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RANEY TRANSDUCERS AND THE LOWEST POINT OF THE n-LAGRANGE SPECTRUM

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Abstract

It is well known that the golden ratio ϕ is the "most irrational" number in the sense that its best rational approximations s/t have error asymptotically $1/(\sqrt{5}t^2)$ and this constant $\sqrt{5}$ is as low as possible. Given a prime p, how can we characterize the reals x such that x and px are both "very irrational"? This is tantamount to finding the lowest point of the *n-Lagrange spectrum* $\mathcal{L}_n$, as previously defined by the third author, for primes $n = p$. We describe an algorithm using Raney transducers that computes $\min \mathcal{L}_p$ if it terminates, which we conjecture it always does. We verify that $\min \mathcal{L}_p$ is the square root of a rational number for primes $p < 2000$. Mysteriously, the highest values of $\min \mathcal{L}_p$ occur for the Heegner primes 67, 3, and 163, listed here in descending order of $\min \mathcal{L}_p$. In addition, for all p, the continued fractions of the corresponding very irrational numbers x and px are in one of three symmetric relations.

1. Introduction

We recall the classical notions of the Lagrange and Markoff spectra. If ξ is an irrational real number, we define its *Lagrange approximability*

$$\lambda(\xi) = \limsup_{\frac{s}{t}\in\mathbb{Q},\, \frac{s}{t}\to\xi} \frac{1}{t^2\left|\frac{s}{t}-\xi\right|} \quad \in \quad \mathbb{R}\cup\infty, \tag{1}$$

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and the *Lagrange spectrum* $\mathcal{L} \subseteq \mathbb{R}$ to be the set of real values attained by $\lambda(\cdot)$. One thinks of $\lambda(\xi)$ as the ease of approximation of ξ by rationals. A classical result usually called Hurwitz's theorem (though it can justly be attributed to Markoff) states that $\lambda(\xi) \geq \sqrt{5}$, with equality when $\xi = \phi$ is the golden ratio, or one of its images under $\mathrm{SL}_2\mathbb{Z}$ (which are hence the "most irrational" numbers). Similarly, if $\xi \neq \xi'$ are two irrationals, then we define their *Markoff approximability*

$$\mu(\xi, \xi') = \sup_{(s,t)\in\mathbb{Z}^2\setminus\{0\}} \frac{|\xi - \xi'|}{|s - t\xi||s - t\xi'|} = \sup_{(s,t)\in\mathbb{Z}^2\setminus\{0\}} \frac{\sqrt{\operatorname{disc} f}}{|f(s,t)|} \quad \in \quad \mathbb{R} \cup \infty \tag{2}$$

where $f(x, y) = a(x - y\xi)(x - y\xi')$ is the quadratic form with roots ξ and ξ' (the scaling a is arbitrary; it is often desirable for f to have integer coefficients). The *Markoff spectrum* $\mathcal{M}$ is then the set of real values of $\mu(\cdot, \cdot)$. This spectrum was first considered by Markoff [15] in the guise of infima of quadratic forms. The Lagrange and Markoff spectra each consist of an initial discrete segment below 3, a mysterious fractal middle region which remains the topic of current research [14, 16], and *Hall's ray* $[F, \infty)$ where $F = 4.5278\ldots$ was computed exactly by Freĭman [8], who also showed that $\mathcal{L} \subsetneq \mathcal{M}$ [7]. See [6] for a comprehensive account of results up to 1989, while new connections and generalizations continue to be unearthed [1, 9].

In this paper, we fix a positive integer n and introduce "n-analogues" of the Lagrange and Markoff spectra by inserting a factor $\gcd(t, n)$ in the numerators of the definitions of approximability in Equations (1) and (2), thus:

$$\lambda_n(\xi) = \limsup_{\frac{s}{t}\in\mathbb{Q},\, \frac{s}{t}\to\xi} \frac{\gcd(t, n)}{t^2\left|\frac{s}{t} - \xi\right|}, \qquad \mathcal{L}_n = \{\lambda_n(\xi) \in \mathbb{R} : \xi \in \mathbb{R}\} \tag{3}$$

$$\mu_n(\xi, \xi') = \sup_{(s,t)\in\mathbb{Z}^2\setminus\{0\}} \frac{\gcd(t, n) \cdot |\xi - \xi'|}{|s - t\xi||s - t\xi'|}, \quad \mathcal{M}_n = \{\lambda_n(\xi, \xi') \in \mathbb{R} : \xi \neq \xi' \in \mathbb{R}\}. \tag{4}$$

These notations were introduced by the third author in [18] and shown to govern the intrinsic approximation of points on conics, generalizing previous work on the unit circle, which gives $\mathcal{L}_2$ [4, 11, 12], and on the conic $x^2 + xy + y^2 = 1$, which gives $\mathcal{L}_3$ [3]. The spectra $\mathcal{M}_n$ appear in the work of Schmidt [23, p. 15] by generalizing the work of Markoff on infima of binary quadratic forms.

Classically, the approximabilities $\lambda(\xi)$ and $\mu(\xi, \xi')$ are invariant under the action of $\mathrm{SL}_2\mathbb{Z}$ (indeed $\mathrm{GL}_2\mathbb{Z}$) by linear fractional transformations on the real projective line. Likewise, it is easy to see that $\lambda_n(\xi)$ and $\mu_n(\xi, \xi')$ are invariant under transformations by the congruence subgroup

$$\Gamma^0(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2\mathbb{Z} : n \mid b \right\}$$

(an $\mathrm{SL}_2\mathbb{Z}$-conjugate of the more familiar $\Gamma_0(n)$, where the divisibility condition is imposed on c instead). More generally, Vulakh [27] defines a notion of Markoff spectrum for any Fuchsian subgroup of $\mathrm{SL}_2\mathbb{R}$.

It is natural to ask if various facts about $\mathcal{L}$ and $\mathcal{M}$ carry over to $\mathcal{L}_n$ and $\mathcal{M}_n$, and whether there are idiosyncratic behaviors for certain values of n. In this paper, we begin to answer these questions as regards the bottom of the spectrum, where we expect to find a countable discrete sequence of isolated points converging to the first limit point. We restrict to $n = p$ prime (a simplification, as the gcd in Equations (3) and (4) can then only take two values). Finding $\min \mathcal{L}_p$ can be thought of as computing a ξ such that ξ and $p\xi$ are both "very irrational," that is, hard to approximate by rationals of small denominator.

Schmidt [22, 23] computes the initial discrete segment of $\mathcal{M}_n$ for $n = 2, 3, 5, 6$. Vulakh does the same for $n = 13$ [27, Theorem 32] and claims that "[t]he results obtained in the preceding sections can be used to find the discrete part of $\mathcal{M}_m$ in those cases" (viz. those not already solved by Schmidt) [27, p. 4090]. However, no general algorithm is given, nor is a general theorem stated on the structure of $\mathcal{M}_n$ for all n. In this paper, we begin to fill this gap. We show that $\mathcal{L}_n$ and $\mathcal{M}_n$ have a common minimum, and we give an algorithm that, if it terminates, computes $\min \mathcal{L}_p$ for a prime p and verifies the following conjecture.

Conjecture 1. For all positive integers n, the lowest point $\min \mathcal{L}_n$ is an isolated point of $\mathcal{L}_n$ and is the irrational square root of a rational number.

Theorem 2. *Conjecture 1 is verified for all prime values $n = p < 2000$.*

The lowest point $\min \mathcal{L}_p$ fluctuates with p (see Section 8) and reaches its highest observed value at $\min \mathcal{L}_{67} = 3.678\ldots$, leading to the following curiosity.

Theorem 3. *The prime $p = 67$ is the unique prime $p < 2000$ with the following property: There is no irrational ξ such that the continued fraction expansions of ξ and $p\xi$ consist after a certain point of only 1's and 2's.*

Along the way, we prove that $\mathcal{L}_n$ and $\mathcal{M}_n$ enjoy certain desirable properties long known for $\mathcal{L}$ and $\mathcal{M}$, specifically:

- $\mathcal{L}_n \subseteq \mathcal{M}_n$ (Proposition 9).
- Any $\alpha \in \mathcal{M}_n$ is realized by a pair (ξ, ξ') for which the supremum in the definition, Equation (4), is attained (Proposition 10).
- $\mathcal{L}_n$ and $\mathcal{M}_n$ are closed (Propositions 19 and 11, respectively).
- $\mathcal{L}_n$ and $\mathcal{M}_n$ contain a Hall's ray $[nF, \infty)$ (Proposition 14).

1.1. Methods

Kim and Sim [11] study and compare $\mathcal{L}_2$ and $\mathcal{M}_2$ using *Romik expansions,* a useful way of simultaneously recording the continued fraction expansions of ξ and 2ξ by words over an alphabet of three digits. Cha, Chapman, Gelb, and Weiss [3] create a suitable analogue of Romik expansions with a five-digit alphabet useful for studying $\mathcal{L}_3$ and $\mathcal{M}_3$. However, for reasons that we will explain below, we expect no analogous code with a finite alphabet to exist for $n > 3$. Hence we need another approach. It is unclear if the geometrical methods of Vulakh can be harnessed for automated computation. Instead, we use *Raney transducers,* a technique for applying a linear fractional transformation to a real number expressed in continued fraction form [20]. A Raney transducer is a finite directed graph whose edges are labeled with segments of a continued fraction. Our algorithm iterates through paths on the Raney transducer with an eye to looking for suitable cycles, representing periodic continued fractions with the desired low approximabilities.

