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INEQUALITIES FOR INCLUSION-EXCLUSION-LIKE SUMS INVOLVING THE CEILING AND THE NEAREST INTEGER FUNCTIONS

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Abstract

We obtain various sharp bounds for inclusion-exclusion-like sums involving the ceiling and the nearest integer functions, which supplement some of the previous results on sums defined by Jacobsthal and Tverberg.

1. Introduction

Let $a, b \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. In 1957, Jacobsthal [6] introduced sums of the form

$$S_{a,b;m}(K) = \sum_{k=0}^{K} f_{a,b;m}(k),$$

where

$$f_{a,b;m}(k) = \left\lfloor \frac{a+b+k}{m} \right\rfloor - \left\lfloor \frac{a+k}{m} \right\rfloor - \left\lfloor \frac{b+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor. \tag{1}$$

In the above equation and throughout this article, unless stated otherwise, $\lfloor x \rfloor$ is the largest integer less than or equal to x, k is an integer, and K is a nonnegative

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integer. So we can consider $f = f_{a,b;m}$ and $S = S_{a,b;m}$ as functions of k and K defined on $\mathbb{Z}$ and on $\mathbb{N} \cup \{0\}$, respectively. Both f and S were also studied by Carlitz [2, 3] and Grimson [5], and were extended in 2012 by Tverberg [13] to the following form.

Definition 1. Let m and ℓ be positive integers, $[1, \ell] = \{1, 2, 3, \ldots, \ell\}$, and C a multiset of ℓ integers $a_1, a_2, \ldots, a_\ell$, that is, $a_i = a_j$ is allowed for some $i \neq j$. Define $f : \mathbb{Z} \to \mathbb{R}$ and $S : \mathbb{N} \cup \{0\} \to \mathbb{R}$ by

$$f(C, m, k) = \sum_{T \subseteq [1,\ell]} (-1)^{\ell - |T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor,$$

$$S(C, m, K) = \sum_{k=0}^{K} f(C, m, k).$$

We sometimes write $f(\{a_1, a_2, \ldots, a_\ell\}, m, k)$ and $S(\{a_1, a_2, \ldots, a_\ell\}, m, K)$ instead of $f(C, m, k)$ and $S(C, m, k)$, respectively. As usual, the empty sum is defined to be zero.

For example, if $C = \{a, b\}$, then $f(C, m, k) = f(\{a, b\}, m, k)$ is the same as that given in (1), and if $C = \{a_1, a_2, a_3\}$, then $f(C, m, k)$ is

$$\begin{aligned} f(\{a_1, a_2, a_3\}, m, k) = &\left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned}$$

Jacobsthal [6] showed that for any $K \in \mathbb{N} \cup \{0\}$, we have

$$0 \leq S(\{a, b\}, m, K) \leq \lfloor m/2 \rfloor, \tag{2}$$

which is a sharp inequality, that is, the lower bound 0 is actually the minimum value and the upper bound $\lfloor m/2 \rfloor$ is the maximum value of $S(\{a, b\}, m, K)$. Tverberg [13] proved (2) in a different way and also gave the extreme values of

$$S(\{a_1, a_2, a_3\}, m, K)$$

without proof. Onphaeng and Pongsriiam [8] then extended Tverberg's result and obtained the maximum and minimum values of f in all cases $\ell \geq 2$. In fact, the extreme values of f are connected with Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n defined, respectively, by the recurrence relations

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geq 2,$$

and

$$j_0 = 2, \quad j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2} \quad \text{for } n \geq 2.$$

The sequences $(J_n)_{n\geq 0}$ and $(j_n)_{n\geq 0}$ are, respectively, A001045 and A014551 in the On-Line Encyclopedia of Integer Sequences (OEIS) [12]. Onphaeng and Pongsriiam [8] also obtained the minimum value of $S(\ell)$ when ℓ is odd and the maximum value of $S(\ell)$ when ℓ is even, where $S(\ell) =: S(\{a_1, a_2, \ldots, a_\ell\}, m, K)$. We remark that they [8] also gave an upper bound of $S(\ell)$ when ℓ is odd, and a lower bound of $S(\ell)$ when ℓ is even, but those bounds were far from being sharp, and so it seemed a totally new method was required to obtained sharp bounds for the missing cases. Using a computer and their intuition, Thanatipanonda and Wong [11] predicted the maximum and minimum values of $S(\ell)$ in the remaining cases with a proof for the maximum of $S(\ell)$ when $\ell = 3$. In addition, Onphaeng and Pongsriiam [8] introduced a new function g and obtained the maximum and minimum values of $g = g_n$ in all cases $n \geq 2$. For the reader's convenience, we recall the definition of g.

Definition 2. Let $g : \mathbb{R}^n \to \mathbb{Z}$ be given by

$$\begin{aligned} g(x_1, x_2, x_3, \ldots, x_n) = \sum_{1\leq i\leq n} \lfloor x_i \rfloor - \sum_{1\leq i_1<i_2\leq n} \lfloor x_{i_1} + x_{i_2} \rfloor \\ + \sum_{1\leq i_1<i_2<i_3\leq n} \lfloor x_{i_1} + x_{i_2} + x_{i_3} \rfloor - \cdots \\ + (-1)^{n-1} \lfloor x_1 + x_2 + x_3 + \cdots + x_n \rfloor. \end{aligned}$$

In other words, we define

$$g(x_1, x_2, x_3, \ldots, x_n) = \sum_{\emptyset \neq T \subseteq [1,n]} (-1)^{|T|-1} \left\lfloor \sum_{i\in T} x_i \right\rfloor.$$

In this article, we extend the results on the functions g, f, and S defined by Tverberg [13] and Onphaeng and Pongsriiam [8] to the case of some equally useful and popular functions, namely, the ceiling and the nearest integer functions r^- and r^+. These functions are closely related to the floor function and appear in many number-theoretic contexts. For example, if D_n is the number of permutations of n distinct letters that have no fixed points (also called derangement of n distinct objects), then

$$D_n = n! \sum_{k=0}^{n} (-1)^k / k! = r^-(n!/e) = r^+(n!/e) \quad \text{for every } n \in \mathbb{N}.$$

Similarly, the nth Fibonacci number is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = r^-(\alpha^n/\sqrt{5}) = r^+(\alpha^n/\sqrt{5}) \quad \text{for all } n \in \mathbb{N},$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, respectively.

Let us now recall the precise definition of the ceiling and the nearest integer functions. For each $x \in \mathbb{R}$, the ceiling of x, denoted by $\lceil x \rceil$, is defined as the smallest integer larger than or equal to x, the number $r^+(x)$ is the nearest integer to x with tie breaking towards positive infinity, and $r^-(x)$ is the nearest integer to x with tie breaking towards negative infinity. Therefore, $r^+(x) = \lfloor x + \frac{1}{2} \rfloor$ and $r^-(x) = \lceil x - \frac{1}{2} \rceil$.

In general, suppose V is a subset of $\mathbb{Z}$ or V is a finite dimensional vector space over $\mathbb{R}$ and $F : V \to \mathbb{R}$, where the choice of V is chosen so that the following extension of g, f, S makes sense. We define the functions $g^{(F)} : V^n \to \mathbb{R}$, $f^{(F)}(C, m, k) : V \to \mathbb{R}$, and $S^{(F)} : V \to \mathbb{R}$ by

$$g^{(F)}(x_1, x_2, \ldots, x_n) = \sum_{\emptyset \neq T \subseteq [1,n]} (-1)^{|T|-1} F\left(\sum_{i \in T} x_i\right),$$

$$f^{(F)}(C, m, k) = f^{(F)}\left(\{a_1, a_2, \ldots, a_\ell\}, m, k\right) = \sum_{T \subseteq [1,\ell]} (-1)^{\ell - |T|} F\left(\frac{k + \sum_{i \in T} a_i}{m}\right),$$

$$S^{(F)}(C, m, K) = S^{(F)}(\{a_1, a_2, \ldots, a_\ell\}, m, K) = \sum_{k=0}^{K} f^{(F)}(C, m, k),$$

where $n, \ell \in \mathbb{Z}^+$, C is a multiset of ℓ integers or ℓ vectors $a_1, a_2, \ldots, a_\ell$, $[1, n] = \{1, 2, 3, \ldots, n\}$, $[1, \ell] = \{1, 2, 3, \ldots, \ell\}$, m is a positive real number, K is a nonnegative integer, and k, $x_1, x_2, \ldots, x_n$ are variables in V. For example, if F is the floor function, $V = \mathbb{R}$, $\mathbb{Z}$, or $\mathbb{N} \cup \{0\}$, then $f^{(F)}$, $S^{(F)}$, and $g^{(F)}$ are the same as f, S, and g in Definitions 1 and 2. In this article, we will study $f^{(F)}$, $S^{(F)}$, and $g^{(F)}$ when F is the ceiling and the nearest integer functions.

Definition 3. Let $g_2 = g^{(\text{ceiling})}$, $g_3 = g^{(r^+)}$, and $g_4 = g^{(r^-)}$ be functions defined on $\mathbb{R}^n$ into $\mathbb{Z}$ given by

$$g_2(x_1, x_2, x_3, \ldots, x_n) = \sum_{\emptyset \neq T \subseteq [1,n]} (-1)^{|T|-1} \left\lceil \sum_{i \in T} x_i \right\rceil,$$

$$g_3(x_1, x_2, x_3, \ldots, x_n) = \sum_{\emptyset \neq T \subseteq [1,n]} (-1)^{|T|-1} \left\lfloor \left(\sum_{i \in T} x_i \right) + \frac{1}{2} \right\rfloor,$$

$$g_4(x_1, x_2, x_3, \ldots, x_n) = \sum_{\emptyset \neq T \subseteq [1,n]} (-1)^{|T|-1} \left\lceil \left(\sum_{i \in T} x_i \right) - \frac{1}{2} \right\rceil.$$

Similarly, we define $f_2 = f^{(\text{ceiling})}$, $f_3 = f^{(r^+)}$, $f_4 = f^{(r^-)}$, $S_2 = S^{(\text{ceiling})}$, $S_3 = S^{(r^+)}$, and $S_4 = S^{(r^-)}$. We sometimes write $f_1 = f = f^{(\text{floor})}$, $g_1 = g = g^{(\text{floor})}$, and $S_1 = S = S^{(\text{floor})}$ too.

We therefore replace the study on f, S, g in Definitions 1 and 2 by f_j, S_j, and g_j for $j = 2, 3, 4$ in Definition 3. For example, we have

$$f_4(C, m, k) = \sum_{T \subseteq [1,\ell]} (-1)^{\ell - |T|} \left\lceil \frac{k + \sum_{i \in T} a_i}{m} - \frac{1}{2} \right\rceil,$$

$$S_4(C, m, K) = \sum_{k=0}^{K} f_4(C, m, k).$$

We may need to change the starting point of the sum in S_2 and S_4 from 0 to 1, so we define similar functions as follows.

Definition 4. With the same meaning of C, m, K, f_2, and f_4 in Definitions 1 and 3, let

$$T_2(C, m, K) = S_2(C, m, K) - f_2(C, m, 0) = \sum_{k=1}^{K} f_2(C, m, k),$$

$$T_4(C, m, K) = S_4(C, m, K) - f_4(C, m, 0) = \sum_{k=1}^{K} f_4(C, m, k).$$

Combining the old results by Onphaeng and Pongsriiam [8] and our new results in this article, we obtain sharp bounds for g_j and f_j for every $j = 1, 2, 3, 4$, and for all $n, \ell \geq 2$. We also obtain some sharp bounds for T_2, S_3, and T_4, but only for certain values of ℓ. In fact, these new results are analogs of those given by Tverberg [13], Onphaeng and Pongsriiam [8], and Thanatipanonda and Wong [11] on the bounds for g_1, f_1, and S_1.

Although the basic idea is the same, the new results do not follow directly from the old ones; we still need some modifications and calculations. For example, if $x \in$

$\mathbb{Z}$, then $\lceil x \rceil = \lfloor x \rfloor$ and the results concerning the floor function can be immediately transferred to those on the ceiling function. However, we have n real variables $x_1, x_2, \ldots, x_n$ in the definition of f, g, S, T and there are 2^n cases to consider depending on the integrality or nonintegrality of each x_i. Furthermore, even though we assume $x_1, x_2, \ldots, x_n \notin \mathbb{Z}$, we still do not know whether or not particular sums such as $x_{i_1} + x_{i_2} + \cdots + x_{i_j}$ are integral. Hence, the new results cannot be obtained directly from the application of the old ones.

The generalizations to other functions may be of interest too. In this paper, we are interested in the ceiling and the nearest integer functions. In previous articles, Munteanu [7] and Phunphayap et al. [9] considered the norm of four vectors and the absolute value of n real numbers with $n \leq 6$, respectively. In the future, we or other researchers may consider sums or products such as

$$f(a) + f(b) + f(c) - f(ab) - f(ac) - f(bc) + f(abc),$$

or more generally

$$\sum_{i=1}^{n} f(a_i) - \sum_{1 \leq i < j \leq n} f(a_i a_j) + \cdots + (-1)^{n-1} f(a_1 a_2 \cdots a_n),$$

where f is a multiplicative function such as the divisor function τ, the sum of positive divisors function σ, or Euler's totient function φ. We do not claim that these problems are important or challenging; we merely hope that our article might interest some readers and give them new ideas for some possible research problems.

We need to apply basic properties of the floor and ceiling functions throughout this article, and we refer the reader to the books by Pongsriiam [10, Chapter 3] and Graham, Knuth, and Patashnik [4, Chapter 3] for more information.

