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ON SMOOTH GAPS BETWEEN PRIMES USING THE MAYNARD-TAO SIEVE

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Abstract

In 1999, Balog, Brüdern, and Wooley showed there are infinitely many prime gaps $p-q$ that are $(\log p)^{\frac{3}{4}}$-smooth, and infinitely many consecutive prime gaps that are $(\log p)^{\frac{7}{8}}$-smooth. Advancements made since then by Zhang, Maynard, and Polymath8btowards resolving the twin prime conjecture have given us the tools to lower the bounds made by Balog, Brüdern, and Wooley to 47. Moreover, we can show there are infinitely many m-tuples of primes whose gaps are all y_m-smooth for a calculable prime y_m.

1. Introduction

The twin prime conjecture posits there are infinitely many pairs of primes $p > q$ such that $p - q = 2$. Whilst unresolved, a significant result from Polymath8b [8, Theorem 1.4], building upon the work of Goldston, Pintz, and Yıldırım [4], Zhang [12], and Maynard [6], asserts there are infinitely many pairs of primes $p > q$ such that $p - q \leq 246$ (a result we elaborate on later).

An alternative way to weaken the twin prime conjecture is to assert there are infinitely many pairs of primes p, q such that $p - q = 2^n$, or, to restate, whose difference is 2-smooth.[1] However, a proof of this conjecture is also beyond current techniques. So is the conjecture stating there are infinitely many pairs of primes $p > q$ whose gap is 3-smooth, or 5-smooth.

We define $\mathcal{S}(y)$ to be the set of all y-smooth integers. In 1999, Balog, Brüdern, and Wooley [1] showed there are infinitely many pairs of primes $p > q$ such that $p-q \in \mathcal{S}\left((\log p)^{\frac{3}{4}}\right)$, and infinitely many *consecutive* primes $p > q$ such that $p-q \in \mathcal{S}\left((\log p)^{\frac{7}{8}}\right)$. Moreover, it follows immediately from the aforementioned results

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[1] An integer is y-smooth if its largest prime factor is less than or equal to y.

m	k_m
2	50
3	35265
4	1624545
5	73807570
6	3340375663

Table 1: Lowest known values of k_m

given by Polymath8b there are infinitely many pairs of primes $p > q$ with gap $p - q \in \mathcal{S}\,(241)$. In this paper, we show the following.

Theorem 1. *There exist infinitely many pairs of primes $p > q$ such that $p - q \in \mathcal{S}(47)$.*

To this effect, we shall use a celebrated theorem due to Maynard [6] (and independently Tao [11]), for which we first require some definitions to state precisely.

Definition 1. A k-tuple of integers H_k is *admissible* if, for every prime p, the elements of H_k do not cover all congruence classes mod p.

Definition 2. The *diameter* of a k-tuple is the difference between its largest and smallest element.

The following version of the Maynard-Tao Theorem was formulated by Banks, Freiberg, and Turnage-Butterbaugh [2].

Theorem 2 ([6])**.** *For any positive integer $m \geq 2$, there exists some $k_m \in \mathbb{N}$ such that, for any admissible k-tuple of integers H_k where $k \geq k_m$, there exist infinitely many integers n such that $n + h$ is prime for at least m elements $h \in H_k$.*

The $m = 2$ case was proved by Zhang in 2013, with $k_2 = 3.5 \cdot 10^6$ [12, Theorem 1]. Polymath8a [7] refined his method such that one could take $k_2 = 632$. Maynard [6, Proposition 4.3] discovered a simpler method that lowered k_2 to 105 and showed k_m to be finite for all m, giving $k_m < cm^2e^{4m}$ for some absolute constant c. Subsequent optimizations made by Polymath8b [8, Theorem 3.2] and Stadlmann [9, Corollary 1] give us the current lowest known values of k_m recorded in Table 1.

Remark 1. For $m > 6$, Stadlmann [9, Theorem 2] showed $k_m < ce^{3.8075m}$ for some absolute constant c.

Remark 2. Computations by Engelsma in an unpublished work [3, Table 5] show that an admissible 50-tuple of integers with diameter 246 exists, and that this diameter is minimal over all admissible 50-tuples.

Remark 3. An immediate consequence of the above fact, and the Maynard-Tao Theorem, is the existence of infinitely many pairs of primes $p > q$ such that $p - q \leq 246$.

