\mathcal{C}(P,P^4+4P^2+1) &= [0,1,P,P^2+1,P^3+2P,-P^2,2P,P^2,\\
&\qquad P^3+2P,-P^2-1,P,-1,0,-1,-P,-P^2-1,\\
&\quad -P^3-2P,P^2,-2P,-P^2,-P^3-2P,P^2+1,-P,1].
\end{aligned}$$

Unsurprisingly, the periods of P, P^2+1, P^2+2, P^2+3, and P^4+4P^2+1 are $2, 12, 8, 6$, and 24, respectively − all integer factors of 24. We plan to use these numbers towards our choices of $q_3, q_4, \ldots$. Recall that $q_1 = 2$ and $q_2 = 3$. One still has to require that the q_i-s be coprime. However, this challenge can be readily circumvented.

Definition 2. For any integer $d > 1$, let $f(d)$ denote the largest positive factor of d that is coprime to 6.

Eventually, we choose $q_1 = 2$, $q_2 = 3$, and the remaining $q_3, q_4, \ldots$ as a subset of

$$\{f(P), f(P^2+1), f(P^2+2), f(P^2+3), f(P^4+4P^2+1)\}. \tag{16}$$

As we shall soon see, if $P > 4$, then at least four of these five quantities are coprime and strictly greater than 1. Together with $q_1 = 2$, $q_2 = 3$, this will allow us to produce the covering claimed earlier.

The following section contains the details of this construction. Naturally, the discussion will be divided into six cases, depending on the value of P modulo 6. We treat the values $1 \leq P \leq 4$ in the final section of the paper.

4. Proof of Theorem 3 for All $P > 4$

We will use $\mathrm{Chrem}([s_1, s_2, s_3, \ldots], [q_1, q_2, q_3, \ldots])$ to represent the system of congruences $K \equiv s_1 \pmod{q_1}$, $K \equiv s_2 \pmod{q_2}$, $K \equiv s_3 \pmod{q_3}$, etc. This notation is inspired by a command in Maple which has a similar syntax, and it will be used extensively in the sequel.

Proof. We prove this in cases.

`Case 1:` Assume that $P \equiv 0 \pmod 6$, $P > 4$. Consider the following four positive integers:

$$q_1 = 2, q_2 = 3, q_3 = P^2 + 1, \text{ and } q_4 = f(P^2 + 3) = \frac{P^2 + 3}{3}. \tag{17}$$

It is easy to see that these numbers are coprime and greater than 1. Let K be a solution of the following system of congruences:

$$\mathrm{Chrem}\left([-1, 0, -1, -2], \left[2, 3, P^2 + 1, \frac{P^2 + 3}{3}\right]\right).$$

One can easily check that $K = (7P^4 + 31P^2 + 18)/6$ is one such solution, and that it is greater than any of the individual q_i. It can be readily verified that this choice of K ensures the desired triple covering. The full details are presented in Table 1 in the Appendix.

`Case 2:` Assume that $P \equiv 1 \pmod 6$, $P > 4$. This case is similar to the previous one, the only difference being that this time we use six numbers instead of four:

$$\begin{aligned} &q_1 = 2, q_2 = 3, q_3 = P, q_4 = f(P^2 + 1) = \frac{P^2 + 1}{2}, q_5 = f(P^2 + 3) = \frac{P^2 + 3}{4}, \\ &q_6 = f(P^4 + 4P^2 + 1) = \frac{P^4 + 4P^2 + 1}{6}. \end{aligned} \tag{18}$$

Again, it can be easily checked that q_i, $i = 1 \ldots 6$ are all greater than 1 and pairwise coprime. Let K be a solution of the following system of congruences:

$$\mathrm{Chrem}\left(\left[0, 0, 0, 1, -1, P^2\right], \left[2, 3, P, \frac{P^2 + 1}{2}, \frac{P^2 + 3}{4}, \frac{P^4 + 4P^2 + 1}{6}\right]\right). \tag{19}$$

One can check that

$$K = P \cdot \frac{P^4 + 4P^2 + 1}{6} \cdot \frac{P^4 + 16P^3 + 4P^2 + 64P + 3}{8} + P^2$$

is one such solution. We claim that this choice of K ensures that each term of any of the three sequences $U_n + K - 1$, $U_n + K$, $U_n + K + 1$ is a proper multiple of at least one of the q_i listed in (18). The details are shown in Table 2 of the Appendix. The verifications are rather straightforward and very similar to the ones presented in the previous case.