2. Lemmas

In this section, we recall Onphaeng and Pongsriiam's results [8], and then provide analogous lemmas that will be used in the proof of our main theorems. We begin with the periodicity and basic relations between functions f_j and g_j.

Lemma 1. *For each $j = 1, 2, 3, 4$, let g_j be the functions given in Definition 3, and let $n \geq 2$. Then the following statements hold.*

(i) $g_3(x_1, x_2, \ldots, x_n) = g_1(x_1, x_2, \ldots, x_n) - g_1(x_1, x_2, \ldots, x_n, 1/2)$.

(ii) $g_4(x_1, x_2, \ldots, x_n) = g_2(x_1, x_2, \ldots, x_n) - g_2(x_1, x_2, \ldots, x_n, -1/2)$.

(iii) *For each $j \in \{1, 2, 3, 4\}$, the function g_j has period 1 in each variable, that is, for $1 \leq j \leq 4$, $1 \leq i \leq n$, and $q \in \mathbb{Z}$, we have*

$$g_j(x_1, x_2, \ldots, x_i + q, \ldots, x_n) = g_j(x_1, x_2, \ldots, x_n). \tag{3}$$

Proof. The fact that g_1 has period 1 was proved by Onphaeng and Pongsriiam [8, Lemma 7]. Since $\lceil q+x\rceil = q+\lceil x\rceil$ for every $q\in\mathbb{Z}$ and $x\in\mathbb{R}$, the proof of (iii) for g_2 is similar: we have $g_2(x_1,x_2,\ldots,x_i+q,\ldots,x_n)$ is equal to

$$\begin{aligned}
&\left(q+\sum_{i=1}^{n}\lceil x_i\rceil\right)-\left(\binom{n-1}{1}q+\sum_{1\le i_1<i_2\le n}\lceil x_{i_1}+x_{i_2}\rceil\right)\\
&\qquad+\left(\binom{n-1}{2}q+\sum_{1\le i_1<i_2<i_3\le n}\lceil x_{i_1}+x_{i_2}+x_{i_3}\rceil\right)\\
&\qquad-\cdots+(-1)^{n-1}\left(\binom{n-1}{n-1}q+\lceil x_1+x_2+\cdots+x_n\rceil\right)\\
&=g_2(x_1,x_2,\ldots,x_n)+q\sum_{0\le k\le n-1}(-1)^k\binom{n-1}{k}. \qquad (4)
\end{aligned}$$

It is well known that the sum in (4) is zero, and so the total sum is $g_2(x_1,x_2,\ldots,x_n)$. Therefore, (iii) is proved when $j\in\{1,2\}$.

Before proving (iii) when $j\in\{3,4\}$, we first prove (i) and (ii). Let $x_{n+1}=1/2$. Then we obtain

$$\begin{aligned}
&g_1(x_1,x_2,\ldots,x_n)-g_1(x_1,x_2,\ldots,x_n,1/2)\\
&\qquad=\sum_{\emptyset\ne T\subseteq[1,n]}(-1)^{|T|-1}\left\lfloor\sum_{i\in T}x_i\right\rfloor-\sum_{\emptyset\ne T\subseteq[1,n+1]}(-1)^{|T|-1}\left\lfloor\sum_{i\in T}x_i\right\rfloor\\
&\qquad=-\sum_{\substack{T\subseteq[1,n+1]\\ n+1\in T}}(-1)^{|T|-1}\left\lfloor\sum_{i\in T}x_i\right\rfloor.
\end{aligned}$$

In the above sum, we can write $T=\{n+1\}\cup T_0$ where $T_0\subseteq[1,n]$, and the sum can be rewritten as

$$\begin{aligned}
&-\sum_{T_0\subseteq[1,n]}(-1)^{|T_0|}\left\lfloor\left(\sum_{i\in T_0}x_i\right)+x_{n+1}\right\rfloor\\
&\qquad=-\sum_{T_0\subseteq[1,n]}(-1)^{|T_0|}\left\lfloor\left(\sum_{i\in T_0}x_i\right)+\frac{1}{2}\right\rfloor\\
&\qquad=\left(\sum_{\emptyset\ne T_0\subseteq[1,n]}(-1)^{|T_0|-1}\left\lfloor\left(\sum_{i\in T_0}x_i\right)+\frac{1}{2}\right\rfloor\right)\\
&\qquad=g_3(x_1,x_2,\ldots,x_n),
\end{aligned}$$

which proves (i). Letting $x_{n+1} = -1/2$ and calculating

$$g_2(x_1, x_2, \ldots, x_n) - g_2(x_1, x_2, \ldots, x_n, x_{n+1})$$

in a similar way, we obtain (ii). By (i) and the fact that g_1 has period 1, we see that the left-hand side of (3) when $j = 3$ is equal to

$$\begin{aligned} &g_1(x_1, x_2, \ldots, x_i + q, \ldots, x_n) - g_1\left(x_1, x_2, \ldots, x_i + q, \ldots, x_n, 1/2\right) \\ &\quad = g_1(x_1, x_2, \ldots, x_i, \ldots, x_n) - g_1\left(x_1, x_2, \ldots, x_i, \ldots, x_n, 1/2\right) \\ &\quad = g_3(x_1, x_2, \ldots, x_n). \end{aligned}$$

Similarly, we obtain (iii) for $j = 4$ from (ii) and the fact that g_2 has period 1. This completes the proof. $\square$

Lemma 2. *Let $\ell \geq 2$. Then the following statements hold for every $j = 1, 2, 3, 4$.*

(i) $f_j\left(\{a_1, a_2, \ldots, a_\ell\}, m, 0\right) = (-1)^{\ell-1} g_j\left(\frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_\ell}{m}\right)$,

(ii) $f_j\left(\{a_1, a_2, \ldots, a_\ell\}, m, k\right) = (-1)^{\ell} g_j\left(\frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_\ell}{m}, \frac{k}{m}\right)$
$+(-1)^{\ell-1} g_j\left(\frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_\ell}{m}\right)$.

Proof. Since the proof for each j is similar, we give a complete proof only for f_2 and g_2 and leave the other cases to the interested reader. When $j = 2$, the left-hand side of (i) is equal to

$$\begin{aligned} \sum_{T \subseteq [1,\ell]} (-1)^{\ell - |T|} \left\lceil \sum_{i \in T} \frac{a_i}{m} \right\rceil &= \sum_{\emptyset \neq T \subseteq [1,\ell]} (-1)^{\ell - |T|} \left\lceil \sum_{i \in T} \frac{a_i}{m} \right\rceil \\ &= (-1)^{\ell-1} \sum_{\emptyset \neq T \subseteq [1,\ell]} (-1)^{1 - |T|} \left\lceil \sum_{i \in T} \frac{a_i}{m} \right\rceil \\ &= (-1)^{\ell-1} g_2\left(\frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_\ell}{m}\right). \end{aligned}$$

This proves (i) when $j = 2$.

Next, let $a_{\ell+1} = k$. Then the right-hand side of (ii) when $j = 2$ is equal to

$$\begin{aligned} &(-1)^{\ell} \left(\sum_{\emptyset \neq T \subseteq [1,\ell+1]} (-1)^{|T|-1} \left\lceil \sum_{i \in T} \frac{a_i}{m} \right\rceil - \sum_{\emptyset \neq T \subseteq [1,\ell]} (-1)^{|T|-1} \left\lceil \sum_{i \in T} \frac{a_i}{m} \right\rceil \right) \\ &\quad = (-1)^{\ell} \sum_{\substack{T \subseteq [1,\ell+1] \\ \ell+1 \in T}} (-1)^{|T|-1} \left\lceil \sum_{i \in T} \frac{a_i}{m} \right\rceil \\ &\quad = (-1)^{\ell} \sum_{T \subseteq [1,\ell]} (-1)^{|T|} \left\lceil \frac{k + \sum_{i \in T} a_i}{m} \right\rceil \\ &\quad = f_2\left(\{a_1, a_2, \ldots, a_\ell\}, m, k\right). \end{aligned}$$

The proof of (i) and (ii) for $j = 1, 3, 4$ is similar. $\square$

Lemma 3. *For each $j = 1, 2, 3, 4$, the function f_j has period m in each variable $a_1, a_2, \dots, a_\ell, k$, that is, for $1 \le i \le \ell$, $1 \le j \le 4$, and $q \in \mathbb{Z}$, we have*

$$\begin{aligned} f_j\left(\{a_1, a_2, \dots, a_i + qm, \dots, a_\ell\}, m, k\right) &= f_j\left(\{a_1, a_2, \dots, a_\ell\}, m, k\right) \\ &= f_j\left(\{a_1, a_2, \dots, a_\ell\}, m, k + qm\right). \end{aligned}$$

Proof. By Lemmas 1 and 2, we see that the left most term in the left-hand side of the above equation is

$$\begin{aligned} &(-1)^\ell g_j\left(\frac{a_1}{m}, \frac{a_2}{m}, \dots, \frac{a_i}{m} + q, \dots, \frac{a_\ell}{m}, \frac{k}{m}\right) + (-1)^{\ell-1} g_j\left(\frac{a_1}{m}, \frac{a_2}{m}, \dots, \frac{a_i}{m} + q, \dots, \frac{a_\ell}{m}\right) \\ &\quad = (-1)^\ell g_j\left(\frac{a_1}{m}, \frac{a_2}{m}, \dots, \frac{a_\ell}{m}, \frac{k}{m}\right) + (-1)^{\ell-1} g_j\left(\frac{a_1}{m}, \frac{a_2}{m}, \dots, \frac{a_\ell}{m}\right) \\ &\quad = f_j\left(\{a_1, a_2, \dots, a_\ell\}, m, k\right). \end{aligned}$$

The other equality is similar. $\square$

We will apply the bounds for f_1 obtained by Onphaeng and Pongsriiam [8] to prove our main results, so we recall them for the reader's convenience.

Lemma 4 ([8], Theorem 8)**.** *For each $\ell \ge 2$, $a_1, a_2, \dots, a_\ell, k \in \mathbb{Z}$, and $m \in \mathbb{Z}^+$, we have*

$$-2^{\ell-2} \le f_1(\{a_1, a_2, \dots, a_\ell\}, m, k) \le 2^{\ell-2}.$$

The lower and upper bounds are best possible in the sense that there are a_1, a_2, $\dots$, a_ℓ, m, k for which equality is achieved. More precisely, the following statements hold.

(i) *If ℓ is odd, m is even, and $a_i = m/2$ for every $i = 1, 2, \dots, \ell$, then*

$$f_1(\{a_1, a_2, \dots, a_\ell\}, m, 0) = -2^{\ell-2} \quad \textit{and} \quad f_1\left(\{a_1, a_2, \dots, a_\ell\}, m, m/2\right) = 2^{\ell-2}.$$

(ii) *If ℓ is even, m is even, and $a_i = m/2$ for every $i = 1, 2, \dots, \ell$, then*

$$f_1(\{a_1, a_2, \dots, a_\ell\}, m, 0) = 2^{\ell-2} \quad \textit{and} \quad f_1\left(\{a_1, a_2, \dots, a_\ell\}, m, m/2\right) = -2^{\ell-2}.$$

Lemma 5 ([8], Theorem 4)**.** *For each $n \ge 2$, the function g_1 has maximum value $2^{n-2} - 1$ and minimum value -2^{n-2}. Furthermore, when $x_k = \frac{1}{2}$ for all $k = 1, 2, \dots, n$, this minimum occurs, and when $x_k = \frac{1}{2} - \frac{1}{n^2}$ for all $k = 1, 2, \dots, n$, the maximum occurs.*

To prove Lemma 7, it is useful to extend the well known Hermite identity to the case of the ceiling function. Recall that for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\sum_{0 \le k \le n-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor nx \rfloor . \tag{5}$$

For the proof of (5) and various generalizations, we refer the reader to Aursukaree, Khemaratchatakumthorn, and Pongsriiam [1]. Hermite's identity, generalizations, and applications are also collected in Pongsriiam's book [10, Section 3.2]. Extending it for the ceiling function, we obtain the next lemma.