1.2. Organization of the Paper

In Section 2, we recall classical results on the spectra $\mathcal{L}$ and $\mathcal{M}$. In Section 3, we prove some elementary results on $\mathcal{L}_n$ and $\mathcal{M}_n$ by working from the definitions. In Section 4, we describe the construction of two types of Raney transducer: a "fast" one, due to Raney, that is useful for computations, and a "slow" one that has certain theoretical advantages. In Section 5, we apply Raney transducers to show that $\mathcal{L}_n$ is closed. In Section 6, we describe an algorithm for computing $\min \mathcal{L}_p$ for prime p and verifying Conjecture 1. In Section 7, we explain certain low-lying points connected with Markoff triples that appear repeatedly as the output of our algorithm. Finally, in Section 8, we display data and muse on patterns found in the outputs for all p.

1.3. Code

The Sage code used in the computational parts of the paper can be found at `https://github.com/sad-ish-cat/DioApprox`. In the comments at the end of the file `raney.sage` are some sample commands as a guide to replicating the computations.

2. Classical Results on Approximability

If ξ is an irrational number, we define the *quality* of an approximation s/t to be the quantity

$$\frac{1}{t^2\left|\dfrac{s}{t} - \xi\right|}$$

appearing in Equation (1). We begin with some classical results pertaining to quality.

Proposition 4 ([6], Appendix 1)**.** *Let $\xi = [a_0, a_1, a_2, \ldots]$ be an irrational number, expressed as an infinite simple continued fraction.*

(a) The quality of the convergent $p_k/q_k = [a_0, \ldots, a_{k-1}]$ is given by

$$\frac{(-1)^k}{q_k^2\left(\frac{p_k}{q_k} - \xi\right)} = a_k + [0, a_{k-1}, a_{k-2}, \ldots, a_1] + [0, a_{k+1}, a_{k+2}, \ldots].$$

(b) Any approximation to ξ of quality at least 2 is a convergent.

(c) The approximability of ξ is the limsup of the qualities of its convergents:

$$\lambda(\xi) = \limsup_{k \to \infty} \left(a_k + [0, a_{k-1}, a_{k-2}, \ldots, a_1] + [0, a_{k+1}, a_{k+2}, \ldots]\right). \tag{5}$$

A *cut* of a continued fraction is a choice of truncation point k as above. A notation like $[a_0, \ldots, a_{k-1} | a_k, \ldots]$ is often used; but this obscures the symmetry between the terms before and after a_k. For this reason, we denote a cut by $[a_0, \ldots, a_{k-1} \boxed{a_k} a_{k+1}, \ldots]$. The *quality* of the cut is the λ-value

$$\lambda\left([a_0, \ldots, a_{k-1} \boxed{a_k} a_{k+1}, \ldots]\right) = a_k + [0, a_{k-1}, a_{k-2}, \ldots, a_1] + [0, a_{k+1}, a_{k+2}, \ldots].$$

Note that the dominant contribution to the quality $\lambda(C)$ of a cut is a_k; the terms $a_{k\pm h}$ lying farther away are progressively less important as h grows. This can be made quantitative.

Lemma 5. *If two simple continued fractions $\xi = [a_0, a_1, \ldots]$ and $\xi' = [a_0', a_1', \ldots]$ have the same initial terms $[a_0, \ldots, a_k] = [a_0', \ldots, a_k']$, then the difference of their values is bounded by*

$$|\xi - \xi'| \leq 2^{1-k}.$$

Proof. The proof is standard and elementary; see [6, Lemma 1]. □

2.1. Approximabilities of Quadratic Irrationals

To compute $\lambda(\xi)$ for a quadratic irrational ξ is a finite task: if

$$\xi = [a_0, \ldots, a_k, \overline{b_1, \ldots, b_\ell}],$$

then the transient terms a_i are of no consequence, and taking the limit of the qualities of cuts at a recurring term b_i, we may write

$$\lambda\left(\overline{b_1, \ldots \boxed{b_i} \ldots, b_\ell}\right) = b_i + [0, \overline{b_{i+1}, \ldots, b_\ell, b_1, \ldots, b_i}] + [0, \overline{b_{i-1}, \ldots, b_1, b_\ell, \ldots, b_i}], \tag{6}$$

so that

$$\lambda(\xi) = \max_{1 \leq i \leq \ell} \left(\overline{b_1, \ldots \boxed{b_i} \ldots, b_\ell}\right). \tag{7}$$

If $\bar{\xi}$ is the algebraic conjugate of ξ, note that $\mu(\xi, \bar{\xi}) = \lambda(\xi)$. The formula for the conjugate of a purely periodic continued fraction,

$$\operatorname{conj}([\overline{b_1, \ldots, b_\ell}]) = -[0, \overline{b_\ell, \ldots, b_1}],$$

implies that the right-hand side of Equation (6) and hence of Equation (7) is the difference of a quadratic irrational and its conjugate, and hence is the irrational square root of a rational number.

3. Elementary Properties of $\mathcal{L}_n$ and $\mathcal{M}_n$

We begin by proving some properties of general interest about the n-spectra $\mathcal{L}_n$ and $\mathcal{M}_n$. In the classical case $n = 1$, these were mostly proved by Cusick [5] and/or Cusick and Flahive [6]. First, we show that the n-approximabilities $\lambda_n(\xi)$ and $\mu_n(\xi, \xi')$ defined in Equations (3) and (4) can be computed in terms of the classical approximabilities $\lambda(\xi)$ and $\mu(\xi, \xi')$, respectively.

Proposition 6. *For each $n \geq 1$, we have*

$$\lambda_n(\xi) = \max_{g|n} \lambda(g\xi) \quad \textit{and} \quad \mu_n(\xi, \xi') = \max_{g|n} \mu(g\xi, g\xi').$$

Proof. We have

$$\begin{aligned}\lambda_n(\xi) &= \limsup_{\frac{s}{t} \to \xi} \frac{\gcd(t,n)}{t^2\left|\frac{s}{t} - \xi\right|} = \limsup_{\frac{s}{t} \to \xi} \max_{g|n, g|t} \frac{g}{t^2\left|\frac{s}{t} - \xi\right|} = \max_{g|n} \limsup_{\frac{s}{t} \to \xi, g|t} \frac{g}{t^2\left|\frac{s}{t} - \xi\right|} \\ &= \max_{g|n} \limsup_{\frac{s}{t'} \to g\xi} \frac{g}{(gt')^2\left|\frac{s}{gt'} - \xi\right|} = \max_{g|n} \limsup_{\frac{s}{t'} \to g\xi} \frac{1}{t'^2\left|\frac{s}{t'} - g\xi\right|} = \max_{g|n} \lambda(g\xi).\end{aligned}$$

Note that in the transformation process, we must allow non-reduced fractions s/t and s/t', but including or excluding such fractions does not affect the supremum because the reduced form always yields a better approximation quality. The proof for μ_n is analogous. $\square$

Corollary 7. *If $m \mid n$, then*

$$\mathcal{L}_n \subseteq \mathcal{L}_m \quad \textit{and} \quad \mathcal{M}_n \subseteq \mathcal{M}_m. \tag{8}$$

In particular,

$$\mathcal{L}_n \subseteq \mathcal{L} \quad \textit{and} \quad \mathcal{M}_n \subseteq \mathcal{M}. \tag{9}$$

Remark 8. We take this corollary as a sign that we have chosen the "correct" scaling of $\mathcal{L}_n$ and $\mathcal{M}_n$. In the literature [3, 4, 11], these spectra have arisen in different contexts and are scaled by a multiplicative constant. With some authors, such as Vulakh [27], the approximabilities are the reciprocals of those given here.