`Case 3:` Assume that $P \equiv -1 \pmod 6$, $P > 4$. This case is almost identical to the previous one. We use exactly the same residues s_i and moduli q_i as in (18), and we require K to satisfy the same system (19). The expression of K is slightly different:

$$K = P \cdot \frac{P^4 + P^2 + 1}{6} \cdot \frac{P^4 + 16P^3 + 4P^2 + 64P + 3}{8} + P^2.$$

The table presenting the covering is also almost unchanged; the only differences appear in the row corresponding to $q_2 = 3$ in the columns for $U_n + K - 1$ and $U_n + K + 1$. We encourage the reader to compare Table 2 to Table 3 in the Appendix.

`Case 4:` Assume that $P \equiv 3 \pmod 6$, $P > 4$. In this case, we still use the following six q_i-s:

$$q_1 = 2, q_2 = 3, q_3 = f(P^2 + 1) = \frac{P^2 + 1}{2}, q_4 = P^2 + 2, q_5 = f(P^2 + 3) = \frac{P^2 + 3}{12},$$
$$q_6 = f(P^4 + 4P^2 + 1) = \frac{P^4 + 4P^2 + 1}{2}. \tag{20}$$

As in the previous cases, it can be verified that all these numbers are greater than 1 and pairwise coprime. Let K be a solution of the following system of congruences:

$$\text{Chrem}\left(\left[0, 0, 1, 0, -1, P^2\right], \left[2, 3, \frac{P^2 + 1}{2}, P^2 + 2, \frac{P^2 + 3}{12}, \frac{P^4 + 4P^2 + 1}{2}\right]\right).$$

One solution of this system is given by

$$K = \frac{P^4 + 4P^2 + 1}{2} \cdot \frac{7P^6 + 26P^4 + 13P^2 - 54}{24} + P^2.$$

We claim that this choice of K ensures that each term of any of the three sequences $U_n + K - 1$, $U_n + K$, $U_n + K + 1$ is a proper multiple of at least one of the q_i defined in Equation (20). This is indeed the case, as it can be checked that every column in Table 4 from the Appendix is a covering.

`Case 5:` Assume that $P \equiv 2 \pmod 6$, $P > 4$. In this case, the following six numbers will be sufficient to achieve the desired triple covering:

$$q_1 = 2, q_2 = 3, q_3 = P^2 + 1, q_4 = f(P^2 + 2), q_5 = P^2 + 3,$$
$$q_6 = f(P^4 + 4P^2 + 1) = \frac{P^4 + 4P^2 + 1}{3}. \tag{21}$$

It is not difficult to check that all these numbers are pairwise coprime. However, one needs to argue that $f(P^2+2) > 1$.

Suppose $P = 6t+2$ for some integer $t \geq 1$. Then $(P^2+2)/6 = 6t^2+4t+1$. This quantity, while odd, may very well be a multiple of 3. Recall that $f(P^2+2)$ is the greatest factor of P^2+2 which is coprime with 6. The only way $f(P^2+2) = 1$ is if $(P^2+2)/2$ is a perfect power of 3, which is equivalent to the diophantine equation

$$1 + 2(3t+1)^2 = 3^r \tag{22}$$

having solutions. However, Ahn et al. (see Lemma 3 in [1]) proved the only nonnegative integer solutions of the equation $1 + 2m^2 = 3^r$ are $(m, r) = (0, 0), (1, 1), (2, 2)$ and $(11, 5)$. Since $t \geq 1$, it follows that equation (22) does not have integer solutions, so $f(P^2+2) > 1$ as claimed.

Let K be a solution of the following system of congruences:

$$\text{Chrem}\left([1, 1, 1, 0, 0, P], \left[2, 3, P^2+1, f(P^2+2), P^2+3, \frac{P^4+4P^2+1}{3}\right]\right).$$

As we do not have an explicit expression for $f(P^2+2)$, we cannot write K in closed form as in the previous cases. However, the existence of infinitely many such solutions is guaranteed by the Chinese Remainder Theorem. We claim that a sufficiently large K will ensure that each term of any of the three sequences $U_n + K - 1$, $U_n + K$, $U_n + K + 1$ is a proper multiple of at least one of the moduli listed in (21). The details are presented in Table 5 of the Appendix.

