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CONNECTING RAMANUJAN'S THETA FUNCTIONS, THE ROGERS-RAMANUJAN CONTINUED FRACTION, AND THE GOLDEN RATIO

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Abstract

We provide new relations involving the Rogers-Ramanujan continued fraction; our proof is based on the ideas of Rogers and a formula for the fifth power of the Jacobi theta function.

1. Introduction

Three months before his death in early 1920, Ramanujan sent a letter to Hardy in which there were 17 new functions, which he called mock theta functions. They are classified into three groups: four of order three, ten of order five, and three of order seven. These mock theta functions are q-series converging for $|q| < 1$ and share certain properties with Jacobi theta functions when q tends to a root of unity. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers, and the set of complex numbers, respectively. Following [3, 4], we let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}.$$

The *q-shifted factorial* $(a; q)_n$ is defined for $|q| < 1$ by

$$(a;q)_n := \begin{cases} 1 & if\ n = 0 \\ \prod_{k=0}^{n-1} (1 - aq^k) & if\ n \in \mathbb{N}, \end{cases}$$

where $a, q \in \mathbb{C}$. We also write

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$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n) = \prod_{n=1}^{\infty}(1-aq^{n-1}), \qquad a,q\in\mathbb{C},\ |q|<1.$$

Whenever $(a;q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be assumed to be satisfied. The following notation will be frequently used in our investigation:

$$(a_1,a_2,a_3\ldots a_k;q)_n = (a_1;q)_n(a_2;q)_n(a_3;q)_n\ldots(a_k;q)_n, \quad n\in\mathbb{N}\cup\{\infty\}.$$

Ramanujan (see [13, p. 13] and [12]) defined the *general theta function* $f(a,b)$ as follows:

$$f(a,b) = 1+\sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}(a^n+b^n) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b,a), \qquad |ab|<1,$$

where a and b are two complex numbers. The three most important special cases of $f(a,b)$ are defined by

$$\phi(q) = f(q,q) = 1+2\sum_{n=1}^{\infty}q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty},$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$\mathfrak{f}(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{+\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

The last equality is known as *Euler's Pentagonal Number Theorem.* Ramanujan [5, 6] also defined the function

$$\chi(q) = (-q;q^2)_\infty.$$

We recall the q-series identity

$$(-q;q)_\infty = \frac{1}{\chi(-q)}.$$

Moreover,

$$\chi(q) = (-q;q^2)_\infty = \frac{1}{(q;q^2)_\infty(-q^2;q^2)_\infty}$$

holds since

$$\begin{aligned}(-q;q^2)_\infty &= \prod_{n=1}^{\infty}(1+q^{2n-1}) = \prod_{n=1}^{\infty}\frac{1+q^n}{1+q^{2n}} = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{(1-q^n)(1+q^{2n})}\\ &= \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})(1+q^{2n})} = \frac{1}{(q;q^2)_\infty(-q^2;q^2)_\infty}.\end{aligned}$$

We recall the Rogers-Ramanujan identities, which are the relations

$$G(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}=\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}, \tag{1}$$

$$H(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}. \tag{2}$$

The Rogers-Ramanujan continued fraction formula (see [1, p. 9]) is the relation

$$R(q)=q^{1/5}\frac{H(q)}{G(q)}=\frac{q^{1/5}(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}, \qquad |q|<1.$$

Remark 1. (a) From Equation (1) it follows that the number of partitions of n such that the adjacent parts differ by at least 2 is the same as the number of partitions of n such that each part is congruent to either 1 or 4 modulo 5.

(b) From Equation (2) it follows that the number of partitions of n such that the adjacent parts differ by at least 2 and such that the smallest part is at least 2 is the same as the number of partitions of n such that each part is congruent to either 2 or 3 modulo 5.