Proposition 9. *For each $n \geq 1$, we have $\mathcal{L}_n \subseteq \mathcal{M}_n$. Moreover, any $\alpha \in \mathcal{L}_n$ can be realized as $\mu_n(\xi, \xi')$ for some $\xi, \xi' \in \mathbb{R}$ for which the supremum in the definition, Equation (4), is attained at some $(s,t) \in \mathbb{Z}^2\backslash\{0\}$.*

Proof. Let $\alpha = \lambda_n(\xi) \in \mathcal{L}_n$. Then there is a sequence of fractions

$$\frac{s_1}{t_1}, \frac{s_2}{t_2}, \ldots$$

converging to ξ such that the "n-qualities," that is, the arguments to the limsup in Equation (3),

$$\alpha_i := \frac{\gcd(t_i, n)}{t_i^2 \left|\frac{s_i}{t_i} - \xi\right|},$$

converge to α. The action of $\Gamma^0(n)$ on $\mathbb{P}^1(\mathbb{Q})$ has finitely many orbits (the *cusps* of the associated modular curve $X_0(n)$). Hence, after passing to a subsequence, we may assume that each s_i/t_i maps by some $\gamma_i \in \Gamma^0(n)$ to a single fraction s/t. Note that $\gcd(t_i, n) = \gcd(t, n)$. The map γ_i is unique up to postcomposition by the stabilizer $\mathcal{Z} = \mathrm{Stab}_{\Gamma^0(n)}(s/t)$, which is isomorphic to $\mathbb{Z}$, generated by one parabolic element with unique fixed point s/t. Thus, fixing a closed fundamental domain $\mathcal{F}$ for $\mathcal{Z}$, containing neither s/t nor (for simplicity) ∞, we can pick γ_i such that $\gamma_i(\infty) \in \mathcal{F}$. Now, after passing to a subsequence, we can find $\xi', \xi'' \in \mathbb{P}^1(\mathbb{R})$ such that

$$\gamma_i(\infty) \to \xi', \quad \gamma_i(\xi) \to \xi''.$$

We claim that (ξ', ξ'') is the desired pair with $\mu_n(\xi', \xi'') = \alpha$. We have

$$\alpha = \lim_{i \to \infty} \frac{\gcd(t_i, n)}{t_i^2 \left|\frac{s_i}{t_i} - \xi\right|} = \lim_{i \to \infty} \frac{\gcd(t, n)|\gamma_i(\xi) - \gamma_i(\infty)|}{|s - t\gamma_i(\xi)||s - t\gamma_i(\infty)|}. \tag{10}$$

First note that if $\xi' = \xi''$, then as $i \to \infty$, the numerator of Equation (10) tends to 0 while the denominator is bounded since $s/t, \infty \notin \mathcal{F}$. So $\alpha = 0$, which is impossible. So $\xi' \neq \xi''$.

For any reduced fraction u/v, let $u_i/v_i = \gamma_i^{-1}(u/v)$ and note that, by the $\Gamma^0(n)$-invariance of the Markoff n-approximability,

$$\begin{aligned} \frac{\gcd(v, n)|\xi' - \xi''|}{|u - v\xi'||u - v\xi''|} &= \lim_{i \to \infty} \frac{\gcd(v, n)|\gamma_i(\infty) - \gamma_i(\xi)|}{|u - v\gamma_i(\infty)||u - v\gamma_i(\xi)|} \\ &= \lim_{i \to \infty} \frac{\gcd(v_i, n)}{v_i|u_i - v_i\xi|}. \end{aligned} \tag{11}$$

We claim that $u_i/v_i \to \xi$, which will complete the proof, since the right-hand side of Equation (11) is bounded above by α, with equality holding when $u/v = s/t$.

Supposing this is not the case, after passing to a subsequence, u_i/v_i converges to some point $\eta \in \mathbb{P}^1(\mathbb{R})$ different from ξ. Now

$$\frac{\gcd(v_i, n)}{v_i|u_i - \xi v_i|} = \frac{\gcd(v, n)|\gamma_i(\xi) - \gamma_i(\infty)|}{|u - \gamma_i(\xi)v||u - \gamma_i(\infty)v|} \to \frac{\gcd(v, n)|\xi' - \xi''|}{|u - \xi' v||u - \xi'' v|} > 0. \tag{12}$$

Since $\eta \neq \xi$, the left-hand side of Equation (12) goes to 0 unless u_i/v_i is infinitely often equal to the same fraction. Passing to a subsequence, we assume that $u_i/v_i =$

$\eta = u_1/v_1$ is constant. Now $\gamma_i = \sigma^{m_i}\gamma_1$ lies in a fixed coset of the stabilizer $\mathcal{Z}_\eta = \sigma^{\mathbb{Z}}$ of η in $\Gamma^0(n)$. Since the γ_i cannot be constant on an infinite subsequence, we must have $|m_i| \to \infty$ and so, for any $x \in \mathbb{P}^1(\mathbb{R})$ different from u/v, we have $\lim_{i\to\infty} \gamma_i(x) = \eta$. In particular, $\xi' = \eta = \xi''$, which is a contradiction. $\square$

Proposition 10. *Any $\alpha \in \mathcal{M}_n$ can be realized by some $\xi, \xi' \in \mathbb{R}$ for which the supremum in the definition, Equation (4), is attained at some $(s,t) \in \mathbb{Z}^2\backslash\{0\}$.*

Proof. Let $\alpha = \mu_n(\xi, \xi')$. If the supremum α in Equation (4) is not attained, then there is an infinite sequence of distinct pairs (s_i, t_i) for which

$$\lim_{i\to\infty} \frac{\gcd(t,n)\cdot|\xi-\xi'|}{|s_i - t_i\xi||s_i - t_i\xi'|} = \alpha. \tag{13}$$

Since $|s_i| + |t_i| \to \infty$, the two factors in the denominator cannot both be bounded, so on passing to a subsequence, one of them tends to ∞ and the other to 0. WLOG $s_i/t_i \to \xi$. Then

$$\alpha = \limsup_{\frac{s}{t}\to\xi} \frac{\gcd(t,n)\cdot|\xi-\xi'|}{|s-t\xi||s-t\xi'|} = \limsup_{\frac{s}{t}\to\xi} \frac{\gcd(t,n)}{t|s-t\xi|} = \lambda_n(\xi).$$

Hence $\alpha \in \mathcal{L}_n$. By the previous proposition, $\alpha \in \mathcal{M}_n$ is realized by a (ξ'', ξ''') for which the supremum is attained. $\square$

Proposition 11. *For each $n \geq 1$, $\mathcal{M}_n$ is closed.*

Proof. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of elements in $\mathcal{M}_n$ tending to a limit α; we shall show that $\alpha \in \mathcal{M}$. By Proposition 10, each $\alpha_i = \mu_n(\xi_i, \xi_i')$ with the supremum being attained at some $s_i/t_i \in \mathbb{P}^1(\mathbb{Q})$. Applying $\Gamma^0(n)$, we can transform each s_i/t_i to one of finitely many values (the cusps of $X_0(n)$, as above), and then, passing to a subsequence, we may assume that $s_i/t_i = s/t$ are all equal. Let $\mathcal{Z} \cong \mathbb{Z}$ be the stabilizer of s/t and $\mathcal{F}$ be a fundamental domain for $\mathcal{Z}$ as in the proof of Proposition 9. Applying elements of $\mathcal{Z}$, we may assume that $\xi_i \in \mathcal{F}$. Passing to a subsequence again, we may assume that $\xi_i \to \xi$ and $\xi_i' \to \xi'$ converge in $\mathbb{P}^1(\mathbb{R})$. For any $(u,v) \in \mathbb{Z}^2\backslash\{0\}$,

$$\begin{aligned} \frac{\gcd(v,n)|\xi-\xi'|}{|u-v\xi||u-v\xi'|} &= \lim_{i\to\infty} \frac{\gcd(v,n)|\xi_i-\xi_i'|}{|u-v\xi_i||u-v\xi_i'|} \\ &\leq \lim_{i\to\infty} \alpha_i \\ &= \alpha, \end{aligned} \tag{14}$$

equality holding when $u/v = s/t$. In particular, $\xi \neq \xi'$ because $\alpha \neq 0$ and $\xi \neq s/t$, as in the proof of Proposition 9. Also, ξ and ξ' are finite and irrational because otherwise there would be a choice of (u,v) for which the left-hand side of Equation (14) tends to infinity. So $\mu_n(\xi, \xi') = \alpha$, as desired. $\square$

The n-Lagrange spectrum $\mathcal{L}_n$ is also closed, but the proof involves Raney transducers and thus will be taken up in the next two sections. Here are a few other elementary facts.

Proposition 12. *For each $n \geq 1$, we have $\min \mathcal{L}_n = \min \mathcal{M}_n$.*

Proof. Since $\mathcal{M}_n$ is closed and bounded below, it has a minimal element $\alpha = \mu_n(\xi, \xi')$. We have $\mu_n(\xi, \xi') \geq \lambda_n(\xi)$. But $\lambda_n(\xi) \in \mathcal{L}_n \subseteq \mathcal{M}_n$, so equality holds and $\alpha = \lambda_n(\xi) \in \mathcal{L}_n$. □

Proposition 13. *For each $n \geq 1$, we have $n\mathcal{L} \subseteq \mathcal{L}_n$ and $n\mathcal{M} \subseteq \mathcal{M}_n$. Here $n\mathcal{L} = \{n\alpha : \alpha \in \mathcal{L}\}$ and likewise for $n\mathcal{M}$.*

Proof. Let $\alpha = \lambda(\xi) \in \mathcal{L}$. Then α is the limit of the n-qualities of a sequence of approximations $s_1/t_1, s_2/t_2, \dots$ tending to ξ. Passing to a subsequence, we may assume that all the $s_i \equiv s$ and all the $t_i \equiv t$ are congruent modulo n. Then, applying a transformation in $\mathrm{GL}_2\mathbb{Z}$ to ξ, we may assume that $t = 0$. Then $\lambda_n(\xi) = n\alpha$, since no sequence of approximations can do better than $s_1/t_1, s_2/t_2, \dots$. This proves that $n\mathcal{L} \subseteq \mathcal{L}_n$.