`Case 6:` Assume that $P \equiv -2 \pmod 6$, $P > 4, P \neq 22$. In this case, we will be using the same six values of q_i as in (21). Again, it can be verified that these numbers are still pairwise coprime. As in the previous case, we need to show that $f(P^2+2) > 1$.

Suppose $P = 6t-2$ for some integer $t \geq 1$. Then, as above, $f(P^2+2) = 1$ if and only if the diophantine equation

$$1 + 2(3t-1)^2 = 3^r \tag{23}$$

has integer solutions. Using Lemma 3 from [1] again, it follows that either $t = 1$ or $t = 4$, which implies that $P = 4$ or $P = 22$. As mentioned earlier, these cases will be studied separately. Assuming for now that $P \notin \{4, 22\}$, it follows that $f(P^2+2) > 1$, as needed.

Let K be a solution of the following system of congruences:

$$\text{Chrem}\left([1, 1, 1, 0, 0, -P], \left[2, 3, P^2+1, f(P^2+2), P^2+3, \frac{P^4+4P^2+1}{3}\right]\right).$$

The existence of such a K is guaranteed by the Chinese Remainder Theorem. We claim that a sufficiently large K will ensure that each term of any of the three sequences $U_n + K - 1$, $U_n + K$, $U_n + K + 1$ is a proper multiple of at least one of the moduli listed in (21). We present the triple covering in Table 6 below, which is almost identical to Table 5 from the previous case. The only differences appear in the rows of $q_2 = 3$ and $q_6 = (P^4 + 4P^2 + 1)/3$.

`Case 7:` Assume that $P = 22$. In this case, we will be using $q_1 = 2, q_2 = 3, q_3 = 5, q_4 = 11, q_5 = 97, q_6 = 131$. The periods of U_n modulo each of these primes are $2, 8, 12, 2, 12$, and 24, respectively.

Let K be a solution of the following system of congruences:

$$\text{Chrem}\left([1, 0, 0, 0, -1, -2], [2, 3, 5, 11, 97, 131]\right).$$

It can be checked that $K = 142395$ is the smallest positive solution.

We claim that for this value of K, each term of any of the three sequences $U_n + K - 1$, $U_n + K$, $U_n + K + 1$ is a proper multiple of at least one of the moduli $2, 3, 5, 11, 97, 131$. We present the triple covering in Table 7 of the Appendix. □

5. Proof of Theorem 3 for $1 \leq P \leq 4$

For proving Theorem 3 for small values of P, we require some modifications. This is due to the fact that at least some of our preferred moduli in (16) become equal to 1 in these cases. For example, if $P = 4$, then $P^2+2 = 18$, and thus $f(P) = f(P^2+2) = 1$. A similar situation takes place when $P = 2$. If $P = 3$, then $P^2 + 3 = 12$, so $f(P) = f(P^2 + 3) = 1$. Finally, if $P = 1$, then all of the last five moduli in (16) are equal to 1. It is therefore reasonable that if one attempts to achieve the same type of triple covering as above, more moduli are going to be needed. We present the details below.

Proof of Theorem 3. We prove this in cases.

`Case 1:` Assume that $P = 4$. In this case, we use the following eleven distinct prime moduli:

$$2, 3, 5, 11, 17, 19, 31, 41, 61, 107, 181. \tag{24}$$

These primes are all divisors of $\gcd(U_{120}, U_{121} - 1)$, and it can be verified that the corresponding periods are $2, 3, 20, 10, 12, 6, 10, 40, 20, 24$, and 30, respectively.

Let K be the smallest positive solution of the following system of congruences:

$$\text{Chrem}\left([1, 0, 3, 4, 4, 0, 27, 17, 0, 91, 47], [2, 3, 5, 11, 17, 19, 31, 41, 61, 107, 181]\right).$$

It turns out that $K = 69411071167563$. We claim that for this choice of K, each term of $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ is divisible by at least one of the primes given in (24). We present the triple covering in Table 8 of the Appendix.