Lemma 6 (An analog of Hermite's identity for the ceiling function)**.** *For each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have*

$$\sum_{1 \le k \le n} \left\lceil x + \frac{k}{n} \right\rceil = \lceil nx \rceil + n.$$

Proof. For $1 \le k \le n$, we have $0 < \frac{k}{n} \le 1$, and so $\left\lceil \frac{k}{n} \right\rceil = 1$. Therefore, if $x \in \mathbb{Z}$, then

$$\sum_{1 \le k \le n} \left\lceil x + \frac{k}{n} \right\rceil = \sum_{1 \le k \le n} (x+1) = nx + n = \lceil nx \rceil + n.$$

So assume that $x \notin \mathbb{Z}$. We write $x = \lfloor x \rfloor + \{x\}$, where $\{x\}$ is the fractional part of x. Then

$$\sum_{1 \le k \le n} \left\lceil x + \frac{k}{n} \right\rceil = \sum_{1 \le k \le n} \left\lceil \lfloor x \rfloor + \{x\} + \frac{k}{n} \right\rceil = n\lfloor x \rfloor + \sum_{1 \le k \le n} \left\lceil \{x\} + \frac{k}{n} \right\rceil . \tag{6}$$

Since $0 < \{x\} < 1$, there exists $j \in \{1, 2, \ldots, n\}$ such that

$$\frac{j-1}{n} < \{x\} \le \frac{j}{n}. \tag{7}$$

Therefore,

$$0 < \{x\} + \frac{k}{n} \le 1 \quad \text{ for } 1 \le k \le n-j$$

and

$$1 < \{x\} + \frac{k}{n} \le 2 \quad \text{ for } n-j+1 \le k \le n.$$

So (6) and (7) imply that

$$\begin{aligned} \sum_{1 \le k \le n} \left\lceil x + \frac{k}{n} \right\rceil &= n\lfloor x \rfloor + \sum_{1 \le k \le n} \left\lceil \{x\} + \frac{k}{n} \right\rceil \\ &= n\lfloor x \rfloor + \sum_{1 \le k \le n-j} \left\lceil \{x\} + \frac{k}{n} \right\rceil + \sum_{n-j+1 \le k \le n} \left\lceil \{x\} + \frac{k}{n} \right\rceil \\ &= n\lfloor x \rfloor + (n-j) + 2j = n\lfloor x \rfloor + n + j. \end{aligned}$$

So it remains to show that $n\lfloor x\rfloor + j = \lceil nx\rceil$. By (7), we see that $j-1 < n\{x\} \le j$, and so $\lceil nx\rceil = \lceil n\lfloor x\rfloor + n\{x\}\rceil = n\lfloor x\rfloor + \lceil n\{x\}\rceil = n\lfloor x\rfloor + j$, as required. $\square$

Lemma 7. *Let K, m, ℓ be integers, $K \ge 0$, $m \ge 1$, $\ell \ge 2$, $[1,\ell] = \{1,2,3,\ldots,\ell\}$, and C a multiset of ℓ integers $a_1, a_2, \ldots, a_\ell$. Then the following statements hold.*

(i) $S_j\left(C, m, m-1\right) = 0$ *for each* $j = 1,2,3,4$.

(ii) $T_j\left(C, m, m\right) = 0$ *for each* $j = 2,4$.

(iii) *For each $j = 1,2,3,4$, the function S_j has period m in the variables a_1, a_2, $\ldots$, a_ℓ, K, that is, for $1 \le j \le 4$, $1 \le i \le \ell$, and $q \in \mathbb{Z}$, we have*

$$S_j\left(C, m, K+qm\right) = S_j\left(C, m, K\right) = S_j\left(\{a_1, a_2, \ldots, a_i + qm, \ldots, a_\ell\}, m, K\right).$$

(iv) *For each $j = 2,4$, the function T_j has period m in the variables a_1, a_2, $\ldots$, a_ℓ, K, that is, for $j = 2,4$, $1 \le i \le \ell$, and $q \in \mathbb{Z}$, we have*

$$T_j\left(C, m, K+qm\right) = T_j\left(C, m, K\right) = T_j\left(\{a_1, a_2, \ldots, a_i + qm, \ldots, a_\ell\}, m, K\right).$$

Proof. The periodicity of S_j in the variables $a_1, a_2, \ldots, a_\ell$ follows from that of f_j, which is proved in Lemma 3. So it remains to prove (i), (ii), and the periodicity of S_j and T_j in the variable K. Since the proof for each j is similar, we give a complete proof only for S_1, S_2, and T_2. By (5), we see that the left-hand side of (i) when $j = 1$ is equal to

$$\begin{aligned}
\sum_{k=0}^{m-1} f_1(C,m,k) &= \sum_{k=0}^{m-1} \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \frac{k+\sum_{i\in T} a_i}{m} \right\rfloor \\
&= \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left(\sum_{k=0}^{m-1} \left\lfloor \frac{k+\sum_{i\in T} a_i}{m} \right\rfloor \right) = \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \sum_{i\in T} a_i \right\rfloor \\
&= \sum_{\emptyset\neq T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left(\sum_{i\in T} a_i \right) = \sum_{\emptyset\neq T\subseteq[1,\ell]} \sum_{i\in T} (-1)^{\ell-|T|} a_i \\
&= \sum_{i=1}^{\ell} \sum_{\substack{T\subseteq[1,\ell] \\ i\in T}} (-1)^{\ell-|T|} a_i = \sum_{i=1}^{\ell} a_i \sum_{\substack{T\subseteq[1,\ell] \\ i\in T}} (-1)^{\ell-|T|}. \qquad (8)
\end{aligned}$$

Next, we will show that the inner sum of the last term in (8) is equal to zero. If we fix $i, n \in [1,\ell]$, then the number of sets T such that $T \subseteq [1,\ell]$, $i \in T$, and

$|T| = n$ is $\binom{\ell-1}{n-1}$. Therefore,

$$\begin{aligned}\sum_{\substack{T\subseteq[1,\ell]\\ i\in T}} (-1)^{\ell-|T|} &= \sum_{n=1}^{\ell} \sum_{\substack{T\subseteq[1,\ell]\\ i\in T,|T|=n}} (-1)^{\ell-|T|} = \sum_{n=1}^{\ell} \binom{\ell-1}{n-1}(-1)^{\ell-n} \\ &= \sum_{i=0}^{\ell-1} \binom{\ell-1}{i}(-1)^{\ell-1-i} = (-1)^{\ell-1} \sum_{i=0}^{\ell-1}(-1)^{i}\binom{\ell-1}{i} = 0, \qquad (9)\end{aligned}$$

where the last equality is a well known identity. Thus, (8) is zero and (i) is proved when $j = 1$. The proof of (i) when $j = 3$ is similar.

Next, we prove (i) when $j = 2$. By Lemma 3, we have

$$f_2(C, m, 0) = f_2(C, m, m).$$

Then by (8), (9), and Lemma 6, we see that the left-hand side of (i) when $j = 2$ is equal to

$$\begin{aligned}\sum_{k=0}^{m-1} f_2(C, m, k) &= \sum_{k=1}^{m} f_2(C, m, k) = \sum_{k=1}^{m} \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lceil \frac{k + \sum_{i\in T} a_i}{m} \right\rceil \\ &= \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left(\sum_{k=1}^{m} \left\lceil \frac{k + \sum_{i\in T} a_i}{m} \right\rceil \right) \\ &= \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left(\left\lceil \sum_{i\in T} a_i \right\rceil + m \right) \\ &= \sum_{\emptyset\neq T\subseteq[1,\ell]} \sum_{i\in T} (-1)^{\ell-|T|} a_i + m \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \\ &= m \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|}. \qquad (10)\end{aligned}$$

Next, we will show that the last sum in (10) is equal to zero. For $0 \le i \le \ell$, the number of sets T such that $T \subseteq [1, \ell]$ and $|T| = i$ is $\binom{\ell}{i}$. Therefore,

$$\sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} = \sum_{i=0}^{\ell} \binom{\ell}{i}(-1)^{\ell-i} = (-1)^{\ell} \sum_{i=0}^{\ell} \binom{\ell}{i}(-1)^{i} = 0.$$

So the last sum in (10) is zero, and (i) is proved when $j = 2$. The proof of (i) when $j = 4$ is similar.

Next, we prove (ii) when $j = 2$. By (i) and Lemma 3, we see that the left-hand side of (ii) when $j = 2$ is equal to

$$\begin{aligned} S_2(C, m, m) - f_2(C, m, 0) &= S_2(C, m, m) - f_2(C, m, m) \\ &= S_2(C, m, m-1) = 0.\end{aligned}$$

This proves (ii) when $j = 2$ and the proof of (ii) when $j = 4$ is similar.

Next, by (i) and Lemma 3, we see that the left-hand side of (iii) when $j = 2$ is equal to

$$\begin{aligned}\sum_{k=0}^{K+qm} f_2(C,m,k) &= \sum_{i=0}^{q-1}\sum_{j=im}^{(i+1)m-1} f_2(C,m,j) + \sum_{k=qm}^{K+qm} f_2(C,m,k)\\ &= \sum_{i=0}^{q-1}\sum_{j=0}^{m-1} f_2(C,m,j+im) + \sum_{k=0}^{K} f_2(C,m,k+qm)\\ &= \sum_{i=0}^{q-1}\sum_{j=0}^{m-1} f_2(C,m,j) + \sum_{k=0}^{K} f_2(C,m,k). \qquad (11)\end{aligned}$$

The inner sum of the first term in (11) is $S_2(C,m,m-1)$, which is zero by (i). The second sum in (11) is $S_2(C,m,K)$, which proves (iii) when $j = 2$. The proof of (iii) when $j = 1, 3, 4$ is similar. By (iii) and Lemma 3, we see that the left-hand side of (iv) when $j = 2$ is equal to

$$S_2(C,m,K+qm) - f_2(C,m,0) = S_2(C,m,K) - f_2(C,m,0) = T_2(C,m,K).$$

The proof of (iv) when $j = 4$ is also similar. So the proof is complete. □

Lemma 8. *For each $q \in \mathbb{Z}$ and $m \in \mathbb{N}$, we have*

$$\left\lfloor \frac{q}{m} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{q + \lfloor m/2 \rfloor}{m} \right\rfloor \quad \textit{and} \quad \left\lceil \frac{q}{m} + \frac{1}{2} \right\rceil = \left\lceil \frac{q + \lceil m/2 \rceil}{m} \right\rceil.$$

Proof. This is clear when m is even. So assume that m is odd. By the division algorithm, there are $q_1, r \in \mathbb{Z}$ such that $2q + m = 2mq_1 + r$ where $0 \le r < 2m$. If $r = 0$, then $2 \mid m$ contradicting the assumption that m is odd. So $1 \le r \le 2m - 1$. Therefore,

$$0 < r/2m < 1, \quad 0 \le (r-1)/2m < 1, \quad \text{and} \quad 0 < (r+1)/2m \le 1.$$

Thus,

$$\begin{aligned}\left\lfloor \frac{q}{m} + \frac{1}{2} \right\rfloor &= \left\lfloor \frac{2q+m}{2m} \right\rfloor = \left\lfloor q_1 + \frac{r}{2m} \right\rfloor = q_1,\\ \left\lfloor \frac{q + \lfloor m/2 \rfloor}{m} \right\rfloor &= \left\lfloor \frac{q + (m-1)/2}{m} \right\rfloor = \left\lfloor \frac{2q+m-1}{2m} \right\rfloor = \left\lfloor q_1 + \frac{r-1}{2m} \right\rfloor = q_1,\\ \left\lceil \frac{q}{m} + \frac{1}{2} \right\rceil &= \left\lceil \frac{2q+m}{2m} \right\rceil = \left\lceil q_1 + \frac{r}{2m} \right\rceil = q_1 + 1,\end{aligned}$$

and

$$\left\lceil \frac{q + \lceil m/2 \rceil}{m} \right\rceil = \left\lceil \frac{q + (m+1)/2}{m} \right\rceil = \left\lceil \frac{2q+m+1}{2m} \right\rceil = \left\lceil q_1 + \frac{r+1}{2m} \right\rceil = q_1 + 1.$$

This completes the proof. □

3. Main Results

We begin with the results for g_2, g_3, g_4, and then prove the inequality for f_2, f_3, and f_4, respectively.

Theorem 1. *For each $n \geq 2$, the function g_2 given in Definition 3 has maximum value 2^{n-2} and minimum value $-2^{n-2}+1$. Furthermore, when $x_k = \frac{1}{2} + \frac{1}{n^2}$ for all $k = 1, 2, \ldots, n$, this minimum occurs, and when $x_k = \frac{1}{2}$ for all $k = 1, 2, \ldots, n$, the maximum occurs.*

Proof. If $n = 2$, then the result is the well known inequality

$$0 \leq \lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil \leq 1, \tag{12}$$

which holds for all $x, y \in \mathbb{R}$. The inequality (12) is sharp: if $x = y = 3/4$, then the left inequality in (12) becomes equality, and if $x = y = 1/2$, then the right inequality in (12) becomes equality. The result when $n \geq 3$ is obtained from the case $n = 2$ and a careful selection of pairs.

For illustration purposes, we first give a proof for the case $n = 3$ and $n = 4$. Recall that

$$g_2(x_1, x_2, x_3) = \lceil x_1 \rceil + \lceil x_2 \rceil + \lceil x_3 \rceil - \lceil x_1 + x_2 \rceil - \lceil x_1 + x_3 \rceil - \lceil x_2 + x_3 \rceil + \lceil x_1 + x_2 + x_3 \rceil.$$

We obtain by (12) that

$$-1 \leq \lceil x_1 + x_2 + x_3 \rceil - \lceil x_1 + x_2 \rceil - \lceil x_3 \rceil \leq 0, \tag{13}$$

$$0 \leq -\lceil x_2 + x_3 \rceil + \lceil x_2 \rceil + \lceil x_3 \rceil \leq 1, \tag{14}$$

$$0 \leq -\lceil x_1 + x_3 \rceil + \lceil x_1 \rceil + \lceil x_3 \rceil \leq 1. \tag{15}$$

Summing (13), (14), and (15), the middle term is $g_2(x_1, x_2, x_3)$. So

$$-1 \leq g_2(x_1, x_2, x_3) \leq 2.$$

Next, we obtain by (12) the following inequalities:

$$0 \leq -\lceil x_1 + x_2 + x_3 + x_4 \rceil + \lceil x_1 + x_2 + x_3 \rceil + \lceil x_4 \rceil \leq 1, \tag{16}$$

$$-1 \leq \lceil x_1 + x_2 + x_4 \rceil - \lceil x_1 + x_2 \rceil - \lceil x_4 \rceil \leq 0, \tag{17}$$

$$-1 \leq \lceil x_1 + x_3 + x_4 \rceil - \lceil x_1 + x_3 \rceil - \lceil x_4 \rceil \leq 0, \tag{18}$$

$$-1 \leq \lceil x_2 + x_3 + x_4 \rceil - \lceil x_2 + x_3 \rceil - \lceil x_4 \rceil \leq 0, \tag{19}$$

$$0 \leq -\lceil x_1 + x_4 \rceil + \lceil x_1 \rceil + \lceil x_4 \rceil \leq 1, \tag{20}$$

$$0 \leq -\lceil x_2 + x_4 \rceil + \lceil x_2 \rceil + \lceil x_4 \rceil \leq 1, \tag{21}$$

$$0 \le -\lceil x_3 + x_4 \rceil + \lceil x_3 \rceil + \lceil x_4 \rceil \le 1. \tag{22}$$

Summing (16) to (22), we see that $-3 \le g_3(x_1, x_2, x_3, x_4) \le 4$.