The proof that $n\mathcal{M} \subseteq \mathcal{M}_n$ is similar but even easier, since by Proposition 10, any $\alpha \in \mathcal{M}$ is achieved by a (ξ, ξ') such that the quality

$$\frac{|\xi - \xi'|}{|s - t\xi||s - t\xi'|}$$

attains its maximum at some (s, t). Applying a $\mathrm{GL}_2\mathbb{Z}$-transformation, we may assume that $(s, t) = (1, 0)$. We then observe that $\mu_n(\xi, \xi') = n\alpha$. □

As an immediate corollary, we get a Hall's ray for the n-spectra.

Proposition 14. *For all $n \geq 1$, we have $[nF, \infty) \subseteq \mathcal{L}_n \subseteq \mathcal{M}_n$, where $F = 4.5278\dots$ is Freĭman's constant, the least F such that $[F, \infty) \subseteq \mathcal{L}$.*

4. Raney Transducers

4.1. *LR*-Sequences

LR-sequences are a beautiful and handy alternative way to think about continued fractions. They appear to have been discovered several times, going back to Hurwitz [10, Section 5], who used the signs $+$ and $-$ instead of R and L, respectively. A pleasant exposition is given by Series [24].

Consider the linear fractional transformations (LFT's)

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

that is,

$$L(x) = \frac{x}{x+1}, \quad R(x) = x+1,$$

which map the real interval $(0,\infty)$ to the subintervals $(0,1)$ and $(1,\infty)$, respectively. Given $\xi \in (0,\infty)$, repeatedly apply L^{-1} or R^{-1}, as needed to keep the value positive, stopping if the value 1 is reached. If ξ is rational, this process yields a finite LR-expansion $R^{a_0}L^{a_1}R^{a_2}\cdots(L \text{ or } R)^{a_k}$ ($a_0 \geq 0$, all other $a_i \geq 1$), characterized by

$$\xi = R^{a_0}L^{a_1}R^{a_2}\cdots(L \text{ or } R)^{a_k}(1) = [a_0, a_1, a_2, \ldots, a_{k-1}, a_k + 1].$$

If ξ is irrational, we instead get an infinite LR-expansion $R^{a_0}L^{a_1}R^{a_2}\cdots$, and

$$\{\xi\} = \bigcap_{k\geq 0} R^{a_0}L^{a_1}R^{a_2}\cdots(L \text{ or } R)^{a_k}[0,\infty], \quad \xi = [a_0, a_1, a_2, \ldots]. \tag{15}$$

Conversely, any infinite LR-sequence represents a unique positive irrational, unless the sequence ends with a constant tail L^∞ or R^∞. For sequences with a constant tail, the intersection point as in Equation (15) is rational; each positive rational has two infinite LR-expansions formed by appending LR^∞ or RL^∞ to its canonical finite LR-expansion.

Although we will not need it in this paper, we would be remiss to omit the following beautiful geometric interpretation of the LR-expansion. Given a positive real number ξ, consider the geodesic from i to ξ in the hyperbolic upper half plane (Figure 1). As it passes through the tessellation formed by applying $\mathrm{SL}_2\mathbb{Z}$ to the geodesic $(0,\infty)$, check whether it exits each successive triangle to the left (L) or the right (R). The resulting sequence is the LR-expansion of ξ, either finite or infinite.

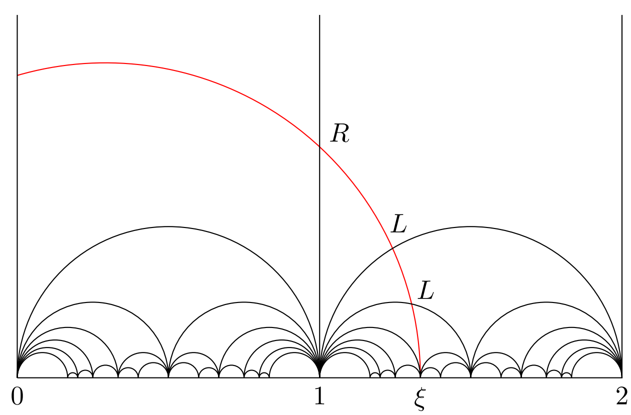


Figure 1: The expansion RLL of a rational number $\xi = 4/3$

4.2. The Slow Raney Transducer

A *Raney transducer* is a finite automaton, introduced by Raney in 1973 [20], associated to an LFT $\gamma \in \mathrm{PGL}_2(\mathbb{Q})$, that computes the LR-expansion of $\gamma(\xi)$ given that of ξ. Such transducers are useful in many questions related to continued fractions; see [13, 19]. See also Sol's expository thesis [26]. In this paper, we are interested in the transformation $\gamma(\xi) = n\xi$.

Here we construct a graph which we call the *slow Raney transducer* because it computes the same transformations, albeit in a simpler and somewhat less efficient way. The naturalness of the construction will be useful for our proofs.

Let
$$\gamma(x) = \frac{ax+b}{cx+d}$$
be an LFT. Assume that the coefficients a, b, c, d are nonnegative integers and the determinant $n = ad - bc$ is positive, which implies that γ maps $[0, \infty]$ into itself in an orientation-preserving way. Scale a, b, c, d to be coprime nonnegative integers. Let $\xi \in [0, \infty]$ be a real number with LR-expansion $S_1 S_2 S_3 \ldots$, where each letter $S_i \in \{L, R\}$. Suppose you want to compute the LR-expansion of $\gamma(\xi)$. When the first letter S_1 is revealed, it may or may not determine the first letter of $\gamma(\xi)$. The operative condition is whether
$$\gamma\big(S_1([0, \infty])\big) \subseteq [0, 1] \quad \text{or} \quad [1, \infty].$$
If it is, then $\gamma(\xi)$ has an LR-expansion starting with $T_1 = L$ or $T_1 = R$, respectively, and the new remaining task is to compute the LR-expansion of
$$\big(T_1^{-1} \circ \gamma \circ S_1\big)(\xi),$$
where the parenthesized LFT again has nonnegative, coprime integer coefficients and determinant n. On the other hand, if $\gamma\big(S_1([0, \infty])\big)$ straddles the point 1, then no letters of the LR-expansion of $\gamma(\xi)$ can be determined. This happens if $\gamma \circ S_1$ is given by a matrix
$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
satisfying the inequalities
$$a > c \quad \text{and} \quad b < d;$$
in other words, M is *row-balanced* in Raney's terminology (i.e. it has no dominant row). We can reduce the computation of general rational LFT's to row-balanced ones. This leads to the following algorithm.

Algorithm 15 (Slow Raney Transducer)**.**

> **Input:** A row-balanced LFT γ, and the first letter S of the LR-expansion of a positive real number $\xi = S(\xi')$.

Output: A new row-balanced LFT γ', and a word W in the alphabet $\{L, R\}$ such that $\gamma(\xi) = W\gamma'(\xi')$.

Method:

1. Start by trying $W_1 \leftarrow ()$ (the empty word) and $\gamma' = \gamma S^{-1}$.
2. Write
$$\gamma' = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
with $a, b, c, d \geq 0$ and $ad - bc = n > 0$.
3. If $a < c$ (implying $b < d$), set $W \leftarrow WL$, $\gamma' \leftarrow L^{-1}\gamma'$ and return to step 2.
4. If $b > d$ (implying $a > c$), set $W \leftarrow WR$, $\gamma' \leftarrow R^{-1}\gamma'$ and return to step 2.
5. Otherwise, γ' is row-balanced. Return W and γ'.

For fixed n, there is a finite set $\mathcal{RB}_n$ of row-balanced matrices, also called *orphans* by Nathanson [17], who produced a theory of continued fraction expansions of LFT's independently from Raney's. The number of orphans of each level n has been tabulated in the OEIS [25]. Hence we can encapsulate the results of Algorithm 15 in a finite directed graph $\mathcal{SRT}_n$ whose nodes are $\mathcal{RB}_n$ and each node γ has two outgoing edges
$$\gamma \xrightarrow{S:W} \gamma'$$
describing the output (W, γ) of Algorithm 15 to the input (γ, S) for $S \in \{L, R\}$. We call this graph the *slow Raney transducer of level* n. Some representative examples are shown in Figures 2 and 3.

Given $\xi \in [0, \infty]$, let $S_1 S_2 S_3 \dots$ be the LR-expansion of ξ (or one of the two LR-expansions if ξ is a positive rational). To compute $\gamma(\xi)$ for $\gamma \in \mathcal{RB}_n$, use the S_i as directions for a walk along the graph,
$$\gamma \xrightarrow{S_1:W_1} \gamma_1 \xrightarrow{S_2:W_2} \gamma_2 \xrightarrow{S_3:W_3} \cdots \tag{16}$$
to obtain a sequence of words W_i. We claim that the concatenation $W_1 W_2 W_3 \dots$ is the LR-expansion of $\gamma(\xi)$. First, we show that the concatenation $W_1 W_2 W_3 \dots$ is indeed an infinite word.

Lemma 16. *Any walk*
$$\gamma_0 \xrightarrow{S_1:W_1} \gamma_1 \xrightarrow{S_2:W_2} \dots \xrightarrow{S_n:W_n} \gamma_n$$
of length n on $\mathcal{SRT}_n$ has at least one nonempty output word $W_i \neq ()$.