`Case 2:` Assume that $P = 3$. In this case, we use the following eleven distinct prime moduli:

$$M := [2, 3, 5, 11, 19, 59, 61, 109, 131, 211, 739]. \tag{25}$$

As in the previous case, all these primes are divisors of $\gcd(U_{120}, U_{121} - 1)$, so the lengths of the periods of $U_n \pmod{M_i}, i = 1 \ldots 12$, are all factors of 120. It can be verified that these lengths are $3, 2, 12, 8, 40, 24, 30, 20, 10, 30$, and 40, respectively.

Let K be the smallest positive solution of the following system of congruences:

$$\text{Chrem}\left([0, 0, 4, 1, 17, 50, 1, 34, 34, 127, 4], [2, 3, 5, 11, 19, 59, 61, 109, 131, 211, 739]\right).$$

The explicit value is $K = 353531805205404$. We claim that for this choice of K, each term of $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ is divisible by at least one of the primes given in (25). We present the triple covering in Table 9. Note that the last five primes are only contributing towards covering the terms $U_n + K - 1$.

`Case 3:` Assume that $P = 2$. The Lucas sequence $U_n(2, -1)$ is known as the *Pell sequence*. In this case, we use the following twelve distinct prime moduli:

$$M = [2, 3, 5, 7, 11, 19, 29, 31, 41, 59, 269, 601]. \tag{26}$$

Like before, all these primes are divisors of $\gcd(U_{120}, U_{121} - 1)$, hence the lengths of the periods of $U_n \pmod{M_i}, i = 1 \ldots 12$, are all factors of 120. It can be verified that these lengths are $2, 8, 12, 6, 24, 40, 20, 30, 10, 40, 60$, and 120, respectively.

Let K be the smallest positive solution of the following system of congruences:

$$\text{Chrem}\left([1, 0, 2, 5, 4, 2, 1, 3, 37, 42, 0, 531], [2, 3, 5, 7, 11, 19, 29, 31, 41, 59, 269, 601]\right).$$

The smallest solution of this system is $K = 100784662201557$. We claim that for this choice of K, each term of $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ is divisible by at least one of the primes given in (26). The details of this triple covering are contained in Table 10.

`Case 4:` Assume that $P = 1$. Finally, we prove Theorem 3 for $U_n(1, -1) = F_n$, the Fibonacci sequence. As noticed earlier, one would expect this case to be the the most challenging, as we have $f(P) = f(P^2 + 1) = f(P^2 + 2) = f(P^2 + 3) = f(P^4 + 4P^2 + 1) = 1$. Nevertheless, a triple covering is possible; we found one such covering using the following 25 primes:

$$\begin{aligned} M = [&2, 3, 5, 7, 11, 17, 19, 23, 31, 41, 47, 61, 107, 181, 241, 541, \\ &1103, 1601, 2161, 2521, 3041, 8641, 20641, 103681, 109441]. \end{aligned} \tag{27}$$

All these primes are divisors of $\gcd(F_{1440}, F_{1441} - 1)$, hence the periods of F_n $\pmod{M_i}, i = 1 \ldots 25$, are all factors of $1440 = 2^5 \cdot 3^2 \cdot 5$. It can be verified that these periods are $3, 8, 20, 16, 10, 36, 18, 48, 30, 40, 32, 60, 72, 90, 240, 90, 96, 160, 80, 120, 160, 720, 240, 144$, and 180, respectively.

Let S be the following list of residues:

$$S = [0, 0, 1, 1, 9, 1, 18, 22, 26, 1, 45, 60, 100, 86, 137, 143,$$
$$145, 1, 55, 1530, 3040, 643, 377, 2591, 7],$$

and let K be the smallest positive solution of the of congruences Chrem(S, M), where M is given in Equation (27). We obtain the following 57-digit solution:

$$K := 305634298839076198110201000141426680099130790997545994766. \quad (28)$$

We claim that for this choice of K, each term of $F_n + K - 1$, $F_n + K$, and $F_n + K + 1$ is divisible by at least one of the primes given in Equation (27). The details of this triple covering are contained in Table 11 of the Appendix. □

6. Final Comments

There are many interesting questions which are worth further study. We mention several possible avenues of future research.