Next, we prove the general case $n \ge 5$. The expression of the form

$$\lceil x_{i_1} + x_{i_2} + \cdots + x_{i_k} \rceil$$

will be called a *k-bracket*. So for each $1 \le k \le n$, there are $\binom{n}{k}$ k-brackets appearing in the sum defining $g_2(x_1, x_2, \ldots, x_n)$. We first pair up the n-bracket with an $(n-1)$-bracket and a 1-bracket as follows:

$$s_1 = (-1)^{n-1}\lceil x_1+x_2+\cdots+x_n \rceil + (-1)^{n-2}\lceil x_1+x_2+\cdots+x_{n-1} \rceil + (-1)^{n-2}\lceil x_n \rceil. \tag{23}$$

We notice that the sign of $\lceil x_n \rceil$ in (23) may or may not be the same as that appearing in the sum defining $g_2(x_1, x_2, \ldots, x_n)$ but it is the same as the sign of $\lceil x_1 + x_2 + \cdots + x_{n-1} \rceil$ so that we can apply (12) to obtain the bound for s_1. Next, we pair up the remaining $(n-1)$-brackets with $(n-2)$-brackets and 1-brackets as follows:

$$(-1)^{n-2}\lceil x_{i_1}+x_{i_2}+\cdots+x_{i_{n-1}} \rceil + (-1)^{n-3}\lceil x_{i_1}+x_{i_2}+\cdots+x_{i_{n-2}} \rceil + (-1)^{n-3}\lceil x_{i_{n-1}} \rceil, \tag{24}$$

where $1 \le i_1 < i_2 < \cdots < i_{n-1} \le n$. We note again that the sign of

$$\lceil x_{i_1} + x_{i_2} + \cdots + x_{i_{n-1}} \rceil \quad \text{and} \quad \lceil x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} \rceil$$

in (24) are the same as those appearing in the sum defining $g_2(x_1, x_2, \ldots, x_n)$ while the sign of $\lceil x_{i_{n-1}} \rceil$ in (24) may or may not be the same, but we can apply (12) to obtain the bound of (24). Since $\lceil x_1 + x_2 + \cdots + x_{n-1} \rceil$ appears in (23), the term $x_{i_{n-1}}$ appearing in the $(n-1)$-brackets in (24) is always x_n. So in fact (24) is

$$(-1)^{n-2}\lceil x_{i_1}+x_{i_2}+\cdots+x_{i_{n-2}}+x_n \rceil + (-1)^{n-3}\lceil x_{i_1}+x_{i_2}+\cdots+x_{i_{n-2}} \rceil + (-1)^{n-3}\lceil x_n \rceil. \tag{25}$$

Then we sum (25) over all possible $1 \le i_1 < i_2 < \cdots < i_{n-2} < n$, and call it s_2. That is

$$\begin{aligned} s_2 = (-1)^{n-2} &\sum_{1 \le i_1 < i_2 < \cdots < i_{n-2} < n} \lceil x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} + x_n \rceil \\ &+ (-1)^{n-3} \sum_{1 \le i_1 < i_2 < \cdots < i_{n-2} < n} \lceil x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} \rceil \\ &+ (-1)^{n-3} \binom{n-1}{n-2} \lceil x_n \rceil. \end{aligned}$$

We continue doing this process as follows. For each $0 \le \ell \le n-1$, let c_ℓ be the sum of all $\lceil x_{i_1} + x_{i_2} + \cdots + x_{i_{n-\ell}} \rceil$ with $1 \le i_1 < i_2 < \cdots < i_{n-\ell} \le n$, a_ℓ

the sum of all such terms with $i_{n-\ell} = n$, and b_ℓ the sum of all such terms with $i_{n-\ell} < n$. Therefore, $c_\ell = a_\ell + b_\ell$. Since b_0 is the empty sum, we have $b_0 = 0$. The number of $(n-\ell)$-brackets appearing in the sum defining c_ℓ is $\binom{n}{n-\ell}$, the number of $(n-\ell)$-brackets appearing in the sum defining a_ℓ is $\binom{n-1}{n-\ell-1}$, and the number of $(n-\ell)$-brackets appearing in the sum defining b_ℓ is $\binom{n-1}{n-\ell}$. In addition, we have

$$\begin{aligned} g_2(x_1, x_2, \dots, x_n) &= c_{n-1} - c_{n-2} + c_{n-3} - \cdots + (-1)^{n-1} c_0 \\ &= \sum_{0 \le \ell \le n-1} (-1)^{n-1-\ell} c_\ell, \\ s_1 &= (-1)^{n-1} a_0 + (-1)^{n-2} b_1 + (-1)^{n-2} \lceil x_n \rceil, \\ s_2 &= (-1)^{n-2} a_1 + (-1)^{n-3} b_2 + (-1)^{n-3} \binom{n-1}{n-2} \lceil x_n \rceil. \end{aligned}$$

In general, for each $1 \le \ell \le n-1$, we let

$$s_\ell = (-1)^{n-\ell} a_{\ell-1} + (-1)^{n-\ell-1} b_\ell + (-1)^{n-\ell-1} \binom{n-1}{n-\ell} \lceil x_n \rceil.$$

Then

$$\begin{aligned} \sum_{1 \le \ell \le n-1} s_\ell = (-1)^{n-1} a_0 &+ \sum_{2 \le \ell \le n-1} (-1)^{n-\ell} a_{\ell-1} + \sum_{1 \le \ell \le n-2} (-1)^{n-\ell-1} b_\ell + b_{n-1} \\ &+ \lceil x_n \rceil \sum_{1 \le \ell \le n-1} (-1)^{n-\ell-1} \binom{n-1}{n-\ell}. \end{aligned} \tag{26}$$

Recall the well known identity $\sum_{0 \le \ell \le n} (-1)^\ell \binom{n}{\ell} = 0$ for all $n \ge 1$. Then the last sum on the right-hand side of (26) is

$$\begin{aligned} -\sum_{1 \le \ell \le n-1} (-1)^{n-\ell} \binom{n-1}{n-\ell} &= -\sum_{1 \le \ell \le n-1} (-1)^\ell \binom{n-1}{\ell} \\ &= -\sum_{0 \le \ell \le n-1} (-1)^\ell \binom{n-1}{\ell} + 1 = 1. \end{aligned}$$

Therefore, the last term in (26) is $\lceil x_n \rceil$. Replacing ℓ by $\ell + 1$ in the first sum on the right-hand side of (26), we see that

$$\begin{aligned} \sum_{1 \le \ell \le n-1} s_\ell &= (-1)^{n-1} a_0 + \sum_{1 \le \ell \le n-2} (-1)^{n-\ell-1} (a_\ell + b_\ell) + b_{n-1} + \lceil x_n \rceil \\ &= (-1)^{n-1} c_0 + \sum_{1 \le \ell \le n-2} (-1)^{n-\ell-1} c_\ell + b_{n-1} + \lceil x_n \rceil \\ &= (-1)^{n-1} c_0 + \sum_{1 \le \ell \le n-2} (-1)^{n-\ell-1} c_\ell + c_{n-1} \qquad (27) \\ &= \sum_{0 \le \ell \le n-1} (-1)^{n-\ell-1} c_\ell = g_2(x_1, x_2, \dots, x_n), \end{aligned}$$

where (27) can be obtained from the definition of c_{n-1}, b_{n-1}, and a_{n-1}. By (12) and (23) we obtain $-1 \leq s_1 \leq 0$ if n is odd, and $0 \leq s_1 \leq 1$ if n is even. Similarly, the sum (25) lies in $[0,1]$ if n is odd, and lies in $[-1,0]$ if n is even. Therefore,

$$0 \leq s_2 \leq \binom{n-1}{n-2} \quad \text{if } n \text{ is odd}$$

and

$$-\binom{n-1}{n-2} \leq s_2 \leq 0 \quad \text{if } n \text{ is even.}$$

In general, for each $1 \leq \ell \leq n-1$, we have

$$0 \leq s_\ell \leq \binom{n-1}{n-\ell} \quad \text{if } n \text{ and } \ell \text{ have different parity}$$

and

$$-\binom{n-1}{n-\ell} \leq s_\ell \leq 0 \quad \text{if } n \text{ and } \ell \text{ have the same parity.}$$

Since $g_2(x_1, x_2, \ldots, x_n) = \sum_{1 \leq \ell \leq n-1} s_\ell$, we obtain, for odd n,

$$-\sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ is odd}}} \binom{n-1}{n-\ell} \leq g_2(x_1, x_2, \ldots, x_n) \leq \sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ is even}}} \binom{n-1}{n-\ell},$$

and for even n,

$$-\sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ is even}}} \binom{n-1}{n-\ell} \leq g_2(x_1, x_2, \ldots, x_n) \leq \sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ is odd}}} \binom{n-1}{n-\ell}.$$

Recall the well known identity

$$\sum_{\substack{0\leq k \leq n \\ k \text{ is even}}} \binom{n}{k} = \sum_{\substack{0\leq k \leq n \\ k \text{ is odd}}} \binom{n}{k} = 2^{n-1}. \tag{28}$$

Therefore, if n is odd, then

$$\sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ is odd}}} \binom{n-1}{n-\ell} = \sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ is even}}} \binom{n-1}{\ell} = 2^{n-2} - 1$$

and

$$\sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ is even}}} \binom{n-1}{n-\ell} = \sum_{\substack{1\leq \ell \leq n-1 \\ \ell \text{ is odd}}} \binom{n-1}{\ell} = \sum_{\substack{0\leq \ell \leq n-1 \\ \ell \text{ is odd}}} \binom{n-1}{\ell} = 2^{n-2}.$$

Similarly, if n is even, then

$$\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}} \binom{n-1}{n-\ell} = 2^{n-2} \quad \text{and} \quad \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}} \binom{n-1}{n-\ell} = 2^{n-2}-1.$$

Hence, $-2^{n-2}+1 \le g_2(x_1, x_2, \ldots, x_n) \le 2^{n-2}$, as required.

Next, we show that the lower bound $-2^{n-2}+1$ and the upper bound 2^{n-2} are actually the minimum and the maximum of $g_2(x_1, x_2, \ldots, x_n)$, respectively. Let $x_k = 1/2$ for every $k = 1, 2, \ldots, n$. Then

$$\begin{aligned} g_2(x_1, x_2, \ldots, x_n) &= \sum_{1\le k\le n} (-1)^{k-1} \left\lceil \frac{k}{2} \right\rceil \binom{n}{k} \qquad (29)\\ &= \sum_{\substack{1\le k\le n\\ k \text{ is even}}} (-1)^{k-1} \frac{k}{2}\binom{n}{k} + \sum_{\substack{1\le k\le n\\ k \text{ is odd}}} (-1)^{k-1} \frac{k+1}{2}\binom{n}{k}\\ &= \frac{1}{2}\sum_{\substack{1\le k\le n\\ k \text{ is even}}} (-1)^{k-1} k\binom{n}{k} + \frac{1}{2}\sum_{\substack{1\le k\le n\\ k \text{ is odd}}} (-1)^{k-1} k\binom{n}{k} + \frac{1}{2}\sum_{\substack{1\le k\le n\\ k \text{ is odd}}} (-1)^{k-1}\binom{n}{k}\\ &= \frac{1}{2}\sum_{1\le k\le n} (-1)^{k-1} k\binom{n}{k} + \frac{1}{2}\sum_{\substack{1\le k\le n\\ k \text{ is odd}}} \binom{n}{k}. \qquad (30) \end{aligned}$$

Recall the well known identity

$$\sum_{k=1}^{n} (-1)^{k-1} k\binom{n}{k} = 0, \quad \text{which holds for all } n \ge 2. \qquad (31)$$

Then (28), (30), and (31) imply that

$$g_2(x_1, x_2, \ldots, x_n) = \frac{1}{2}\left(2^{n-1}\right) = 2^{n-2}. \qquad (32)$$

Next, let $x_k = \frac{1}{2} + \frac{1}{n^2}$ for every $k = 1, 2, \ldots, n$. Then

$$g_2(x_1, x_2, \ldots, x_n) = \sum_{1\le k\le n} (-1)^{k-1} \left\lceil \frac{k}{2} + \frac{k}{n^2} \right\rceil \binom{n}{k}. \qquad (33)$$

Recall that $n \ge 2$. If $1 \le k \le n$ and k is even, then $\left\lceil \frac{k}{2} + \frac{k}{n^2} \right\rceil = \frac{k}{2} + 1$. If $1 \le k \le n$ and k is odd, then $\left\lceil \frac{k}{2} + \frac{k}{n^2} \right\rceil = \frac{k+1}{2}$. Therefore, the left-hand side of (33) is equal to

$$\begin{aligned}\sum_{\substack{1\le k\le n\\ k\text{ is even}}}(-1)^{k-1}\left(\frac{k}{2}+1\right)\binom{n}{k}+\sum_{\substack{1\le k\le n\\ k\text{ is odd}}}(-1)^{k-1}\frac{k+1}{2}\binom{n}{k}\\ =&\frac{1}{2}\sum_{\substack{1\le k\le n\\ k\text{ is even}}}(-1)^{k-1}k\binom{n}{k}-\sum_{\substack{1\le k\le n\\ k\text{ is even}}}\binom{n}{k}+\frac{1}{2}\sum_{\substack{1\le k\le n\\ k\text{ is odd}}}(-1)^{k-1}k\binom{n}{k}\\ &+\frac{1}{2}\sum_{\substack{1\le k\le n\\ k\text{ is odd}}}\binom{n}{k}\\ =&\frac{1}{2}\sum_{1\le k\le n}(-1)^{k-1}k\binom{n}{k}-\sum_{\substack{1\le k\le n\\ k\text{ is even}}}\binom{n}{k}+\frac{1}{2}\sum_{\substack{1\le k\le n\\ k\text{ is odd}}}\binom{n}{k}.\end{aligned}\tag{34}$$

Then (28), (31), and (34) imply that

$$\begin{aligned}g_2(x_1,x_2,\ldots,x_n)&=-\sum_{\substack{1\le k\le n\\ k\text{ is even}}}\binom{n}{k}+\frac{1}{2}\sum_{\substack{1\le k\le n\\ k\text{ is odd}}}\binom{n}{k}\\ &=-\left(2^{n-1}-1\right)+2^{n-2}=-2^{n-2}+1.\end{aligned}$$

This completes the proof. □

The proof of the next theorem follows the same idea as that of Theorem 1, but we still need to adjust some calculations.