Here, further investigation of the Rogers-Ramanujan continued fraction is undertaken, based on formulas of the classical theory of Jacobi theta functions. The properties of this fraction are closely related to combinatorial analysis through the Rogers-Ramanujan identities. The Rogers-Ramanujan formulas [9, p. 214-215] are certainly among the most amazing results in the field of mathematical analysis. The British mathematician Leonard J. Rogers (1862-1933) and, independently of him, the Indian mathematician Srinivasa Ramanujan (1887-1920), discovered these interesting and important identities. Note that Rogers' work became famous thanks to Ramanujan after an Indian mathematician found it in 1917 while looking through old volumes of the Proceedings of the London Mathematical Society [14].

An overview of many results on the theory of the continued fraction of Rogers-Ramanujan $R(q)$ is given in [17]. This source also provides an extensive bibliography on the subject. The proofs of two Ramanujan formulas for $R(q)$ based on the properties of infinite products of the classical theory of Jacobi theta functions are presented in [10]. A number of important results of the theory are proved in the source [11]. The connection between continued fractions and modular functions is detailed in [7] and [16].

2. Main Results

In this section, we state and prove a set of two theorems, which are given below.

Theorem 1. *Let* $q := e^{\pi i \tau}$, $\Im(\tau) > 0$. *Further let* $q' := e^{\pi i \tau'}$, $\tau\tau' = -1$. *Then*

$$\left\{\sqrt{\Phi} - \frac{1}{\sqrt{\Phi}}\right\}\left\{U + \frac{1}{U}\right\} + \left\{\sqrt{\Phi} + \frac{1}{\sqrt{\Phi}}\right\}\left\{V - \frac{1}{V}\right\} = \frac{1}{\sqrt{q}} \cdot \frac{g(0)}{\{(q;q)_\infty (q^5;q^5)_\infty\}^2}, \tag{3}$$

where $\Phi := (\sqrt{5}+1)/2$ *is the golden ratio, where*

$$U := \left\{R(q)R(q'^{4/5})\right\}^{5/2}, \qquad V := \left\{\frac{R(q)}{R(q'^{4/5})}\right\}^{5/2},$$

and the coefficient $g(0)$ *is given by*

$$g(0) = \frac{16\psi^5(q) - \tau(q)\left\{\tau^4(q) + 5\tau^2(q)\rho(q) + 5\rho^2(q)\right\}}{5\psi(q^5)},$$

with

$$\tau(q) = \psi(q) - 5q^3\psi(q^{25}),$$
$$\rho(q) = \psi^2(q) - 5q\psi^2(q^5).$$

Proof. The assertion of Theorem 3 [8, p. 488] can be rewritten using the formulas

$$\begin{aligned} \vartheta_1^5(z, \sqrt{q}) = &g(0)\vartheta_1(5z, q^{5/2}) \\ &+ 2g(1)q^{5/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+3)/2} \sin[(10n+3)z] \\ &+ 2g(2)q^{5/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)/2} \sin[(10n+1)z], \end{aligned} \tag{4}$$

$$g(1) = \frac{\Delta(q)q^{1/10}}{R^{1/2}(q)} \left\{ \sqrt{\frac{5+\sqrt{5}}{2}} R^{5/2}(q'^{4/5}) - \sqrt{\frac{5-\sqrt{5}}{2}} R^{-5/2}(q'^{4/5}) \right\}, \tag{5}$$

and

$$g(2) = \frac{\Delta(q)R^{1/2}(q)}{q^{1/10}} \left\{ \sqrt{\frac{5-\sqrt{5}}{2}} R^{5/2}(q'^{4/5}) + \sqrt{\frac{5+\sqrt{5}}{2}} R^{-5/2}(q'^{4/5}) \right\}, \tag{6}$$

where the coefficient $g(0)$ is determined in [8, p. 490].

Here $\Delta(q) := \sqrt[4]{5}\{(q;q)_\infty(q^5;q^5)_\infty\}^2$ and for $z \in \mathbb{C}$, the classical Jacobi theta function is defined by the equality

$$\vartheta_1(z,q) := 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin[(2n+1)z],$$

(see [18]).