- Are there any pairs of coprime integers P, Q, other than the ones listed in Theorem 4 and Theorem 1, such that $U_n(P, Q)$ is composite for all but finitely many n? In their report, the referee noted that one may obtain additional examples if one considers Aurifeuillean factorizations. In particular, they point out that if $2xy$ is a perfect square, then $x^{4n+2} + y^{4n+2} = (x^{2n+1} + y^{2n+1})^2 - 2xy(xy)^{2n}$ is factorable. Presumably, one might be able to construct infinitely many counterexamples to the conjecture in [2].

- Is is possible to extend Theorem 3 for all integer values of Q? We certainly expect the answer to be affirmative. In fact, we propose the following stronger conjecture.

  Conjecture 1. Let P, Q be coprime integers so that $P \geq 1$ and $P^2 - 4Q > 0$. Then, for every $d \geq 1$, there exists a positive integer K such that

  $$|U_n(P, Q) + K - p| \geq d, \text{ for every } n \geq 0 \text{ and every prime } p.$$

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Appendix. We include several tables containing the triple coverings relevant to the proof of Theorem 3.

	U_n+K-1	U_n+K	U_n+K+1
2	$n\equiv 0 \pmod 2$	$n\equiv 1 \pmod 2$	$n\equiv 0 \pmod 2$
3	$n\equiv 1 \pmod 2$	$n\equiv 0 \pmod 2$	
P^2+1		$n\equiv 1,11 \pmod{12}$	$n\equiv 0 \pmod 3$
$(P^2+3)/3$			$n\equiv 1,5 \pmod 6$

Table 1: A triple covering when $P\equiv 0 \pmod 6, P>4$. Each cell contains the values of n for which U_n+K-1, U_n+K, and U_n+K+1 are proper multiples of the moduli 2, 3, P^2+1, and $(P^2+3)/3$, respectively.

	U_n+K-1	U_n+K	U_n+K+1
2	$n\equiv 1,2 \pmod 3$	$n\equiv 0 \pmod 3$	$n\equiv 1,2 \pmod 3$
3	$n\equiv 1,2,7 \pmod 8$	$n\equiv 0 \pmod 4$	$n\equiv 3,5,6 \pmod 8$
P	$n\equiv 1 \pmod 2$	$n\equiv 0 \pmod 2$	
$(P^2+1)/2$	$n\equiv 0 \pmod 3$	$n\equiv 5,7 \pmod{12}$	
$(P^2+3)/4$		$n\equiv 1,5 \pmod 6$	$n\equiv 0 \pmod 6$
$(P^4+4P^2+1)/6$		$n\equiv 5,19 \pmod{24}$	$n\equiv 9,15 \pmod{24}$

Table 2: A triple covering when $P\equiv 1 \pmod 6, P>4$. Each cell contains the values of n for which U_n+K-1, U_n+K, and U_n+K+1 are proper multiples of the moduli 2, 3, P, $(P^2+1)/2$, $(P^2+3)/4$, and $(P^4+4P^2+1)/6$, respectively.

	U_n+K-1	U_n+K	U_n+K+1
2	$n\equiv 1,2 \pmod 3$	$n\equiv 0 \pmod 3$	$n\equiv 1,2 \pmod 3$
3	$n\equiv 1,6,7 \pmod 8$	$n\equiv 0 \pmod 4$	$n\equiv 2,3,5 \pmod 8$
P	$n\equiv 1 \pmod 2$	$n\equiv 0 \pmod 2$	
$(P^2+1)/2$	$n\equiv 0 \pmod 3$	$n\equiv 5,7 \pmod{12}$	
$(P^2+3)/4$		$n\equiv 1,5 \pmod 6$	$n\equiv 0 \pmod 6$
$(P^4+4P^2+1)/6$		$n\equiv 5,19 \pmod{24}$	$n\equiv 9,15 \pmod{24}$

Table 3: A triple covering when $P\equiv -1 \pmod 6, P>4$. Each cell contains the values of n for which U_n+K-1, U_n+K, and U_n+K+1 are proper multiples of the moduli 2, 3, P, $(P^2+1)/2$, $(P^2+3)/4$, and $(P^4+4P^2+1)/6$, respectively.