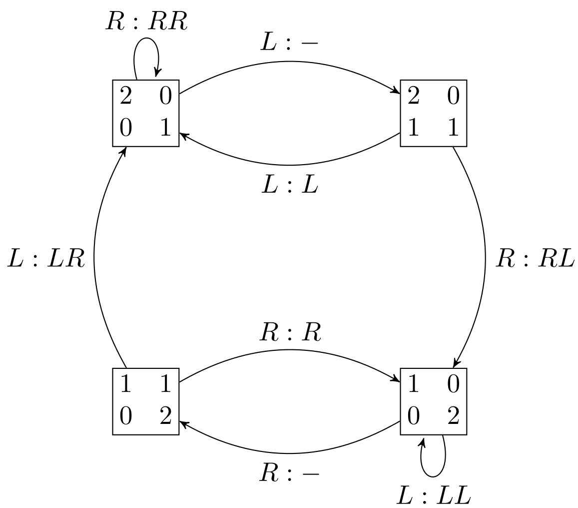


Figure 2: The slow Raney transducer $\mathcal{SRT}_2$.

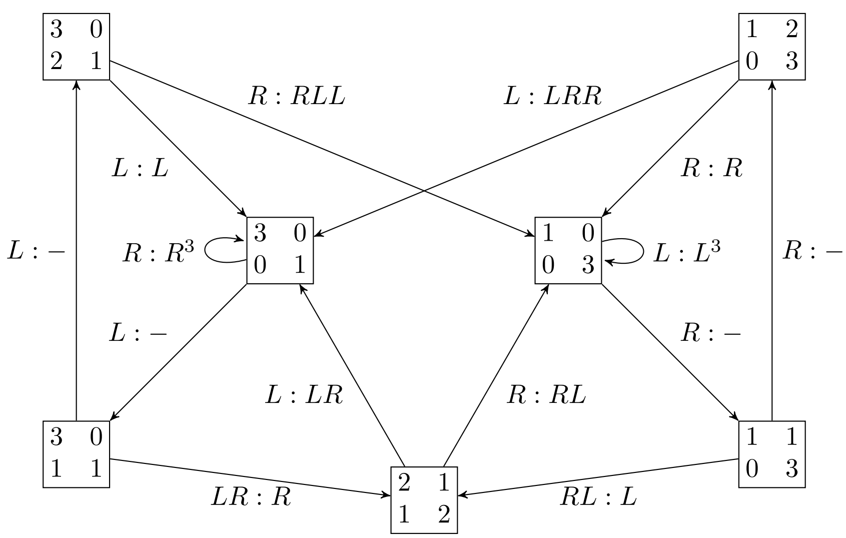


Figure 3: The slow Raney transducer $\mathcal{SRT}_3$.

Proof. Suppose that $W_1, \ldots, W_n$ are all empty. This means that

$$1 \in \gamma_n(0, \infty) = \gamma_0 S_1 S_2 \cdots S_n(0, \infty).$$

In particular, the number $\alpha = \gamma_0^{-1}(1)$ has unique LR-expansion starting with $S_1 S_2 \cdots S_n$. But, letting $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, the rational number $\alpha = \frac{d-b}{a-c}$ has numerator and denominator at most n and hence a finite LR-expansion of length at most $n-1$, which is a contradiction. $\square$

Now, with W_i as in Equation (16), the number η with LR-expansion $W_1 W_2 W_3 \ldots$ is characterized by

$$\begin{aligned}\{\eta\} &= \bigcap_i W_1 W_2 \cdots W_i[0, \infty] \subseteq \bigcap_i W_1 W_2 \cdots W_i \gamma_i [0, \infty] \\ &= \bigcap_i \gamma S_1 S_2 \cdots S_i[0, \infty] = \gamma\left(\bigcap_i S_1 S_2 \cdots S_i[0, \infty]\right) = \gamma(\{\xi\}) = \{\gamma(\xi)\}.\end{aligned}$$

4.3. The Fast Raney Transducer

As n grows, the number of nodes grows rather quickly. A more compact alternative is a graph $\mathcal{T}_n$, which we call the *(fast) Raney transducer,* whose nodes are the *doubly balanced* matrices $\mathcal{DB}_n$ satisfying the inequalities

$$a > b, \quad c < d, \quad a > c, \quad b < d$$

(that is, neither row and neither column is dominant). Each edge has a label $V : W$ where both the input V and the output W are finite words in the alphabet $\{L, R\}$. As before, an edge

$$\gamma \xrightarrow{V:W} \gamma'$$

has the property that $\gamma \circ V = W \circ \gamma'$. The input words V are no longer just one letter, but the input words emanating from any node form a *base* for LR-sequences, that is, any infinite word in the letters L and R starts with exactly one of them, such as $\{L, RL, RRL, RRR\}$. Further details are found in Raney [20]. The set $\mathcal{DB}_n$ is of manageable size; for $n = p$ prime, we have $|\mathcal{DB}_n| = p$. Examples are shown in Figures 4-6 (further examples can be found in [20]). The reader is invited to prove the following construction of the fast transducer from the slow one; we will not need it in the sequel.

Proposition 17.

(a) A node $\gamma \in \mathcal{RB}_n$ of $\mathcal{SRT}_n$ has indegree 2 or more if and only if $\gamma \in \mathcal{DB}_n$.

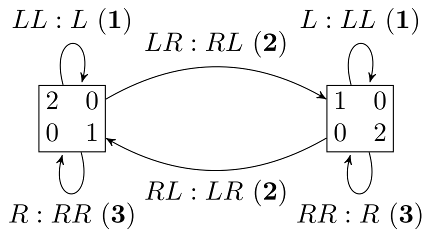


Figure 4: The Raney transducer $\mathcal{T}_2$. The parenthesized boldface numbers are the corresponding *Romik digits*, used in [11, 21]. The lowest point $\min \mathcal{L}_2 = 2\sqrt{2}$ arises by following the 2-cycle of edges marked (**2**).

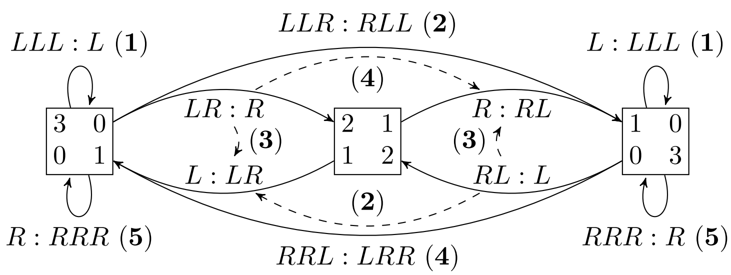


Figure 5: The Raney transducer $\mathcal{T}_3$. The lowest point $\min \mathcal{L}_3 = 2\sqrt{3}$ arises by following either of the two small 2-cycles involving the middle node.

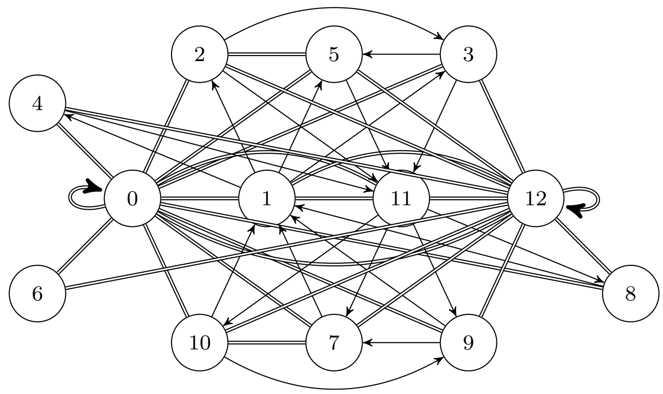
(b) *Starting with $\mathcal{SRT}_n$, we repeatedly perform the following operation: Pick any node γ with indegree* 1*, merge it into its predecessor γ', and update the labels of the edges out of γ according to the following rule.*

$$\gamma' \xrightarrow{V_1:W_1} \gamma \xrightarrow{V_2:W_2} \delta \quad \rightsquigarrow \quad \gamma' \xrightarrow{V_1V_2:W_1W_2} \delta.$$

When no nodes remain with indegree 1*, the resulting graph is $\mathcal{T}_n$.*

4.4. Connection to Romik Expansions

In [11] and [3], respectively, *Romik expansions* of real numbers are used to understand the spectra here denoted $\mathcal{L}_2$ and $\mathcal{L}_3$. This notion is closely related to Raney transducers of small level. For $\mathcal{T}_2$ (Figure 4), the three outgoing edges of each node correspond to the Romik digits **1**, **2**, and **3**. For $\mathcal{T}_3$ (Figure 5), the generalized Romik digits **1**–**5** of [3] correspond either to edges or to two-edge walks (the latter indicated by the dashed arrows) starting and ending at one of the two hub nodes, either $\left[\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1&0\\0&3\end{smallmatrix}\right]$. Any walk on $\mathcal{T}_3$ can be translated into a sequence of Romik