Theorem 2. *For each $n\ge 2$, the function g_3 given in Definition 3 has maximum value 2^{n-2} and minimum value -2^{n-2}. Furthermore, when $x_k=\frac{1}{2}-\frac{1}{n^2}$ for all $k=1,2,\ldots,n$, this minimum occurs, and when $x_k=\frac{1}{2}$ for all $k=1,2,\ldots,n$, the maximum occurs.*

Proof. By Lemma 1, we can assume that $0\le x_k<1$ for every $1\le k\le n$. First, we give a proof for the case $n=2$. Recall the well known inequality that

$$-1\le\lfloor x\rfloor+\lfloor y\rfloor-\lfloor x+y\rfloor\le 0,$$

which holds for all $x,y\in\mathbb{R}$. Therefore,

$$-1\le\left\lfloor x+\frac{1}{2}\right\rfloor+\lfloor y\rfloor-\left\lfloor\left(x+\frac{1}{2}\right)+y\right\rfloor\le 0.$$

Since $0\le y<1$, we obtain

$$-1\le\left\lfloor x+\frac{1}{2}\right\rfloor-\left\lfloor x+y+\frac{1}{2}\right\rfloor\le 0\tag{35}$$

and

$$0 \leq \left\lfloor y + \frac{1}{2} \right\rfloor \leq 1. \tag{36}$$

Summing (35) and (36), the middle term is $g_3(x, y)$, and so

$$-1 \leq g_3(x, y) \leq 1. \tag{37}$$

The inequality (37) is sharp: if $x = y = 1/4$, then the left inequality in (37) becomes equality, and if $x = y = 1/2$, then the right inequality in (37) becomes equality. The result when $n \geq 3$ is based on a careful selection of pairs and the case $n = 2$.

For illustration purposes, we first give a proof for the case $n = 3$ and $n = 4$. By a similar idea as in the proof of Theorem 1, we have

$$0 \leq \left\lfloor x_1 + x_2 + x_3 + \frac{1}{2} \right\rfloor - \left\lfloor x_1 + x_2 + \frac{1}{2} \right\rfloor \leq 1, \tag{38}$$

$$-1 \leq -\left\lfloor x_2 + x_3 + \frac{1}{2} \right\rfloor + \left\lfloor x_2 + \frac{1}{2} \right\rfloor \leq 0, \tag{39}$$

$$-1 \leq -\left\lfloor x_1 + x_3 + \frac{1}{2} \right\rfloor + \left\lfloor x_1 + \frac{1}{2} \right\rfloor \leq 0, \tag{40}$$

$$0 \leq \left\lfloor x_3 + \frac{1}{2} \right\rfloor \leq 1. \tag{41}$$

Summing (38), (39), (40), and (41) gives $-2 \leq g_3(x_1, x_2, x_3) \leq 2$. Similarly, we have

$$-1 \leq -\left\lfloor x_1 + x_2 + x_3 + x_4 + \frac{1}{2} \right\rfloor + \left\lfloor x_1 + x_2 + x_3 + \frac{1}{2} \right\rfloor \leq 0, \tag{42}$$

$$0 \leq \left\lfloor x_1 + x_2 + x_4 + \frac{1}{2} \right\rfloor - \left\lfloor x_1 + x_2 + \frac{1}{2} \right\rfloor \leq 1, \tag{43}$$

$$0 \leq \left\lfloor x_1 + x_3 + x_4 + \frac{1}{2} \right\rfloor - \left\lfloor x_1 + x_3 + \frac{1}{2} \right\rfloor \leq 1, \tag{44}$$

$$0 \leq \left\lfloor x_2 + x_3 + x_4 + \frac{1}{2} \right\rfloor - \left\lfloor x_2 + x_3 + \frac{1}{2} \right\rfloor \leq 1, \tag{45}$$

$$-1 \leq -\left\lfloor x_1 + x_4 + \frac{1}{2} \right\rfloor + \left\lfloor x_1 + \frac{1}{2} \right\rfloor \leq 0, \tag{46}$$

$$-1 \leq -\left\lfloor x_2 + x_4 + \frac{1}{2} \right\rfloor + \left\lfloor x_2 + \frac{1}{2} \right\rfloor \leq 0, \tag{47}$$

$$-1 \leq -\left\lfloor x_3 + x_4 + \frac{1}{2} \right\rfloor + \left\lfloor x_3 + \frac{1}{2} \right\rfloor \leq 0, \tag{48}$$

$$0 \le \left\lfloor x_4 + \frac{1}{2} \right\rfloor \le 1. \tag{49}$$

Summing (42) to (49), we see that $-4 \le g_3(x_1, x_2, x_3, x_4) \le 4$.

In general, we let $n \ge 5$, call the expression of the form $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_\ell} + \frac{1}{2} \rfloor$ an *ℓ-bracket*, and follow closely the method used in the proof of Theorem 1. For each $1 \le \ell \le n$, let c_ℓ be the sum of all ℓ-brackets with $1 \le i_1 < i_2 < \cdots < i_\ell \le n$, a_ℓ the sum of all such terms with $i_\ell = n$, and b_ℓ the sum of all such terms with $i_\ell < n$. Therefore, $c_\ell = a_\ell + b_\ell$, the number of summands of c_ℓ is $\binom{n}{\ell}$, the number of summands of a_ℓ is $\binom{n-1}{\ell-1}$, and the number of summands of b_ℓ is $\binom{n-1}{\ell}$. As usual, the empty sum is defined to be zero, so $b_n = 0$. Let

$$s_\ell = (-1)^{n-\ell} a_{n-\ell+1} + (-1)^{n-\ell-1} b_{n-\ell}$$

for each $1 \le \ell \le n-1$, and let $s_n = a_1$. Then for $1 \le \ell \le n-1$, we have

$$0 \le s_\ell \le \binom{n-1}{n-\ell} \quad \text{if } n-\ell \text{ is even}$$

and

$$-\binom{n-1}{n-\ell} \le s_\ell \le 0 \quad \text{if } n-\ell \text{ is odd}.$$

In addition, we have

$$\begin{aligned}
\sum_{1 \le \ell \le n} s_\ell &= (-1)^{n-1} a_n + \sum_{2 \le \ell \le n-1} (-1)^{n-\ell} a_{n-\ell+1} \\
&\quad + \sum_{1 \le \ell \le n-2} (-1)^{n-\ell-1} b_{n-\ell} + b_1 + s_n \\
&= (-1)^{n-1} a_n + \sum_{1 \le \ell \le n-2} (-1)^{n-\ell-1} (a_{n-\ell} + b_{n-\ell}) + b_1 + a_1 \\
&= (-1)^{n-1} c_n + \sum_{1 \le \ell \le n-2} (-1)^{n-\ell-1} c_{n-\ell} + c_1 \\
&= \sum_{0 \le \ell \le n-1} (-1)^{n-\ell-1} c_{n-\ell} \\
&= g_3(x_1, x_2, \ldots, x_n).
\end{aligned}$$

Therefore,

$$-\sum_{\substack{1 \le \ell \le n \\ n-\ell \text{ is odd}}} \binom{n-1}{n-\ell} \le g_3(x_1, x_2, \ldots, x_n) \le \sum_{\substack{1 \le \ell \le n \\ n-\ell \text{ is even}}} \binom{n-1}{n-\ell}.$$

Replacing ℓ by $\ell + 1$, we see that

$$\sum_{\substack{1 \le \ell \le n \\ n-\ell \text{ is even}}} \binom{n-1}{n-\ell} = \sum_{\substack{0 \le \ell \le n-1 \\ n-\ell \text{ is odd}}} \binom{n-1}{n-1-\ell} = 2^{n-2}.$$

Similarly, we have

$$-\sum_{\substack{1\le \ell\le n\\ n-\ell \text{ is odd}}}\binom{n-1}{n-\ell}=-2^{n-2}.$$

Hence,

$$-2^{n-2}\le g_3(x_1,x_2,\ldots,x_n)\le 2^{n-2}, \tag{50}$$

as required.

Next we show that the lower and upper bounds in (50) are the minimum and the maximum of $g_3(x_1,x_2,\ldots,x_n)$. Let $x_k=1/2$ for every $k=1,2,\ldots,n$. By Lemmas 1 and 5, we obtain

$$\begin{aligned}g_3(x_1,x_2,\ldots,x_n)&=g_1(x_1,x_2,\ldots,x_n)-g_1(x_1,x_2,\ldots,x_n,1/2)\\&=-2^{n-2}-\left(-2^{n-1}\right)=2^{n-2}.\end{aligned}$$

This shows that 2^{n-2} is the maximum value of g_3. Next let $x_k=\frac{1}{2}-\frac{1}{n^2}$ for every $k=1,2,\ldots,n$. Then

$$g_3(x_1,x_2,\ldots,x_n)=\sum_{1\le k\le n}(-1)^{k-1}\left\lfloor\frac{k+1}{2}-\frac{k}{n^2}\right\rfloor\binom{n}{k}. \tag{51}$$

If $1\le k\le n$ and k is even, then $\left\lfloor\frac{k+1}{2}-\frac{k}{n^2}\right\rfloor=\frac{k}{2}=\left\lfloor\frac{k}{2}\right\rfloor$. If $1\le k\le n$ and k is odd, then $\left\lfloor\frac{k+1}{2}-\frac{k}{n^2}\right\rfloor=\frac{k-1}{2}=\left\lfloor\frac{k}{2}\right\rfloor$. Writing $\left\lfloor\frac{k}{2}\right\rfloor=\frac{k}{2}+\left\{\frac{k}{2}\right\}$, we see that (51) is

$$\begin{aligned}g_3(x_1,x_2,\ldots,x_n)&=\frac{1}{2}\sum_{1\le k\le n}(-1)^{k-1}k\binom{n}{k}-\sum_{1\le k\le n}(-1)^{k-1}\left\{\frac{k}{2}\right\}\binom{n}{k}\\&=\frac{1}{2}\sum_{1\le k\le n}(-1)^{k-1}k\binom{n}{k}-\frac{1}{2}\sum_{\substack{1\le k\le n\\ k\text{ is odd}}}\binom{n}{k},\end{aligned}$$

where the last equality is obtained from the fact that $\left\{\frac{k}{2}\right\}=0$ if k is even and $\left\{\frac{k}{2}\right\}=\frac{1}{2}$ if k is odd. By (28) and (31), we obtain $g_3(x_1,x_2,\ldots,x_n)=-2^{n-2}$. This completes the proof. □

Theorem 3. *For each $n\ge 2$, the function g_4 given in Definition 3 has maximum value 2^{n-2} and minimum value -2^{n-2}. Furthermore, when $x_k=\frac{1}{2}$ for all $k=1,2,\ldots,n$, this minimum occurs, and when $x_k=\frac{1}{2}+\frac{1}{n^2}$ for all $k=1,2,\ldots,n$, the maximum occurs.*

Proof. Since the proof of this theorem is similar to those of Theorems 1 and 2, we give less details. By Lemma 1, we can assume that $0<x_k\le 1$ for every $1\le k\le n$. For $n=2$, we let $x_1=x$, $x_2=y$, and recall that

$$g_4(x,y)=\left\lceil x-\frac{1}{2}\right\rceil+\left\lceil y-\frac{1}{2}\right\rceil-\left\lceil x+y-\frac{1}{2}\right\rceil.$$

By (12) and the inequality $0 < y \leq 1$, we obtain

$$-1 \leq \left\lceil x - \frac{1}{2} \right\rceil - \left\lceil x + y - \frac{1}{2} \right\rceil \leq 0 \tag{52}$$

and

$$0 \leq \left\lceil y - \frac{1}{2} \right\rceil \leq 1. \tag{53}$$

Summing (52) and (53), the middle term is $g_4(x, y)$. Therefore,

$$-1 \leq g_4(x, y) \leq 1. \tag{54}$$

The inequality (54) is sharp: if $x = y = 1/2$, then the left inequality in (54) becomes equality, and if $x = y = 3/4$, then the right inequality in (54) becomes equality.