	$U_n + K - 1$	$U_n + K$	$U_n + K + 1$
2	$n \equiv 1, 2 \pmod 3$	$n \equiv 0 \pmod 3$	$n \equiv 1, 2 \pmod 3$
3	$n \equiv 1 \pmod 2$	$n \equiv 0 \pmod 2$	
$(P^2 + 1)/2$	$n \equiv 0 \pmod 3$	$n \equiv 5, 7 \pmod{12}$	
$P^2 + 2$	$n \equiv 1, 7 \pmod 8$	$n \equiv 0 \pmod 4$	$n \equiv 3, 5 \pmod 8$
$(P^2 + 3)/12$		$n \equiv 1, 5 \pmod 6$	$n \equiv 0 \pmod 6$
$(P^4 + 4P^2 + 1)/2$		$n \equiv 5, 19 \pmod{24}$	$n \equiv 9, 15 \pmod{24}$

Table 4: A triple covering when $P \equiv 3 \pmod 6, P > 4$. Each cell contains the values of n for which $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ are proper multiples of 2, 3, $(P^2 + 1)/2$, $P^2 + 2$, $(P^2 + 3)/12$, and $(P^4 + 4P^2 + 1)/2$, respectively.

	$U_n + K - 1$	$U_n + K$	$U_n + K + 1$
2	$n \equiv 0 \pmod 2$	$n \equiv 1 \pmod 2$	$n \equiv 0 \pmod 2$
3	$n \equiv 0 \pmod 4$	$n \equiv 2, 3, 5 \pmod 8$	$n \equiv 1, 6, 7 \pmod 8$
$P^2 + 1$	$n \equiv 0 \pmod 3$	$n \equiv 5, 7 \pmod{12}$	
$f(P^2 + 2)$	$n \equiv 1, 7 \pmod 8$	$n \equiv 0 \pmod 4$	$n \equiv 3, 5 \pmod 8$
$P^2 + 3$	$n \equiv 1, 5 \pmod 6$	$n \equiv 0 \pmod 6$	
$(P^4 + 4P^2 + 1)/3$		$n \equiv 14, 22 \pmod{24}$	

Table 5: A triple covering when $P \equiv 2 \pmod 6$. Each cell contains the values of n for which $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ are proper multiples of 2, 3, $P^2 + 1$, $f(P^2 + 2)$, $P^2 + 3$, and $(P^4 + 4P^2 + 1)/3$, respectively.

	$U_n + K - 1$	$U_n + K$	$U_n + K + 1$
2	$n \equiv 0 \pmod 2$	$n \equiv 1 \pmod 2$	$n \equiv 0 \pmod 2$
3	$n \equiv 0 \pmod 4$	$n \equiv 3, 5, 6 \pmod 8$	$n \equiv 1, 2, 7 \pmod 8$
$P^2 + 1$	$n \equiv 0 \pmod 3$	$n \equiv 5, 7 \pmod{12}$	
$f(P^2 + 2)$	$n \equiv 1, 7 \pmod 8$	$n \equiv 0 \pmod 4$	$n \equiv 3, 5 \pmod 8$
$P^2 + 3$	$n \equiv 1, 5 \pmod 6$	$n \equiv 0 \pmod 6$	
$(P^4 + 4P^2 + 1)/3$		$n \equiv 2, 10 \pmod{24}$	

Table 6: A triple covering when $P \equiv -2 \pmod 6$, $P \notin \{4, 22\}$. Each cell contains the values of n for which $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ are proper multiples of 2, 3, $P^2 + 1$, $f(P^2 + 2)$, $P^2 + 3$, and $(P^4 + 4P^2 + 1)/3$, respectively.

	$U_n + K - 1$	$U_n + K$	$U_n + K + 1$
2	$n \equiv 0 \pmod 2$	$n \equiv 1 \pmod 2$	$n \equiv 0 \pmod 2$
3	$n \equiv 1, 2, 7 \pmod 8$	$n \equiv 0 \pmod 4$	$n \equiv 3, 5, 6 \pmod 8$
5	$n \equiv 1, 11 \pmod{12}$	$n \equiv 0 \pmod 3$	$n \equiv 5, 7 \pmod{12}$
11	$n \equiv 1 \pmod 2$	$n \equiv 0 \pmod 2$	
97		$n \equiv 1, 11 \pmod{12}$	$n \equiv 0 \pmod 3$
131			$n \equiv 1, 23 \pmod{24}$

Table 7: A triple covering when $P = 22$. Each cell contains the values of n for which $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ are proper multiples of the moduli $2, 3, 5, 11, 97$, and 131, respectively.