Remark 4. Maynard [6] showed k_2 can be lowered to 5 assuming the Elliott-Halberstam Conjecture.

The twin prime conjecture belongs to a larger group of problems on the distribution of patterns of small prime gaps. Such patterns can be represented by a k-tuple of integers H_k. In generalizing the twin prime conjecture, one can ask: for how many integers n is $n+h$ prime for all $h \in H_k$? The case $H_2 = (0, 2)$ corresponds to the twin prime conjecture. In 1923, Hardy and Littlewood [5], through probabilistic reasoning, made the following conjecture (the formulation we present here follows Tao [10]).

Conjecture 1 (Hardy-Littlewood k-tuples Conjecture)**.** Given an admissible k-tuple H_k, we define

$$\mathfrak{G} = \mathfrak{G}\left(H_k\right) := \prod_{p \in \mathcal{P}} \frac{1 - \frac{v_p}{p}}{\left(1 - \frac{1}{p}\right)^k}$$

where $\mathcal{P}$ is the set of all primes and v_p is the number of congruence classes mod p covered by the elements in H_k. Then, the number of natural numbers $n < x$ such that $n + H_k$ consists entirely of primes is asymptotic to $\mathfrak{G}\frac{x}{(\log x)^k}$.

This conjecture is generally believed to be true and appears consistent with experimental data. Moreover, the Maynard-Tao Theorem is a significant stride towards it. We will be applying the Maynard-Tao Theorem to the question of smooth gaps between primes, with the following results.

Theorem 3. *Let y_m represent the largest prime less than or equal to k_m, where k_m is as in Table 1, Remark 1, and Remark 4. Then, there exists infinitely many m-tuples of primes $P_m = (p_1, p_2, p_3, ..., p_m)$ such that $p_i - p_j \in \mathcal{S}(y_m)$ for all $1 \leq j < i \leq m$.*

Note that Theorem 1 is simply a special case of Theorem 3, where $m = 2$, $k_m = 50$, and $y_m = 47$.

Remark 5. Improvements on the bound on k_m would improve upon Theorem 3, lowering the bound y_m.

We give an example of Remark 5 below.

Theorem 4. *Assuming the Elliott-Halberstam Conjecture, there exist infinitely many pairs of primes $p > q$ such that $p - q \in \mathcal{S}(5)$.*

Proof. The theorem follows directly from Remark 4 and Theorem 3. □

Definition 3. A tuple H is *difference y-smooth* if $h_i - h_j \in \mathcal{S}(y)$ for every pair of elements $h_i, h_j \in H$.

Theorem 5. *For any positive integer k, there exists a k-tuple of integers H_k such that H_k is admissible and H_k is difference k-smooth.*

Theorem 3 follows from Theorem 5, which we will show in Section 2.

Remark 6. Balog, Brüdern, and Wooley [1] distinguished between smooth prime gaps and consecutive smooth gaps. However, Banks, Freiberg, and Turnage-Butterbaugh [2, Theorem 1] established the admissible k-tuples in the Maynard-Tao Theorem also infinitely often represent consecutive prime gaps. Hence, there is no need to distinguish between consecutive and non-consecutive smooth prime gaps.

2. Proofs of Theorem 3 and Theorem 5

To prove Theorem 3, we apply the Maynard-Tao theorem to a difference y_m-smooth k_m-tuple (whose existence is established in the proof of Theorem 5).

Proof of Theorem 3. Assuming Theorem 5, let H be a k_m-tuple that is admissible and difference k_m-smooth. Since H is admissible and contains at least k_m elements, then, by the Maynard-Tao Theorem, there are infinitely many $n \in \mathbb{N}$ such that $n+h$ is prime for at least m elements $h \in H$. Let $P_m(n) = (p_1, p_2, p_3, ..., p_m)$ be such an m-tuple of primes. For any two elements of p_i, $p_j \in P_m(n)$, $p_i - p_j = h_s - h_t$, where $h_s, h_t \in H$. Recall H is difference k_m-smooth, so, $p_i - p_j = h_s - h_t \in \mathcal{S}(y_m)$. Therefore, there are infinitely many m-tuples of primes $P_m(n)$ with gaps $p_i - p_j \in \mathcal{S}(k_m) = \mathcal{S}(y_m)$ for all pairs of elements p_i, $p_j \in P_m(n)$. □

We prove Theorem 5 by an explicit construction of a difference k-smooth k-tuple of integers.