The result when $n \geq 3$ is based on a careful selection of pairs and the case $n = 2$. The illustration for the case $n = 3$ and $n = 4$ is left to the reader.

Let $n \geq 5$, call the expression of the form $\lceil x_{i_1} + x_{i_2} + \cdots + x_{i_\ell} - \frac{1}{2} \rceil$ an ℓ-*bracket*, and follow closely the method used in the proof of Theorem 2. For each $1 \leq \ell \leq n$, let c_ℓ be the sum of all ℓ-brackets with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, a_ℓ the sum of all such terms with $i_\ell = n$, b_ℓ the sum of all such terms with $i_\ell < n$, and let

$$s_\ell = (-1)^{n-\ell} a_{n-\ell+1} + (-1)^{n-\ell-1} b_{n-\ell}$$

for $1 \leq \ell \leq n - 1$ and $s_n = a_1$. Then

$$0 \leq s_\ell \leq \binom{n-1}{n-\ell} \qquad \text{if } n - \ell \text{ is even,}$$
$$-\binom{n-1}{n-\ell} \leq s_\ell \leq 0 \quad \text{if } n - \ell \text{ is odd,}$$

and

$$\sum_{1 \leq \ell \leq n} s_\ell = g_4(x_1, x_2, \ldots, x_n).$$

Therefore,

$$-\sum_{\substack{1 \leq \ell \leq n \\ n-\ell \text{ is odd}}} \binom{n-1}{n-\ell} \leq g_4(x_1, x_2, \ldots, x_n) \leq \sum_{\substack{1 \leq \ell \leq n \\ n-\ell \text{ is even}}} \binom{n-1}{n-\ell}.$$

This implies

$$-2^{n-2} \leq g_4(x_1, x_2, \ldots, x_n) \leq 2^{n-2},$$

as required. If $x_k = 1/2$ for every $k = 1, 2, \ldots, n$, then we apply Lemma 1 and Theorem 1 to obtain

$$\begin{aligned} g_4(x_1, x_2, \ldots, x_n) &= g_2(x_1, x_2, \ldots, x_n) - g_2(x_1, x_2, \ldots, x_n, -1/2) \\ &= g_2(x_1, x_2, \ldots, x_n) - g_2(x_1, x_2, \ldots, x_n, 1/2) \\ &= 2^{n-2} - 2^{n-1} = -2^{n-2}. \end{aligned}$$

Next, let $x_k = \frac{1}{2} + \frac{1}{n^2}$ for every $k = 1, 2, \ldots, n$. Then

$$g_4(x_1, x_2, \ldots, x_n) = \sum_{1 \le k \le n} (-1)^{k-1} \left\lceil \frac{k-1}{2} + \frac{k}{n^2} \right\rceil \binom{n}{k}. \tag{55}$$

If $1 \le k \le n$ and k is even, then $\left\lceil \frac{k-1}{2} + \frac{k}{n^2} \right\rceil = \frac{k}{2} = \left\lceil \frac{k}{2} \right\rceil$. If $1 \le k \le n$ and k is odd, then $\left\lceil \frac{k-1}{2} + \frac{k}{n^2} \right\rceil = \frac{k+1}{2} = \left\lceil \frac{k}{2} \right\rceil$. Therefore, (55) becomes

$$g_4(x_1, x_2, \ldots, x_n) = \sum_{1 \le k \le n} (-1)^{k-1} \left\lceil \frac{k}{2} \right\rceil \binom{n}{k}.$$

By (29) and (32), we obtain $g_4(x_1, x_2, \ldots, x_n) = 2^{n-2}$. This completes the proof. □

Collecting Theorems 1 to 3 and that by Onphaeng and Pongsriiam [8, Theorem 4], we obtain the following corollary.

Corollary 1. *Let g_1, g_2, g_3, g_4 be the functions defined in Definitions 2 and 3, and $n \ge 2$. Then the following inequalities hold:*

$$-2^{n-2} \le g_1 \le 2^{n-2} - 1,$$
$$-2^{n-2} + 1 \le g_2 \le 2^{n-2},$$

and

$$-2^{n-2} \le g_3, g_4 \le 2^{n-2}.$$

In addition, if $x_k = 1/2$ for every $k \le n$, then

$$g_1 = -2^{n-2} = g_4 \text{ and } g_2 = 2^{n-2} = g_3;$$

if $x_k = \frac{1}{2} - \frac{1}{n^2}$ for every $k \le n$, then

$$g_1 = 2^{n-2} - 1 = -g_3 - 1;$$

if $x_k = \frac{1}{2} + \frac{1}{n^2}$ for every $k \le n$, then

$$g_2 = -2^{n-2} + 1 = -g_4 + 1.$$

Next, we give a sharp inequality for f_2, f_3, and f_4, respectively. The proof for f_2 closely follows those of Theorems 1 to 3 while the proofs for f_3 and f_4 are much shorter.

Theorem 4. *For each $\ell \ge 2$, $a_1, a_2, \ldots, a_\ell, k \in \mathbb{Z}$, and $m \in \mathbb{Z}^+$, we have*

$$-2^{\ell-2} \le f_2(\{a_1, a_2, \ldots, a_\ell\}, m, k) \le 2^{\ell-2}.$$

Moreover, the upper and lower bounds are best possible in the sense that there are $a_1, a_2, \ldots, a_\ell$, m, k for which equality is achieved. More precisely, the following statements hold.

(i) *If* ℓ *is odd,* m *is even, and* $a_i = m/2$ *for every* $i = 1, 2, \ldots, \ell$*, then*

$$f_2(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = 2^{\ell-2} \quad \textit{and} \quad f_2\left(\{a_1, a_2, \ldots, a_\ell\}, m, m/2\right) = -2^{\ell-2}.$$

(ii) *If* ℓ *is even,* m *is even, and* $a_i = m/2$ *for every* $i = 1, 2, \ldots, \ell$*, then*

$$f_2(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = -2^{\ell-2} \quad \textit{and} \quad f_2\left(\{a_1, a_2, \ldots, a_\ell\}, m, m/2\right) = 2^{\ell-2}.$$

Proof. By Lemma 3, we can assume that $0 \le a_i < m$ for every $1 \le i \le \ell$. It is not difficult to see that for $x, y \in \mathbb{R}$, if $0 \le y < 1$, then $0 \le \lceil x + y \rceil - \lceil x \rceil \le 1$. So if $\ell = 2$, then

$$0 \le \left\lceil \frac{a_1 + a_2 + k}{m} \right\rceil - \left\lceil \frac{a_1 + k}{m} \right\rceil \le 1 \tag{56}$$

and

$$-1 \le -\left\lceil \frac{a_2 + k}{m} \right\rceil + \left\lceil \frac{k}{m} \right\rceil \le 0. \tag{57}$$

Summing (56) and (57), we obtain $-1 \le f_2(\{a_1, a_2\}, m, k) \le 1$. The result when $\ell \ge 3$ is based on a careful selection of pairs and the case $\ell = 2$. Let $\ell \ge 3$, call the expression of the form $\left\lceil \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_r} + k}{m} \right\rceil$ an r-*bracket*, and follow closely the method used in the proof of Theorems 1, 2, and 3. For each $1 \le r \le \ell$, let c_r be the sum of all r-brackets with $1 \le i_1 < i_2 < \cdots < i_r \le \ell$, a_r the sum of all such terms with $i_r = \ell$, b_r the sum of all such terms with $i_r < \ell$, and

$$s_r = (-1)^{r+1} a_{\ell-r+1} + (-1)^r b_{\ell-r} \text{ for } r < \ell, \text{ and } s_\ell = (-1)^{\ell+1} a_1 + (-1)^\ell \left\lceil \tfrac{k}{m} \right\rceil.$$

Then, for $1 \le r \le \ell$, we have $c_r = a_r + b_r$, $b_\ell = 0$,

$$0 \le s_r \le \binom{\ell-1}{\ell-r} \quad \text{if } r \text{ is odd}$$

and

$$-\binom{\ell-1}{\ell-r} \le s_r \le 0 \quad \text{if } r \text{ is even}.$$

In addition, we obtain

$$\begin{aligned}\sum_{1\le r\le \ell} s_r &= a_\ell + \sum_{2\le r\le \ell-1}(-1)^{r+1}a_{\ell-r+1} + \sum_{1\le r\le \ell-2}(-1)^r b_{\ell-r} + (-1)^{\ell-1}b_1 + s_\ell \\ &= a_\ell + \sum_{1\le r\le \ell-2}(-1)^r(a_{\ell-r}+b_{\ell-r}) + (-1)^{\ell-1}b_1 + (-1)^{\ell+1}a_1 + (-1)^\ell\left\lceil\frac{k}{m}\right\rceil \\ &= c_\ell + \sum_{1\le r\le \ell-2}(-1)^r c_{\ell-r} + (-1)^{\ell-1}c_1 + (-1)^\ell\left\lceil\frac{k}{m}\right\rceil \\ &= \sum_{0\le r\le \ell-1}(-1)^r c_{\ell-r} + (-1)^\ell\left\lceil\frac{k}{m}\right\rceil \\ &= f_2(\{a_1,a_2,\ldots,a_\ell\},m,k).\end{aligned}$$

Therefore,

$$-\sum_{\substack{1\le r\le \ell\\ r\text{ is even}}}\binom{\ell-1}{\ell-r} \le f_2(\{a_1,a_2,\ldots,a_\ell\},m,k) \le \sum_{\substack{1\le r\le \ell\\ r\text{ is odd}}}\binom{\ell-1}{\ell-r}.$$

This implies

$$-2^{\ell-2} \le f_2(\{a_1,a_2,\ldots,a_\ell\},m,k) \le 2^{\ell-2}, \tag{58}$$

as required. If ℓ is odd, m is even, and $a_i = m/2$ for every $1\le i\le \ell$, we obtain by Lemma 2 and Theorem 1 that

$$f_2(\{a_1,a_2,\ldots,a_\ell\},m,0) = g_2\left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right) = 2^{\ell-2}$$

and

$$\begin{aligned}f_2(\{a_1,a_2,\ldots,a_\ell\},m,m/2) &= (-1)^\ell g_2\left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right) + (-1)^{\ell-1}g_2\left(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right) \\ &= -2^{\ell-2}.\end{aligned}$$

If ℓ is even, m is even, and $a_i = m/2$ for every $1\le i\le \ell$, we obtain similarly that

$$f_2(\{a_1,a_2,\ldots,a_\ell\},m,0) = -2^{\ell-2} \quad\text{and}\quad f_2(\{a_1,a_2,\ldots,a_\ell\},m,m/2) = 2^{\ell-2}.$$

So $2^{\ell-2}$ and $-2^{\ell-2}$ in (58) cannot be improved. This completes the proof. □

Theorem 5. *For each $\ell\ge 2$, $a_1,a_2,\ldots,a_\ell,k\in\mathbb{Z}$, and $m\in\mathbb{Z}^+$, we have*

$$-2^{\ell-2} \le f_3(\{a_1,a_2,\ldots,a_\ell\},m,k) \le 2^{\ell-2}.$$

Moreover, the lower and upper bounds are best possible in the sense that there are $a_1,a_2,\ldots,a_\ell$, m, k for which equality is achieved. More precisely, the following statements hold.

(i) *If ℓ is odd, m is even, and $a_i = m/2$ for every $i = 1, 2, \ldots, \ell$, then*

$$f_3(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = 2^{\ell-2} \quad \textit{and} \quad f_3(\{a_1, a_2, \ldots, a_\ell\}, m, m/2) = -2^{\ell-2}.$$

(ii) *If ℓ is even, m is even, and $a_i = m/2$ for every $i = 1, 2, \ldots, \ell$, then*

$$f_3(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = -2^{\ell-2} \quad \textit{and} \quad f_3(\{a_1, a_2, \ldots, a_\ell\}, m, m/2) = 2^{\ell-2}.$$

Proof. By the definition of f_1 and f_3, we have

$$\begin{aligned} f_3(\{a_1, a_2, \ldots, a_\ell\}, m, k) &= \sum_{T \subseteq [1,\ell]} (-1)^{\ell-|T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} + \frac{1}{2} \right\rfloor \\ &= \sum_{T \subseteq [1,\ell]} (-1)^{\ell-|T|} \left\lfloor \frac{m + 2k + \sum_{i \in T} 2a_i}{2m} \right\rfloor \\ &= f_1(\{2a_1, 2a_2, \ldots, 2a_\ell\}, 2m, 2k + m). \end{aligned}$$

From this and Lemma 4, we obtain the desired lower and upper bounds for f_3. For (i), we obtain by Lemma 2 and Theorem 2 that

$$f_3(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = 2^{\ell-2} \text{ and } f_3(\{a_1, a_2, \ldots, a_\ell\}, m, m/2) = -2^{\ell-2}.$$

For (ii), we obtain similarly that

$$f_3(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = -2^{\ell-2} \text{ and } f_3(\{a_1, a_2, \ldots, a_\ell\}, m, m/2) = 2^{\ell-2}.$$

So $2^{\ell-2}$ and $-2^{\ell-2}$ are best possible. This completes the proof. □

Theorem 6. *For each $\ell \geq 2$, $a_1, a_2, \ldots, a_\ell, k \in \mathbb{Z}$, and $m \in \mathbb{Z}^+$, we have*

$$-2^{\ell-2} \leq f_4(\{a_1, a_2, \ldots, a_\ell\}, m, k) \leq 2^{\ell-2}.$$

Moreover, the bounds are best possible in the sense that there are $a_1, a_2, \ldots, a_\ell$, m, k for which equality is achieved. More precisely, the following statements hold.