From	In : Out	To
0	$L^{13} : L$	0
0	$R : R^{13}$	0
0	$LR : R^6$	1
0	$L^7R : RLR$	2
0	$L^9R : RLL$	3
0	$L^8R : RL$	4
0	$L^{10}R : RLLL$	5
0	$L^6R : R$	6
0	$LLR : R^4$	7
0	$L^4R : RR$	8
0	$LLLR : RRR$	9
0	$L^5R : RRL$	10
0	$L^{11}R : RL^5$	11
0	$L^{12}R : RL^{12}$	12
1	$L^6 : LR$	0
1	$R : RR$	2
1	$LLR : RL$	3
1	$LR : R$	4
1	$LLLR : RLL$	5
1	$L^4R : RL^4$	11
1	$L^5R : RL^{11}$	12
2	$L^4 : LRR$	0
2	$R : R$	3
2	$LR : RL$	5
2	$LLR : RLLL$	11
2	$LLLR : RL^{10}$	12

From	In : Out	To
3	$LLL : LRRR$	0
3	$R : R$	5
3	$LR : RLL$	11
3	$LLR : RL^9$	12
4	$LL : LR^4$	0
4	$R : RL$	11
4	$LR : RL^8$	12
5	$RLL : LR^5$	0
5	$L : L$	2
5	$RR : R$	11
5	$RLR : RL^7$	12
6	$L : LR^6$	0
6	$R : RL^6$	12
7	$LRL : LR^7$	0
7	$LL : L$	1
7	$R : R$	10
7	$LRR : RL^5$	12
8	$RL : LR^8$	0
8	$L : LR$	1
8	$RR : RL^4$	12
9	$RRL : LR^9$	0
9	$RL : LRR$	1
9	$L : L$	7
9	$RRR : RLLL$	12

From	In : Out	To
10	$RRRL : LR^{10}$	0
10	$RRL : LRRR$	1
10	$RL : LR$	7
10	$L : L$	9
10	$R^4 : RLL$	12
11	$R^5L : LR^{11}$	0
11	$R^4L : LR^4$	1
11	$RRRL : LRR$	7
11	$RL : L$	8
11	$RRL : LR$	9
11	$L : LL$	10
11	$R^6 : RL$	12
12	$R^{12}L : LR^{12}$	0
12	$R^{11}L : LR^5$	1
12	$R^5L : LLR$	2
12	$RRRL : LLL$	3
12	$R^4L : LL$	4
12	$RRL : L^4$	5
12	$R^6L : L$	6
12	$R^{10}L : LRRR$	7
12	$R^8L : LR$	8
12	$R^9L : LRR$	9
12	$R^7L : LRL$	10
12	$RL : L^6$	11
12	$L : L^{13}$	12
12	$R^{13} : R$	12

Label	0	1	2	3	4	5	6	7	8	9	10	11	12
Matrix	$\begin{bmatrix}13&0\\0&1\end{bmatrix}$	$\begin{bmatrix}7&1\\1&2\end{bmatrix}$	$\begin{bmatrix}5&2\\1&3\end{bmatrix}$	$\begin{bmatrix}4&3\\1&4\end{bmatrix}$	$\begin{bmatrix}5&4\\3&5\end{bmatrix}$	$\begin{bmatrix}3&2\\1&5\end{bmatrix}$	$\begin{bmatrix}7&6\\6&7\end{bmatrix}$	$\begin{bmatrix}5&1\\2&3\end{bmatrix}$	$\begin{bmatrix}5&3\\4&5\end{bmatrix}$	$\begin{bmatrix}4&1\\3&4\end{bmatrix}$	$\begin{bmatrix}3&1\\2&5\end{bmatrix}$	$\begin{bmatrix}2&1\\1&7\end{bmatrix}$	$\begin{bmatrix}1&0\\0&13\end{bmatrix}$

Figure 6: The Raney transducer $\mathcal{T}_{13}$.

digits thanks to the fact that the walk must return at least every two steps to one of these two hub nodes.

For larger n, the picture is quite different, as shown in Figure 6. Here there are many infinite walks not meeting the two hub nodes labeled 0 and 12. In particular, the lowest point $\min \mathcal{L}_{13} = \sqrt{221}/5$ corresponds to a (non-simple!) cycle

$$2 \xrightarrow{R:R} 3 \xrightarrow{R:R} 5 \xrightarrow{L:L} 2 \xrightarrow{LR:RL} 5 \xrightarrow{L:L} 2$$

(or to its mirror image $10 \to 9 \to 7 \to 10 \to 7 \to 10$) that does not use either hub node 0 or 12 at all. Consequently, we do not expect a system of Romik digits to shed light on $\mathcal{L}_n$ for $n > 3$.

Remark 18. The dichotomy between the $n \leq 3$ and $n \geq 4$ cases can also be explained geometrically in terms of the images of the geodesic 0—$i\infty$ under $\Gamma_0(n)$. For $n = 1, 2, 3$, the resulting geodesics tessellate the hyperbolic plane by congruent regular 3-, 4-, and 6-gons, respectively (see Figures 1, 7, 8), suggesting encodings with 2, 3, and 5 symbols, respectively (because a line entering a tile at one side can exit on any of the other sides). For $n \geq 4$, the resulting geodesics do not form regions with finitely many sides (see Figure 9).

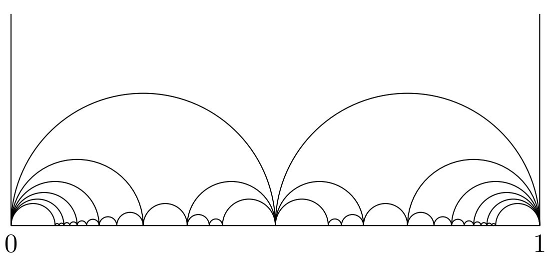


Figure 7: Tiling by hyperbolic squares corresponding to $\Gamma^0(2)$

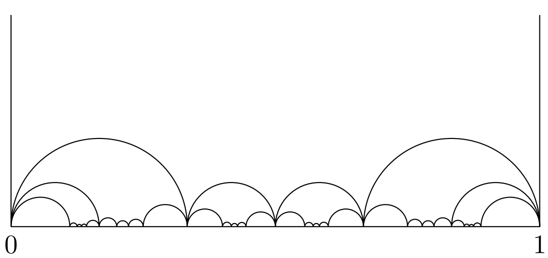


Figure 8: Tiling by hyperbolic regular hexagons corresponding to $\Gamma^0(3)$

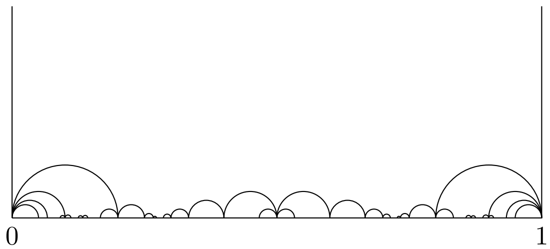


Figure 9: Tiling corresponding to $\Gamma^0(5)$. The tiles have infinitely many sides.

5. Closedness of the n-Lagrange Spectrum

Proposition 19. *For each $n \geq 1$, $\mathcal{L}_n$ is closed.*

The proof given below closely follows the proof of closedness of the classical Lagrange spectrum given by Cusick [5], and in particular shows the following stronger result.

Proposition 20. *For each $n \geq 1$, $\mathcal{L}_n$ is the closure of the set $\mathcal{P}_n$ of n-approximabilities of quadratic irrationals.*

Proof. Let G be a finite graph that computes, for given ξ, the LR-expansion of $d\xi$ for all divisors $d \mid n$ simultaneously. Such a graph can be constructed as follows. Take for the nodes $V(G)$ the cartesian product $\prod_{d|n} V(\mathcal{SRT}_d)$ of the nodes of the slow Raney transducers corresponding to the divisors of n. Give each node $(\gamma_d)_d$ two outgoing edges

$$(\gamma_d)_d \xrightarrow{S:(W_d)_d} (\gamma'_d)_d$$

whose targets and labels are derived from those of the edges

$$\gamma_d \xrightarrow{S:W_d} \gamma'_d$$

emanating from the respective node γ_d in each graph $\mathcal{S}_d$.

Now, any infinite LR-sequence can be encoded as a walk on G starting from the start node $\left(\left[\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right]\right)_{d|n}$. As we walk, we assemble the words W_d for each divisor $d \mid n$ to get the LR-expansion of $d\xi$. We can then compute $\lambda_n(\xi)$ by taking the limsup of the approximation qualities of each cut of each $d\xi$ via Proposition 4.

We now proceed to prove that

$$\mathcal{L}_n = \text{Closure}(\mathcal{P}_n). \tag{17}$$

Observe that $\mathcal{P}_n$ is the subset of approximabilities derived from (eventually) *periodic* walks on G. We prove the $(\subseteq)$ and $(\supseteq)$ directions separately.

We first prove the containment, $\mathcal{L}_n \subseteq \text{Closure}(\mathcal{P}_n)$. Let $\alpha = \lambda_n(\xi)$. Then $\alpha = \lambda(d_0\xi)$ for some $d_0 \mid n$ by Proposition 6. View ξ as an infinite walk

$$\mathcal{W}: \quad v_0 \xrightarrow[e_0]{S_1:(W_{1,d})_d} v_1 \xrightarrow[e_1]{S_2:(W_{2,d})_d} v_2 \xrightarrow[e_2]{S_3:(W_{3,d})_d} \cdots .$$

Inasmuch as an LR-sequence is another form of a continued fraction, we define a *cut* of an LR-sequence to be a choice of a (maximal) consecutive block of L's or R's, and the *quality* of the cut to be the quality of the corresponding cut of a continued fraction, as explained in Section 2.