Proof of Theorem 5. Let ω be the product of all positive primes up to k inclusive.

Consider the arithmetic progression $H_k = (0, \omega, 2\omega, 3\omega, ..., (k-1)\omega)$.

For every prime $p \leq k$, $p \mid \omega$. So, every element of H_k is $0 \bmod p$. For every prime $p > k$, there are at least $k+1$ congruence classes mod p, but only k elements in H_k, and therefore, the elements of H_k cannot cover every congruence class mod p. Hence, H_k is admissible.

The difference between any two elements of H_k is $a\omega$ for some integer $0 < a < k$. We define y to be the largest prime not exceeding k. Then, both a and $\omega \in \mathcal{S}(y)$, so $a\omega \in \mathcal{S}(y)$. Therefore, H_k is difference y-smooth.

Thus, given any positive integer k, H_k is an admissible, y-smooth k-tuple, as desired. □

3. Optimality of y_m

A natural question arises from the above formulation: Is the bound y_m optimal, or can it be lowered?

Proposition 1. *Let z_k represent the largest prime less than or equal to k. Then, H_k is admissible implies H_k is not difference ℓ-smooth for any $\ell < z_k$.*

Proof. Consider the elements of H_k mod z_k. Because H_k is admissible, its elements cannot cover all congruence classes mod z_k. At most, its k elements can cover $z_k - 1$ congruence classes. But $k > z_k - 1$, so by the pigeonhole principle, there exists some pair of elements $h_i, h_j \in H_k$ in the same congruence class mod z_k. Then, $h_i - h_j \equiv 0 \bmod z_k$, so $z_k \mid h_i - h_j$. But $z_k > \ell$ and z_k is prime, so z_k is a witness for H_k not being difference ℓ-smooth for any $\ell < z_k$. □

Remark 7. As a consequence of Proposition 1, improving Theorem 3 requires a different method, one avoiding the Maynard-Tao sieve.

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References

[1] A. Balog, J. Brüdern, and T. D. Wooley, On smooth gaps between consecutive prime numbers, *Mathematika* **46** (1) (1999), 57–75.

[2] W. D. Banks, T. Freiberg, and C. L. Turnage-Butterbaugh, Consecutive primes in tuples, *Acta Arith.* **167** (3) (2015), 261-266.

[3] T. J. Engelsma, Permissible patterns of primes, preprint, available at `https://www.opertech.com/ primes/k-tuples.html`

[4] D. A. Goldston, J. Pintz, and C. Y. Yıldırım, Primes in tuples, *Ann. of Math. (2)* **170** (2) (2009), 819-862.

[5] G.H. Hardy and J.E. Littlewood, Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes, *Acta Math.* **44** (1) (1923), 1-70.

[6] J. Maynard, Small gaps between primes, *Ann. of Math. (2)* **181** (1) (2015), 383-413.

[7] D. H. J. Polymath, New equidistribution estimates of Zhang type, *Algebra Number Theory* **8** (9) (2014), 2067-2199.

[8] D. H. J. Polymath, Variants of the Selberg sieve, and bounded intervals containing many primes, *Res. Math. Sci.* **1** (1) (2014), Article 12.

[9] J. Stadlmann, On primes in arithmetic progressions and bounded gaps between many primes, preprint, arXiv: 2309.00425.

[10] T. Tao, The prime tuples conjecture, sieve theory, and the work of Goldston-Pintz-Yıldırım, Motohashi-Pintz, and Zhang, blog post, https://terrytao.wordpress.com/2013/05/19/the-prime-tuples-conjecture-sieve-theory-and-the-work-of-goldston-pintz-yildirim-motohashi-pintz-and-zhang/.

[11] T. Tao, Polymath8b: Bounded intervals with many primes, after Maynard, blog post, https://terrytao.wordpress.com/2013/11/19/polymath8b-bounded-intervals-with-many-primes-after-maynard/.

[12] Y. Zhang, Bounded gaps between primes, *Ann. of Math. (2)* **179** (3) (2014), 1121-1174.