(i) *If ℓ is odd, m is even, and $a_i = m/2$ for every $i = 1, 2, \ldots, \ell$, then*

$$f_4(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = -2^{\ell-2} \quad \textit{and} \quad f_4(\{a_1, a_2, \ldots, a_\ell\}, m, m/2) = 2^{\ell-2}.$$

(ii) *If ℓ is even, m is even, and $a_i = m/2$ for every $i = 1, 2, \ldots, \ell$, then*

$$f_4(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = 2^{\ell-2} \quad \textit{and} \quad f_4(\{a_1, a_2, \ldots, a_\ell\}, m, m/2) = -2^{\ell-2}.$$

Proof. By Lemma 3 and the definition of f_2 and f_4, we have

$$\begin{aligned} f_4(\{a_1, a_2, \ldots, a_\ell\}, m, k) &= f_4(\{a_1, a_2, \ldots, a_\ell\}, m, k+m) \\ &= \sum_{T \subseteq [1,\ell]} (-1)^{\ell-|T|} \left\lceil \frac{k+m+\sum_{i\in T} a_i}{m} - \frac{1}{2} \right\rceil \\ &= \sum_{T \subseteq [1,\ell]} (-1)^{\ell-|T|} \left\lceil \frac{2k+m+\sum_{i\in T} 2a_i}{2m} \right\rceil \\ &= f_2(\{2a_1, 2a_2, \ldots, 2a_\ell\}, 2m, 2k+m). \end{aligned}$$

By Theorem 4, we obtain the desired lower and upper bounds for f_4.

For (i), we obtain by Lemma 2 and Theorem 3 that

$$f_4(\{a_1, a_2, \ldots, a_\ell\}, m, 0) = -2^{\ell-2} \quad \text{and} \quad f_4(\{a_1, a_2, \ldots, a_\ell\}, m, m/2) = 2^{\ell-2}.$$

For (ii), we obtain similarly that $f_4(\{a_1, a_2, \ldots, a_\ell\}, m, 0)$ is equal to $2^{\ell-2}$ and $f_4(\{a_1, a_2, \ldots, a_\ell\}, m, m/2)$ is $-2^{\ell-2}$. So $2^{\ell-2}$ and $-2^{\ell-2}$ are best possible. This completes the proof. $\square$

Recall that $T_2(C, m, K) = \sum_{k=1}^{K} f_2(C, m, k)$. In the next two theorems, we give sharp bounds for T_ℓ for $\ell = 2$, and then for $\ell \geq 3$, respectively.

Theorem 7. *For each $a, b \in \mathbb{Z}$, and $m, K \in \mathbb{N}$, we have*

$$0 \leq T_2(\{a, b\}, m, K) \leq \lfloor m/2 \rfloor .$$

The inequality is sharp in the sense that there are a, b, m, K for which equality is achieved. More precisely, if m is even and $a = b = m/2$, then

$$T_2(\{a, b\}, m, m) = 0 \quad \textit{and} \quad T_2(\{a, b\}, m, m/2) = \lfloor m/2 \rfloor .$$

Proof. Since the permutation of a, b does not change the value of T_2, we can assume that $a \leq b$. In addition, by Lemma 7, we can assume that $0 \leq a, b < m$ and $1 \leq K \leq m$. Let

$$A = T_2(\{a, b\}, m, K), \qquad X_1 = \sum_{k=1}^{K} \left\lceil \frac{a+b+k}{m} \right\rceil, \qquad X_2 = \sum_{k=1}^{K} \left\lceil \frac{a+k}{m} \right\rceil,$$

$$X_3 = \sum_{k=1}^{K} \left\lceil \frac{b+k}{m} \right\rceil, \qquad X_4 = \sum_{k=1}^{K} \left\lceil \frac{k}{m} \right\rceil.$$

Then $A = X_1 - X_2 - X_3 + X_4$. Since $K \leq m$, we have $\lceil k/m \rceil = 1$ for $k = 1, 2, \ldots, K$, and so $X_4 = K$. If $a + K \leq m$, then $\lceil \frac{a+k}{m} \rceil = 1$ for $k = 1, 2, \ldots, K$, and so

$$X_2 = \sum_{k=1}^{K} \left\lceil \frac{a+k}{m} \right\rceil = K.$$

Similarly, if $b + K \leq m$, then $X_3 = K$. For convenience, we record this as follows.

$$\text{if } a + K \leq m, \text{ then } X_2 = K; \text{ if } b + K \leq m, \text{ then } X_3 = K. \tag{59}$$

Since $a < m$, we see that $K \leq m < 2m - a$. Therefore, if $a + K > m$, then

$$\left\lceil \frac{a+k}{m} \right\rceil = \begin{cases} 1, & \text{for } 1 \leq k \leq m - a; \\ 2, & \text{for } m - a < k \leq K, \end{cases}$$

and thus

$$X_2 = \sum_{k=1}^{K} \left\lceil \frac{a+k}{m} \right\rceil = (m - a) + 2(K + a - m) = 2K + a - m. \tag{60}$$

Similarly,

$$\text{if } b + K > m, \text{ then } X_3 = 2K + b - m. \tag{61}$$

To obtain the value of X_1, we separate the calculation into several cases.

`Case 1:` $a + b < m$. Since $a \leq b$, we have $a < m/2$. Therefore,

$$a \leq \lfloor m/2 \rfloor . \tag{62}$$

If $a+b+K \leq m$, then similar to (59), we obtain $X_1 = X_2 = X_3 = K$, which implies $A = 0$ and we are done. So we assume that $a + b + K > m$. Since $a + b < m$ and $K \leq m$, we see that $K \leq 2m - a - b$. Then

$$\left\lceil \frac{a+b+k}{m} \right\rceil = \begin{cases} 1, & \text{for } 1 \leq k \leq m - a - b; \\ 2, & \text{for } m - a - b < k \leq K. \end{cases}$$

Therefore, we obtain

$$X_1 = (m - a - b) + 2(K - m + a + b) = 2K - m + a + b. \tag{63}$$

If $b+K \leq m$, then (59), (62), and (63) imply that $X_2 = X_3 = K$, $A = a+b+K-m$, and $0 \leq A \leq a \leq \lfloor m/2 \rfloor$. If $a+K \leq m < b+K$, then (59), (61), (62), and (63) imply that $X_2 = K$, $X_3 = 2K+b-m$, $A = a$, and so $0 \leq A \leq \lfloor m/2 \rfloor$. If $m < a+K$, then (60), (61), (62), and (63) imply that $A = m-K$, and therefore $0 \leq A < a \leq \lfloor m/2 \rfloor$. In any case, we obtain $0 \leq A \leq \lfloor m/2 \rfloor$.

`Case 2:` $a + b \geq m$. Since $a \leq b$, we have $b \geq \lceil m/2 \rceil$, and therefore

$$m - b \leq \lfloor m/2 \rfloor . \tag{64}$$

We separate the calculation into two subcases, namely, $a + b + K \leq 2m$ and $a + b + K > 2m$.

`Case 2.1:` $a+b+K \le 2m$. Then $\lceil \frac{a+b+k}{m} \rceil = 2$ for $k = 1, 2, \ldots, K$, and so

$$X_1 = 2K. \tag{65}$$

If $b+K \le m$, then we obtain by (59), (64), and (65) that $A = K \le m-b \le \lfloor m/2 \rfloor$. If $a+K \le m < b+K$, then by (59), (61), (64), and (65), we obtain $A = m-b \le \lfloor m/2 \rfloor$. If $m < a+K$, then (60), (61), (64), and (65) give

$$A = m - (a+K) + m - b < m - b \le \lfloor m/2 \rfloor$$

and $A = 2m - (a+b+K) \ge 0$ by the assumption of this case.

`Case 2.2:` $a+b+K > 2m$. Then we obtain

$$\left\lceil \frac{a+b+k}{m} \right\rceil = \begin{cases} 2, & \text{for } 1 \le k \le 2m-a-b; \\ 3, & \text{for } 2m-a-b < k \le K. \end{cases}$$

Then

$$X_1 = 2(2m-a-b) + 3(a+b+K-2m) = 3K - 2m + a + b. \tag{66}$$

Since $m < a+K \le b+K$, we obtain from (60), (61), and (66) that $A = 0$.

Thus, in any case, we have

$$0 \le T_2(\{a,b\}, m, K) \le \lfloor m/2 \rfloor.$$

If $m \ge 4$ and $a = b = K = 1$, then $T_2(\{a,b\}, m, K) = 0$. If m is even and $a = b = K = m/2$, we obtain

$$T_2(\{a,b\}, m, K) = \sum_{k=1}^{m/2} \left\lceil 1 + \frac{k}{m} \right\rceil - 2 \sum_{k=1}^{m/2} \left\lceil \frac{1}{2} + \frac{k}{m} \right\rceil + \sum_{k=1}^{m/2} \left\lceil \frac{k}{m} \right\rceil.$$

Since $\lceil 1 + \frac{k}{m} \rceil = 2$, $\lceil \frac{1}{2} + \frac{k}{m} \rceil = 1$, and $\lceil \frac{k}{m} \rceil = 1$ for $k = 1, 2, \ldots, m/2$, we have

$$T_2(\{a,b\}, m, K) = 2\,(m/2) - 2\,(m/2) + (m/2) = \lfloor m/2 \rfloor.$$

This completes the proof. $\square$

Theorem 8. *For each $\ell \ge 2$, $a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$, and $m, K \in \mathbb{N}$, we have*

$$-2^{\ell-2} \lfloor m/2 \rfloor \le T_2(\{a_1, a_2, \ldots, a_\ell\}, m, K) \le 2^{\ell-2} \lfloor m/2 \rfloor. \tag{67}$$

The inequality is sharp in the sense that there are $a_1, a_2, \ldots, a_\ell$, m, k for which equality is achieved. More precisely, the following statements hold.

(i) *If ℓ is odd, m is even, and $a_i = m/2$ for every $i = 1, 2, \ldots, \ell$, then*

$$T_2(\{a_1, a_2, \ldots, a_\ell\}, m, K) = -2^{\ell-2} \lfloor m/2 \rfloor.$$

(ii) *If ℓ is even, m is even, and $a_i = m/2$ for every $i = 1, 2, \ldots, \ell$, then*

$$T_2(\{a_1, a_2, \ldots, a_\ell\}, m, K) = 2^{\ell-2} \lfloor m/2 \rfloor .$$

Proof. If $\ell = 2$, then the result follows immediately from Theorem 7. For easy reference, we state it again here as follows:

$$0 \le T_2(\{a, b\}, m, K) \le \lfloor m/2 \rfloor . \tag{68}$$

The result when $\ell \ge 3$ is based on the case $\ell = 2$ and a careful selection of pairs, and we first illustrate the idea by giving the proof for the case $\ell = 3$. Recall that

$$\begin{aligned} f_2(\{a_1, a_2, a_3\}, m, k) = {} & \left\lceil \frac{a_1 + a_2 + a_3 + k}{m} \right\rceil - \left\lceil \frac{a_1 + a_2 + k}{m} \right\rceil - \left\lceil \frac{a_1 + a_3 + k}{m} \right\rceil \\ & - \left\lceil \frac{a_2 + a_3 + k}{m} \right\rceil + \left\lceil \frac{a_1 + k}{m} \right\rceil + \left\lceil \frac{a_2 + k}{m} \right\rceil \\ & + \left\lceil \frac{a_3 + k}{m} \right\rceil - \left\lceil \frac{k}{m} \right\rceil . \end{aligned}$$

We have

$$\begin{aligned} f_2(\{a_1 + a_2, a_3\}, m, k) = {} & \left\lceil \frac{a_1 + a_2 + a_3 + k}{m} \right\rceil - \left\lceil \frac{a_1 + a_2 + k}{m} \right\rceil \\ & - \left\lceil \frac{a_3 + k}{m} \right\rceil + \left\lceil \frac{k}{m} \right\rceil , \end{aligned} \tag{69}$$

$$-f_2(\{a_1, a_3\}, m, k) = -\left\lceil \frac{a_1 + a_3 + k}{m} \right\rceil + \left\lceil \frac{a_1 + k}{m} \right\rceil + \left\lceil \frac{a_3 + k}{m} \right\rceil - \left\lceil \frac{k}{m} \right\rceil , \tag{70}$$

$$-f_2(\{a_2, a_3\}, m, k) = -\left\lceil \frac{a_2 + a_3 + k}{m} \right\rceil + \left\lceil \frac{a_2 + k}{m} \right\rceil + \left\lceil \frac{a_3 + k}{m} \right\rceil - \left\lceil \frac{k}{m} \right\rceil . \tag{71}$$

Summing (69), (70), and (71), we see that

$$\begin{aligned} f_2(\{a_1, a_2, a_3\}, m, k) = {} & f_2(\{a_1 + a_2, a_3\}, m, k) - f_2(\{a_1, a_3\}, m, k) \\ & - f_2(\{a_2, a_3\}, m, k). \end{aligned} \tag{72}$$

Summing (72) over $k = 1, 2, \ldots, K$ and applying (68), we obtain that

$$\begin{aligned} T_2(\{a_1, a_2, a_3\}, m, K) &= T_2(\{a_1 + a_2, a_3\}, m, K) - T_2(\{a_1, a_3\}, m, K) \\ &\quad - T_2(\{a_2, a_3\}, m, K) \\ &\ge -2 \lfloor m/2 \rfloor . \end{aligned}$$