There is a sequence of cuts $C_1, C_2, \ldots$ of the LR-expansion of $d_0\xi$ whose qualities converge to α. A cut C arises from a block of consecutive L's or R's on the output words $W_{i,d}$ in G giving the LR-expansion of $d\xi$ for some $d \mid n$. Define the *cut edge* to be the edge e_C of G on which the first L or R of this block is written. Since G is a finite graph, there is for each $k \geq 1$ and $\epsilon > 0$ a pair of cuts $C^{(1)}, C^{(2)}$ such that:

- $C^{(1)}$ has quality at least $\alpha - \epsilon$.
- All cuts of all $d\xi$ for some $d \mid n$, whose cut edge are at or after the cut edge e_i of $C^{(1)}$ along $\mathcal{W}$, have quality at most $\alpha + \epsilon$.
- $C^{(2)}$ has the same cut edge $e_j = e_i$ as $C^{(1)}$ and corresponds to the same block of L's or R's on the label $W_{i,d}$ of this edge, but occurring later on in $\mathcal{W}$: $j > i$.
- Moreover, the k steps before and after the cut edge agree for $C^{(1)}$ and $C^{(2)}$: $e_{i+h} = e_{j+h}$, $-k \leq h \leq k$.

Also, since $\alpha = \lambda_n(\xi)$ is finite, the terms of the continued fractions of all $d\xi$ are eventually bounded by $\lfloor \alpha \rfloor$, and we can assume that all the terms from e_{i-k} onward obey this bound.

Let ξ' be the irrational number corresponding to the following walk $\mathcal{W}'$ on G: Begin as for $\mathcal{W}$ until reaching the cut edge e_j, and then repeat the portion of $\mathcal{W}$ following e_i up to e_j infinitely. Note that $\mathcal{W}'$ is a periodic walk, so ξ' is a quadratic irrational. By construction, for any cut C' of $\mathcal{W}'$, there is a cut C of $\mathcal{W}$ such that C' and C agree for at least k edges before and after the cut edge. By Lemma 16, the corresponding LR-sequences agree before and after the first letter of the cut for at least k/n letters, yielding at least $k/(n\lfloor \alpha \rfloor) - 1$ common terms of the continued fraction on each side of the cut term. By Lemma 5, we have

$$\lambda(C') \leq \lambda(C) + 2^{3-\frac{k}{n\lfloor \alpha \rfloor}} \leq \alpha + \epsilon + 2^{3-\frac{k}{n\lfloor \alpha \rfloor}}.$$

Also, the recurring appearance of e_i yields an infinite sequences of cuts C' for which

$$\lambda(C') \geq \lambda(C^{(1)}) - 2^{3-\frac{k}{n\lfloor \alpha \rfloor}} \geq \alpha - \epsilon - 2^{3-\frac{k}{n\lfloor \alpha \rfloor}}.$$

Consequently,

$$|\lambda_n(\xi') - \alpha| \le \epsilon + 2^{3-\frac{k}{n\lceil\alpha\rceil}}. \tag{18}$$

Taking $k \to \infty$ and $\epsilon \to 0$, the right-hand side of Equation (18) goes to 0, so $\alpha \in \text{Closure}(\mathcal{P}_n)$ as desired.

We now prove the reverse containment, $\mathcal{L}_n \supseteq \text{Closure}(\mathcal{P}_n)$. Let $\xi_1, \xi_2, \ldots$ be a sequence of quadratic irrationals whose n-approximabilities α_i converge to $\alpha \in \text{Closure}(\mathcal{P}_n)$. For each i, α_i is the quality of a cut C_i in the periodic part of $d_i\xi_i$, according to Equation (7). Also, ξ_i corresponds to an eventually periodic walk $\mathcal{W}_i$ on G, and C_i has a particular cut edge e_i in the periodic part of $\mathcal{W}_i$. Passing to a subsequence, we may assume that:

- All $d_i = d$ are equal.
- All $e_i = e$ are the same edge, and the cuts C_i are at the same block of L's or R's starting on this edge.
- The walks $\mathcal{W}_k$ and $\mathcal{W}_{k+1}$ agree for k steps before and after e.

Now construct a walk $\mathcal{W}$ as follows. Follow $\mathcal{W}_1$ until the cut edge e, then walk one period of $\mathcal{W}_2$ starting and ending at e, then one period of $\mathcal{W}_3$ starting and ending at e, and so on. This $\mathcal{W}$ corresponds to an irrational number ξ. Similar to the preceding part, we check that $\lambda_n(\xi) = \alpha$, establishing that $\alpha \in \mathcal{L}_n$. □

6. An Algorithm to Compute $\min \mathcal{L}_p$

Raney transducers are ideally suited for computing elements of the n-Lagrange and n-Markoff spectra, especially if $n = p$ is prime, to which case we now specialize.

Recall that an LR-sequence can be viewed as a continued fraction, where each term of the continued fraction corresponds to a run of consecutive L's or R's in the sequence. Given a finite LR-sequence

$$W = L^{a_0}R^{a_1}L^{a_2}\cdots(L \text{ or } R)^{a_k} \quad \text{or} \quad R^{a_0}L^{a_1}R^{a_2}\cdots(L \text{ or } R)^{a_k},$$

let $\underline{\lambda}(W)$ be the minimum possible approximability of a completion of W, as evidenced by cuts in W, namely,

$$\underline{\lambda}(W) = \max_{1\le i\le k}\big(a_i + [0, a_{i-1}, a_{i-2}, \ldots, a_{i\%2}] + [0, a_{i+1}, a_{i+2}, \ldots, a_{k-((k-i)\%2)}]\big),$$

where $i \,\%\, 2$ denotes the least nonnegative residue of i modulo 2 (either 0 or 1).

Let $\beta > 0$. We say that an LR-sequence S is *β-good* if $\underline{\lambda}(S) \le \beta$, and *$\beta$-bad* otherwise. We say that a path P of a (slow or fast) Raney transducer is *β-good* if the input and output LR-sequences I_P and O_P formed by walking this path are both β-good; and *β-bad* otherwise.

We can now state our algorithm for computing $\min \mathcal{L}_n$, currently implemented only for primes $n = p$.

Algorithm 21.

Input:

- A prime p

Output, if the algorithm terminates:

- The minimal value $\alpha = \min \mathcal{L}_p$
- A bound β such that $\mathcal{L}_p \cap [0, \beta) = \{\alpha\}$
- A listing of representatives of all $\Gamma^0(p)$-classes of irrationals ξ achieving $\lambda_p(\xi) = \alpha$

Running variables:

- k, the lengths of paths to be considered (initially 1)
- α, the lowest known value in $\mathcal{L}_p$ (initially ∞)
- β, the lowest $\underline{\lambda}$-value found for α-bad paths (initially ∞)
- A list of all good paths of length $k - 1$ (initially, all paths of length 0 are good)

Iteration step, for $k = 1, 2, \ldots$ **:**

1. List all paths of length k on the Raney transducer $\mathcal{T}_p$ whose subpaths of length $k - 1$ are both α-good.
2. For each such path P, check whether P is α-good, that is, its input and output words V and W satisfy $\underline{\lambda}(P) = \max\{\underline{\lambda}(V), \underline{\lambda}(W)\} \leq \alpha$.
3. If P is α-good, add P to the list of good paths.
4. Additionally, if P is an α-good cycle, update $\alpha \leftarrow \min\{\alpha, \lambda(P^\infty)\}$, where $\lambda(P^\infty) = \max\{\lambda(V^\infty), \lambda(W^\infty)\}$ denotes the approximability of P extended periodically as in Equation (7).
5. If P is α-bad, let $\beta \leftarrow \min\{\beta, \underline{\lambda}(P)\}$.

Stopping condition: We stop if, for some positive integers m and t, the good paths have the following property: every path of length m occurring as the middle segment of a good path of length $m + 2t$ has a unique forward extension to a path of length $m+1$ occurring as the middle segment of a good path of length $m + 2t + 1$. If this happens, we deduce that any infinite good path repeats (after at most t transient initial moves) around one of finitely many cycles $P_1, \ldots, P_r$. We compute their approximabilities $\alpha_i = \lambda(P_i^\infty)$ and arrange them so that $\alpha_1 \leq \cdots \leq \alpha_r$. Let α_2' be the smallest α_i that exceeds the minimum α_1 ($\alpha_2' = \infty$ if no such α_i is found).

Return values: Once the stopping condition is achieved, we return

- $\alpha = \alpha_1$: the best λ_p-value of a cycle found
- $\beta \leftarrow \min\{\beta, \alpha_2'\}$
- an irrational ξ corresponding to each optimal cycle, which may be retrieved by picking a path from the start node $\left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right]$ to the cycle and converting the resulting eventually periodic LR-expansion to an irrational number ξ. (This is possible because the Raney transducer is strongly connected, as Raney proves [20, 8.3].)

Remark 22. The choice of t and hence m in the stopping condition is unimportant; it is easy to see that if the stopping condition holds for some m and t, it holds for all sufficiently large m and t. In our implementation we take $t \approx \sqrt{k}$, $m = k - 2t - 1$ which seems to be good enough.