	$U_n + K - 1$	$U_n + K$	$U_n + K + 1$
2	$n \equiv 0 \pmod 2$	$n \equiv 1 \pmod 2$	$n \equiv 0 \pmod 2$
3	$n \equiv 1, 2, 7 \pmod 8$	$n \equiv 0 \pmod 4$	$n \equiv 3, 5, 6 \pmod 8$
5	$7, 13, 14, 16 \pmod{20}$	$3, 4, 6, 17 \pmod{20}$	$1, 12, 18, 19 \pmod{20}$
11	$n \equiv 5 \pmod{10}$	$n \equiv 8 \pmod{10}$	$n \equiv 3, 4, 7 \pmod{10}$
17		$n \equiv 8, 10 \pmod{12}$	
19	$n \equiv 1, 5 \pmod 6$	$n \equiv 0 \pmod 6$	
31		$n \equiv 2 \pmod{10}$	
41		$n \equiv 17, 23 \pmod{40}$	$n \equiv 15, 25 \pmod{40}$
61	$n \equiv 1, 19 \pmod{20}$	$n \equiv 0 \pmod 5$	$n \equiv 9, 11 \pmod{20}$
107	$n \equiv 3, 21 \pmod{24}$	$n \equiv 7, 17 \pmod{24}$	
181		$n \equiv 14 \pmod{30}$	

Table 8: A triple covering when $P = 4$. Each cell contains the values of n for which $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ are proper multiples of the moduli $2, 3, 5, 11, 17, 19, 31, 41, 61, 107$, and 181, respectively.

	U_n+K-1	U_n+K	U_n+K+1
2	$n \equiv 1,2 \pmod 3$	$n \equiv 0 \pmod 3$	$n \equiv 1,2 \pmod 3$
3	$n \equiv 1 \pmod 2$	$n \equiv 0 \pmod 2$	
5	$n \equiv 8,10 \pmod{12}$	$n \equiv 1,11 \pmod{12}$	$n \equiv 0 \pmod 3$
11	$n \equiv 0 \pmod 4$	$n \equiv 3,5 \pmod 8$	
19	$n \equiv 2,18 \pmod{40}$		$n \equiv 1,26,34,39 \pmod{40}$
59	$n \equiv 3,21 \pmod{24}$	$n \equiv 7,17 \pmod{24}$	
61	$n \equiv 0 \pmod{30}$		
109	$n \equiv 14,16 \pmod{20}$		
131	$n \equiv 6 \pmod{10}$		
211	$n \equiv 18 \pmod{30}$		
739	$n \equiv 22,38 \pmod{40}$		

Table 9: A triple covering when $P = 3$. Each cell contains the values of n for which $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ are proper multiples of the moduli $2, 3, 5, 11, 19, 59, 61, 109, 131, 211$, and 739, respectively.

	U_n+K-1	U_n+K	U_n+K+1
2	$n \equiv 0 \pmod 2$	$n \equiv 1 \pmod 2$	$n \equiv 0 \pmod 2$
3	$n \equiv 1,6,7 \pmod 8$	$n \equiv 0 \pmod 4$	$n \equiv 2,3,5 \pmod 8$
5	$n \equiv 5,7 \pmod{12}$	$n \equiv 8,10 \pmod{12}$	$n \equiv 2,4 \pmod{12}$
7		$n \equiv 2 \pmod 6$	$n \equiv 1,5 \pmod 6$
11		$n \equiv 5,18,19 \pmod{24}$	$n \equiv 9,15 \pmod{24}$
19	$n \equiv 19,21 \pmod{40}$	$n \equiv 7,22,33,38 \pmod{40}$	$n \equiv 9,30,31 \pmod{40}$
29	$n \equiv 0 \pmod 5$	$n \equiv 9,11 \pmod{20}$	$n \equiv 12,18 \pmod{20}$
31	$n \equiv 5,25,28 \pmod{30}$	$n \equiv 18 \pmod{30}$	$n \equiv 14 \pmod{30}$
41	$n \equiv 3,7 \pmod{10}$		
59	$n \equiv 10,11,29 \pmod{40}$		
269	$n \equiv 1,59 \pmod{60}$	$n \equiv 0 \pmod{15}$	$n \equiv 29,31 \pmod{60}$
601		$n \equiv 6,54 \pmod{120}$	