Similarly, we have

$$T_2(\{a_1, a_2, a_3\}, m, K) \le \lfloor m/2 \rfloor \le 2 \lfloor m/2 \rfloor .$$

In general, we let $\ell \geq 4$, call the expression of the form $\left\lceil \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rceil$ an r-*bracket*, and follow closely the method used in the proof of Theorems 1, 2, 3, and 4. The well known identities previously recalled will be applied without reference. For each $1 \leq r \leq \ell$, let $c_r(k)$ be the sum of all r-*brackets* with $1 \leq i_1 < i_2 < \cdots < i_r \leq \ell$, $a_r(k)$ the sum of all such terms with $i_r = \ell$, and $b_r(k)$ the sum of all such terms with $i_r < \ell$, and for $1 \leq r \leq \ell - 1$, let

$$s_r(k) = (-1)^{r+1} a_{\ell-r+1}(k) + (-1)^r b_{\ell-r}(k) + (-1)^r \binom{\ell-1}{\ell-r} a_1(k) + (-1)^{r+1} \binom{\ell-1}{\ell-r} \left\lceil \frac{k}{m} \right\rceil .$$

For convenience, we write c_r, a_r, b_r, s_r instead of $c_r(k)$, $a_r(k)$, $b_r(k)$, $s_r(k)$, respectively. Then $b_\ell = 0$ and $c_\ell = a_\ell$. In addition, for $1 \leq r \leq \ell - 1$, we have $c_r = a_r + b_r$,

$$s_r = (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_{\ell-r} < \ell} f_2(\{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-r}}, a_\ell\}, m, k),$$

and

$$\sum_{k=1}^{K} s_r = (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_{\ell-r} < \ell} T_2(\{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-r}}, a_\ell\}, m, K).$$

So by (68), we see that

$$0 \leq \sum_{k=1}^{K} s_r \leq \binom{\ell-1}{\ell-r} \lfloor m/2 \rfloor \quad \text{ if } r \text{ is odd}$$

and

$$-\binom{\ell-1}{\ell-r} \lfloor m/2 \rfloor \leq \sum_{k=1}^{K} s_r \leq 0 \quad \text{ if } r \text{ is even.}$$

In addition, we obtain

$$\begin{aligned}\sum_{1\le r\le \ell-1} s_r &= a_\ell + \sum_{2\le r\le \ell-1}(-1)^{r+1}a_{\ell-r+1} + \sum_{1\le r\le \ell-2}(-1)^r b_{\ell-r} + (-1)^{\ell-1}b_1 \\ &\qquad + (-1)^{\ell+1}a_1 + (-1)^\ell \left\lceil \frac{k}{m}\right\rceil \\ &= a_\ell + \sum_{1\le r\le \ell-2}(-1)^r(a_{\ell-r}+b_{\ell-r}) + (-1)^{\ell-1}b_1 + (-1)^{\ell+1}a_1 + (-1)^\ell\left\lceil \frac{k}{m}\right\rceil \\ &= c_\ell + \sum_{1\le r\le \ell-2}(-1)^r c_{\ell-r} + (-1)^{\ell-1}c_1 + (-1)^\ell \left\lceil \frac{k}{m}\right\rceil \\ &= \sum_{0\le r\le \ell-1}(-1)^r c_{\ell-r} + (-1)^\ell\left\lceil \frac{k}{m}\right\rceil \\ &= f_2(\{a_1,a_2,\dots,a_\ell\},m,k).\end{aligned}$$

Therefore,

$$\begin{aligned}-\sum_{\substack{1\le r\le \ell-1\\ r \text{ is even}}}\binom{\ell-1}{\ell-r}\lfloor m/2\rfloor &\le \sum_{k=1}^{K} f_2(\{a_1,a_2,\dots,a_\ell\},m,k) \\ &\le \sum_{\substack{1\le r\le \ell-1\\ r \text{ is odd}}}\binom{\ell-1}{\ell-r}\lfloor m/2\rfloor .\end{aligned}$$

This implies

$$-2^{\ell-2}\lfloor m/2\rfloor \le T_2(\{a_1,a_2,\dots,a_\ell\},m,K) \le 2^{\ell-2}\lfloor m/2\rfloor ,$$

as required.

Next, we prove (i) and (ii). Let $C=\{a_1,a_2,\dots,a_\ell\}$. If ℓ is odd, m is even, and $a_i=m/2$ for every $1\le i\le \ell$, we obtain by Lemma 2 and Theorem 1 that

$$f_2(C,m,m/2) = (-1)^\ell g_2\left(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right) + (-1)^{\ell-1}g_2\left(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right) = -2^{\ell-2}.$$

Let $1\le k\le m/2$. By the definition of $f_2(C,m,k)$, we see that

$$f_2(C,m,k) = \sum_{T\subseteq[1,\ell]}(-1)^{\ell-|T|}\left\lceil \frac{k}{m}+\frac{|T|}{2}\right\rceil = \sum_{r=0}^{\ell}(-1)^{\ell-r}\binom{\ell}{r}\left\lceil \frac{k}{m}+\frac{r}{2}\right\rceil .$$

Since $1\le k\le m/2$, we have $\frac{r}{2}<\frac{k}{m}+\frac{r}{2}\le\frac{r+1}{2}$. So if r is even, then

$$\left\lceil \frac{k}{m}+\frac{r}{2}\right\rceil = \frac{r}{2}+1 = \left\lceil \frac{r+1}{2}\right\rceil$$

and if r is odd, then

$$\left\lceil \frac{k}{m} + \frac{r}{2} \right\rceil = \frac{r+1}{2} = \left\lceil \frac{r+1}{2} \right\rceil.$$

In any case, we obtain

$$\left\lceil \frac{k}{m} + \frac{r}{2} \right\rceil = \left\lceil \frac{r+1}{2} \right\rceil = \left\lceil \frac{m/2}{m} + \frac{r}{2} \right\rceil.$$

This implies that $f_2(C, m, k) = f_2(C, m, m/2)$ for every $k = 1, 2, \ldots, m/2$. Then

$$T_2\left(C, m, m/2\right) = \sum_{k=1}^{m/2} f_2(C, m, k) = (m/2) f_2(C, m, m/2) = -2^{\ell-2} \left\lfloor m/2 \right\rfloor.$$

So $-2^{\ell-2} \lfloor m/2 \rfloor$ in (67) cannot be improved when ℓ is odd. If ℓ is even, m is even, and $a_i = m/2$ for every $1 \leq i \leq \ell$, we obtain similarly that

$$f_2(C, m, k) = f_2(C, m, m/2) = 2^{\ell-2}$$

for every $k = 1, 2, \ldots, m/2$. Then $T_2\left(C, m, m/2\right) = 2^{\ell-2} \lfloor m/2 \rfloor$. So $2^{\ell-2} \lfloor m/2 \rfloor$ in (67) cannot be improved when ℓ is even. This completes the proof. □

Next, we give upper and lower bounds for $S_3(\{a, b\}, m, K)$ and $T_4(\{a, b\}, m, K)$.

Theorem 9. *For each $a, b \in \mathbb{Z}$, $m \in \mathbb{N}$, and $K \in \mathbb{N} \cup \{0\}$, we have*

$$-\left\lfloor m/2 \right\rfloor \leq S_3(\{a, b\}, m, K) \leq \left\lfloor m/2 \right\rfloor.$$

Moreover, the lower bound $-\lfloor m/2 \rfloor$ is best possible in the sense that there are a, b, m, K such that $S_3(\{a, b\}, m, K) = -\lfloor m/2 \rfloor$.

Proof. If $m = 1$, then $S_3(\{a, b\}, m, K) = 0$ and the result follows immediately. So assume that $m \geq 2$. Recall that

$$f_3(\{a, b\}, m, k) = \left\lfloor \frac{a+b+k}{m} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a+k}{m} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b+k}{m} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{k}{m} + \frac{1}{2} \right\rfloor.$$

By Lemma 8, we obtain that $f_3(\{a, b\}, m, k)$ is equal to

$$\begin{aligned} &\left\lfloor \frac{a+b+k+\lfloor m/2 \rfloor}{m} \right\rfloor - \left\lfloor \frac{a+k+\lfloor m/2 \rfloor}{m} \right\rfloor - \left\lfloor \frac{b+k+\lfloor m/2 \rfloor}{m} \right\rfloor + \left\lfloor \frac{k+\lfloor m/2 \rfloor}{m} \right\rfloor \\ &\qquad = f_1\left(\{a, b\}, m, k + \lfloor m/2 \rfloor\right). \end{aligned}$$

In addition, we have

$$\begin{aligned}
S_3(\{a,b\},m,K) &= \sum_{0\le k\le K} f_3(\{a,b\},m,k) \\
&= \sum_{0\le k\le K} f_1\left(\{a,b\},m,k+\lfloor m/2\rfloor\right) \\
&= \sum_{\lfloor m/2\rfloor\le k\le K+\lfloor m/2\rfloor} f_1\left(\{a,b\},m,k\right) \\
&= \sum_{0\le k\le K+\lfloor m/2\rfloor} f_1\left(\{a,b\},m,k\right) - \sum_{0\le k<\lfloor m/2\rfloor} f_1\left(\{a,b\},m,k\right) \\
&= S_1\left(\{a,b\},m,K+\lfloor m/2\rfloor\right) - S_1\left(\{a,b\},m,\lfloor m/2\rfloor-1\right). \qquad (73)
\end{aligned}$$

By (2), we obtain

$$-\lfloor m/2\rfloor \le S_3(\{a,b\},m,K) \le \lfloor m/2\rfloor. \qquad (74)$$

Next, we show that the lower bound in (74) cannot be improved. Let m be even, $K=(m/2)-1$, and $a=b=m/2$. Then, for $0\le k\le K$, we have

$$f_1\left(\{a,b\},m,k\right) = 1-2\left\lfloor \frac{1}{2}+\frac{k}{m}\right\rfloor + \left\lfloor \frac{k}{m}\right\rfloor = 1.$$

By Lemma 7 and (73), we have

$$\begin{aligned}
S_3\left(\{a,b\},m,K\right) &= S_1\left(\{a,b\},m,m-1\right) - S_1\left(\{a,b\},m,(m/2)-1\right) \\
&= 0-(m/2) = -\lfloor m/2\rfloor.
\end{aligned}$$

This completes the proof. □

Theorem 10. *For each $a,b\in\mathbb{Z}$ and $m,K\in\mathbb{N}$, we have*

$$-\lfloor m/2\rfloor \le T_4(\{a,b\},m,K) \le \lfloor m/2\rfloor.$$

The lower bound $-\lfloor m/2\rfloor$ is best possible in the sense that there are a, b, m, K such that $T_4(\{a,b\},m,K) = -\lfloor m/2\rfloor$.

Proof. Recall that

$$f_4(\{a,b\},m,k) = \left\lceil \frac{a+b+k}{m}-\frac{1}{2}\right\rceil - \left\lceil \frac{a+k}{m}-\frac{1}{2}\right\rceil - \left\lceil \frac{b+k}{m}-\frac{1}{2}\right\rceil + \left\lceil \frac{k}{m}-\frac{1}{2}\right\rceil.$$

By Lemmas 3 and 8, we obtain $f_4(\{a,b\},m,k) = f_4(\{a,b\},m,k+m)$, which is equal to

$$\begin{aligned}
&\left\lceil \frac{a+b+k+\lceil m/2\rceil}{m}\right\rceil - \left\lceil \frac{a+k+\lceil m/2\rceil}{m}\right\rceil - \left\lceil \frac{b+k+\lceil m/2\rceil}{m}\right\rceil + \left\lceil \frac{k+\lceil m/2\rceil}{m}\right\rceil \\
&\qquad = f_2\left(\{a,b\},m,k+\lceil m/2\rceil\right).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
T_4(\{a,b\},m,K) &= \sum_{1\le k\le K} f_4(\{a,b\},m,k) \\
&= \sum_{1\le k\le K} f_2\left(\{a,b\},m,k+\lceil m/2\rceil\right) \\
&= \sum_{1+\lceil m/2\rceil\le k\le K+\lceil m/2\rceil} f_2\left(\{a,b\},m,k\right) \\
&= \sum_{1\le k\le K+\lceil m/2\rceil} f_2\left(\{a,b\},m,k\right) - \sum_{1\le k\le \lceil m/2\rceil} f_2\left(\{a,b\},m,k\right) \\
&= T_2\left(\{a,b\},m,K+\lceil m/2\rceil\right) - T_2\left(\{a,b\},m,\lceil m/2\rceil\right). \qquad (75)
\end{aligned}$$

By Theorem 7, we obtain

$$-\lfloor m/2\rfloor \le T_4(\{a,b\},m,K) \le \lfloor m/2\rfloor. \qquad (76)$$

Next, we show that the lower bound in (76) cannot be improved. Let m be even and $K=a=b=m/2$. Then, for $1\le k\le K$, we have

$$f_2\left(\{a,b\},m,k\right) = 2-2\left\lceil \frac{1}{2}+\frac{k}{m}\right\rceil + \left\lceil \frac{k}{m}\right\rceil = 1.$$

By Lemma 7, Theorem 7, and (75), we have

$$T_4\left(\{a,b\},m,K\right) = T_2\left(\{a,b\},m,m\right) - T_2\left(\{a,b\},m,K\right) = -\lfloor m/2\rfloor.$$

This completes the proof. □

Unlike f_j and g_j for $1\le j\le 4$, we currently do not have a complete result for S_j and T_j. We hope we will obtain a better result for S_j and T_j in the future, and we encourage readers to try to find better results or expand the scope of the topic.

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