Dividing both sides of Equation (4) by z, we pass to the limit as $z \to 0$. Further, after using the Jacobi formula for the cube of q-shifted factorial, Equation (5), Equation (6) and [1, p. 19], we obtain the relation

$$\begin{aligned}&\left\{\sqrt{\frac{5+\sqrt{5}}{2}} - 3\sqrt{\frac{5-\sqrt{5}}{2}}\right\}\left\{U + \frac{1}{U}\right\}\\ &+ \left\{\sqrt{\frac{5-\sqrt{5}}{2}} + 3\sqrt{\frac{5+\sqrt{5}}{2}}\right\}\left\{\frac{1}{V} - V\right\}\\ &+ \left\{\frac{5^{3/4}}{\sqrt{q}}\cdot\frac{g(0)}{\{(q;q)_\infty(q^5;q^5)_\infty\}^2}\right\} = 0.\end{aligned} \tag{7}$$

After elementary transformations, Equation (7) becomes (3). We thus have completed our proof of the theorem. □

Remark 2. Since it follows from [8, (26) and (27), p. 489] that

$$R(q'^{4/5}) = \frac{\sqrt{\sqrt{5}-1} - \sqrt{\sqrt{5}+1}R(q^5)}{\sqrt{\sqrt{5}+1} + \sqrt{\sqrt{5}-1}R(q^5)} = \frac{1-\Phi R(q^5)}{\Phi + R(q^5)},$$

Equation (3) actually expresses a complex relation between $R(q)$ and $R(q^5)$.

Next, we prove a formula in which the coefficient $g(0)$ of the theta expansion is expressed only in terms of the q-shifted factorial. From this formula one can obtain a twenty-fifth order modular equation. The derivation of the formula is somewhat cumbersome, but quite elementary, and is based on Theorem 1.

Remark 3. The formula displays how the number 5 plays a unique role in the theory. This is also seen from the Rogers-Ramanujan formulas and Euler's pentagonal number theorem.

Theorem 2. *Let $E_1 := (q;q)_\infty$, $E_5 := (q^5;q^5)_\infty$ and $E_{25} := (q^{25};q^{25})_\infty$. Then*

$$\begin{aligned}g(0) = \frac{{E_5}^5}{E_{25}} + 20q\frac{{E_1}^4E_{25}}{E_5} + 110q^2\frac{{E_1}^3{E_{25}}^2}{E_5}\\ +350q^3\frac{{E_1}^2{E_{25}}^3}{E_5} + 600q^4\frac{E_1{E_{25}}^4}{E_5} + 625q^5\frac{{E_{25}}^5}{E_5}.\end{aligned}$$

Proof. Squaring both sides of Equation (3), and simplifying the expression using Theorem 5.5 from [2, p. 19] and the remark following Theorem 1, we get

$$\frac{\delta^2}{u(1-u-u^2)^5(u^4+2u^3+4u^2+3u+1)(u^4-3u^3+4u^2-2u+1)}$$

$$= \frac{g^2(0)}{q\{(q;q)_\infty(q^5;q^5)_\infty\}^4},$$

where

$$\delta := u^{10}-20u^9+30u^8-60u^7+60u^6-114u^5-60u^4-60u^3-30u^2-20u-1, \ u := R(q^5).$$

Hence, we have

$$\frac{\delta^2}{u(1-u-u^2)^4(1-11u^5-u^{10})} = \frac{g^2(0)}{q\{(q;q)_\infty(q^5;q^5)_\infty\}^4}.$$

Using the Ramanujan formulas [1, (1.1.10) and (1.1.11), p. 11], we obtain

$$g(0) = \pm\frac{q^5}{u^5}\cdot\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty}\delta.$$

Dividing both sides of Equation (4) by $\vartheta_1(5z, q^{5/2})$, and passing first to the limit at $z \to 0$ and then to the limit at $q \to 0$ in exactly this order, we get $g(0) \to 1$, since $R(q'^{4/5}) \to 1/\Phi$. Hence, we have

$$g(0) = -\frac{q^5}{u^5}\cdot\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty}\delta. \tag{8}$$

Now the assertion of the theorem follows from Equation (8) and Ramanujan's formulas mentioned above. We thus have completed our proof of the theorem. □

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