Table 10: A triple covering when $P = 2$. Each cell contains the values of n for which $U_n + K - 1$, $U_n + K$, and $U_n + K + 1$ are proper multiples of the moduli $2, 3, 5, 7, 11, 19, 29, 31, 41, 59, 269$, and 601, respectively.

	F_n+K-1	F_n+K	F_n+K+1
2	$n \equiv 1,2 \pmod 3$	$n \equiv 0 \pmod 3$	$n \equiv 1,2 \pmod 3$
3	$n \equiv 1,2,7 \pmod 8$	$n \equiv 0 \pmod 4$	$n \equiv 3,5,6 \pmod 8$
5	$n \equiv 0 \pmod 5$	$n \equiv 9,11,12,18 \pmod{20}$	$n \equiv 4,6,7,13 \pmod{20}$
7	$n \equiv 0 \pmod 8$	$n \equiv 7,9,10,14 \pmod{16}$	$n \equiv 5,11 \pmod{16}$
11	$n \equiv 4 \pmod{10}$	$n \equiv 3,7 \pmod{10}$	$n \equiv 1,2,9 \pmod{10}$
17	$n \equiv 0 \pmod 9$	$n \equiv 17,19,20,34 \pmod{36}$	$n \equiv 15,21 \pmod{36}$
19	$n \equiv 3,8,15 \pmod{18}$	$n \equiv 1,2,17 \pmod{18}$	$n \equiv 0 \pmod{18}$
23	$n \equiv 3,32,40,45 \pmod{48}$	$n \equiv 1,2,22,47 \pmod{48}$	$n \equiv 0 \pmod{24}$
31		$n \equiv 5,14,25 \pmod{30}$	
41	$n \equiv 0 \pmod{20}$	$n \equiv 19,21,22,38 \pmod{40}$	$n \equiv 17,23 \pmod{40}$
47	$n \equiv 4,12 \pmod{32}$	$n \equiv 3,29 \pmod{32}$	$n \equiv 1,2,14,31 \pmod{32}$
61	$n \equiv 3,57 \pmod{60}$	$n \equiv 1,2,28,59 \pmod{60}$	$n \equiv 0 \pmod{15}$
107	$n \equiv 6,19,30,53 \pmod{72}$		
181			$n \equiv 48 \pmod{90}$
241	$n \equiv 134,226 \pmod{240}$	$n \equiv 34,86 \pmod{240}$	$n \equiv 158,202 \pmod{240}$
541	$n \equiv 22 \pmod{90}$		$n \equiv 78 \pmod{90}$
1103	$n \equiv 60,84 \pmod{96}$		
1601	$n \equiv 0 \pmod{80}$	$n \equiv 79,81,82,158 \pmod{160}$	$n \equiv 77,83 \pmod{160}$
2161		$n \equiv 50,70 \pmod{80}$	
2521			$n \equiv 24,36 \pmod{120}$
3041	$n \equiv 3,157 \pmod{160}$	$n \equiv 1,2,78,159 \pmod{160}$	$n \equiv 0 \pmod{80}$
8641			$n \equiv 474,606 \pmod{720}$
20641		$n \equiv 134,226 \pmod{240}$	
103681			$n \equiv 102,114 \pmod{144}$
109441			$n \equiv 96,174 \pmod{180}$

Table 11: A triple covering of F_n. Here, K is the 57-digit number given in Equality (28). Each cell contains the values of n for which F_n+K-1, F_n+K, and F_n+K+1 are proper multiples of the moduli $2,3,5,7,11,17,19,23,31,41,47,$ $61,107,181,241,541,1103,1601,$ $2161,2521,3041,8641,20641,103681,$ and 109441, respectively.