Theorem 23. *If this algorithm terminates, it shows that*

(a) $\alpha = \min \mathcal{L}_p$ *is the square root of a rational number.*

(b) $\mathcal{L}_p \cap [-\infty, \beta) = \{\alpha\}$*; in particular,* α *is an isolated point of* $\mathcal{L}_n$ *and* $\mathcal{M}_n$*.*

Proof. If the stopping condition is satisfied for some t and m, then already a cycle P has been found of a finite approximability $\alpha = \lambda(P^\infty)$. Its approximability $\lambda_n(P^\infty)$ is always the square root of a rational number for the reasons pointed out in Section 2.1. Let β' be the value of β just before the final step, where we return $\beta = \min\{\beta', \alpha_2'\}$. Let $\gamma = \lambda_n(\xi) \in \mathcal{L}_n$ be a value. We may assume that ξ is positive and thus corresponds to an infinite walk W on $\mathcal{T}_p$. There are two cases:

1. There are infinitely many paths of length m in the walk W that are *not* the middle segment of a good path of length $m + 2t$. Each such path corresponds to an approximation to ξ whose n-quality is at least β, implying $\gamma \geq \beta'$.

2. After a certain point, every path of length m in the walk W is the middle segment of a good path of length $m + 2t$. By the stopping condition, W is eventually periodic, with period being one of the cycles enumerated at the last step. So γ is one of $\alpha_1, \ldots, \alpha_t$.

Hence $\mathcal{L}_n \cap [0, \beta') = \{\alpha_1, \ldots, \alpha_t\} \cap [0, \beta')$. In particular, the lowest point α_1 of $\mathcal{L}_n$ is isolated, and the next lowest point is at least $\min\{\alpha_2', \beta'\} = \beta$, as desired.

For each good cycle α, the corresponding irrational ξ is unique up to changing finitely many initial steps, which corresponds to a transformation in $\Gamma^0(p)$ for the following reason. If we have two eventually identical paths

$$\begin{tikzcd} \gamma_0 \arrow[r, bend left, "V_1 : W_1"] \arrow[r, bend right, "V_2 : W_2"'] & \gamma \arrow[r, "V : W"] & \cdots \end{tikzcd}$$

then the corresponding real numbers ξ_1 and ξ_2 with LR-expansions V_1V and V_2V, respectively, differ by the LFT

$$\xi_2 = V_2V_1^{-1}\xi_1,$$

where, by the defining relation of the Raney transducer,

$$\begin{aligned} V_2V_1^{-1} &= (\gamma_0^{-1}W_2\gamma)(\gamma_0^{-1}W_1\gamma)^{-1} = \gamma_0^{-1}W_2W_1^{-1}\gamma_0 \\ &\in \mathrm{SL}_2\mathbb{Z} \cap \gamma_0^{-1} \cdot \mathrm{SL}_2\mathbb{Z} \cdot \gamma_0 = \Gamma^0(p). \end{aligned}$$ □

Remark 24. Together with the computations in the attached code, this shows Theorems 2 and 3. It would be attractive to prove that, conversely, if the lowest point is isolated in one or both spectra then the algorithm terminates; but this does not seem easy.

7. Markoff Triples and Low-Lying Points in $\mathcal{L}_p$

Before presenting our data on the lowest point in $\mathcal{L}_p$, we must explain certain points in $\mathcal{L}$ that recur in many $\mathcal{L}_p$.

As is classically known (see [2, Chapter 2, Theorems II and III]), the points $\alpha \in \mathcal{L}$ (equivalently $\alpha \in \mathcal{M}$) in the discrete portion below 3 are parametrized by *Markoff triples,* positive integer solutions (x, y, z) to the *Markoff equation* $x^2+y^2+z^2 = 3xyz$. For each Markoff triple with $x \le y \le z$, there is an attached irrational number ξ with approximability $\lambda_z = \sqrt{9 - 4/z^2}$. The famous longstanding *Markoff uniqueness conjecture* states that any positive integer z occurs at most once as the largest element of a Markoff triple; it implies that the numbers ξ with approximability values $\lambda_z < 3$ are uniquely determined by their approximability λ_z up to $\mathrm{GL}_2\mathbb{Z}$-transformation. We would like to propose a strengthening of this conjecture:

Conjecture 25 (Strong Markoff Uniqueness Conjecture)**.** Each of the badly approximable irrationals ξ corresponding to a Markoff triple (x, y, z) has a different field of definition $\mathbb{Q}\big[\sqrt{9z^2 - 4}\big]$. In other words, if (x, y, z) and (x', y', z') are distinct Markoff triples, then

$$\frac{9z^2 - 4}{9z'^2 - 4}$$

is not a square in $\mathbb{Q}$.

To our knowledge, this conjecture has not appeared before in the literature. It is easily checked numerically, and we have computed that it holds for $z, z' \le 10^{30}$. This conjecture implies a handy characterization for when one of the Markoff points $\lambda_z \in \mathcal{L}$ appears in one of the p-spectra $\mathcal{L}_p$.

Proposition 26. *Let (x, y, z) be a Markoff triple, let $\lambda_z = \sqrt{9 - 4/z^2} \in \mathcal{L}$ be the corresponding approximability, and let*

$$D = \begin{cases} 9z^2 - 4, & z \text{ odd} \\ \dfrac{9z^2 - 4}{4}, & z \text{ even} \end{cases}$$

be the discriminant of the associated quadratic form. For a prime p, we have:

(a) *If p is the product of two principal primes in the quadratic order $\mathcal{O}_D$ of discriminant D, then $\lambda_p \in \mathcal{L}_p$.*

(b) *Conversely, if $\lambda_p \in \mathcal{L}_p$ and the Strong Markoff Uniqueness Conjecture holds for $z' \leq z$, then p in the product of two principal primes in $\mathcal{O}_D$.*

Proof. The quadratic form $f(x, y)$ of discriminant D corresponds to an ideal class $[\mathfrak{a}]$ of $\mathcal{O}_D$, invertible because f is primitive, and also ambiguous (that is, 2-torsion) [2, Lemma 9]. We have $\lambda(\xi) = \lambda_z$ for only one $\mathrm{GL}_2\mathbb{Z}$-class of real numbers ξ, namely those for which $\mathbb{Z}\langle 1, \xi\rangle$ is a representative of $[\mathfrak{a}]$.

If p is the product of two principal primes, necessarily of the form $\mathfrak{p}\bar{\mathfrak{p}}$, then $\mathfrak{a} \subset \mathfrak{p}\mathfrak{a}$ give two ideals of class $[\mathfrak{a}]$. We may scale the ideals so that 1 is a primitive element of both of them and write $\mathfrak{a} = \langle 1, \xi\rangle$, $\mathfrak{p}\mathfrak{a} = \langle 1, p\xi\rangle$. Then $\lambda_p(\xi) = \max\{\lambda(\xi), \lambda(p\xi)\} = \max\{\lambda_z, \lambda_z\} = \lambda_z$.

Conversely, suppose $\lambda_z \in \mathcal{L}_p$, so there is an element ξ such that

$$\lambda_z = \lambda_p(\xi) = \max\{\lambda(\xi), \lambda(p\xi)\}.$$

By flipping $\xi \mapsto p/\xi$, we may assume that $\lambda(\xi) = \lambda_z$. Then $\lambda(p\xi) \leq \lambda_z < 3$, so $p\xi$ is also an irrational of Markoff type, corresponding to a Markoff triple (x', y', z') with $z' \leq z$. But ξ and $p\xi$ are defined over the same quadratic field. By the Strong Markoff Uniqueness Conjecture, this implies that $(x', y', z') = (x, y, z)$. Then the ideals $\mathfrak{a} = \langle 1, \xi\rangle$ and $\mathfrak{a}' = \langle 1, p\xi\rangle$ are two representatives $\mathfrak{a} \supset \mathfrak{a}'$ of the ideal class $[\mathfrak{a}]$ corresponding to the Markoff form f. We have $\mathfrak{a}/\mathfrak{a}'$ cyclic of order p. Since $[\mathfrak{a}]$ is invertible, we may form the quotient ideal $\mathfrak{p} = \mathfrak{a}\mathfrak{a}'^{-1}$, a principal prime ideal of norm p. We have $p = \mathfrak{p}\bar{\mathfrak{p}}$, a product of two principal primes as desired. □

If $\mathrm{Cl}(\mathcal{O}_D)$ is pure 2-torsion, then Gauss's genus theory shows that the primes p that split into two principal primes in $\mathcal{O}_D$ are cut out by congruence conditions mod D. For the first six Markoff triples, this holds, and we get the following.

Corollary 27. *The presence of points $\alpha \leq 10\sqrt{26}/17 = 2.999423\ldots$ in $\mathcal{L}_p$ is governed by congruence conditions:*

(a) $(1, 1, 1)$: $\sqrt{5} \in \mathcal{L}_p$ *if and only if* $p \equiv 0, \pm 1 \mod 5$.

(b) $(1, 1, 2)$: $2\sqrt{2} \in \mathcal{L}_p$ *if and only if* $p \equiv 2, \pm 1 \mod 8$.

(c) $(1,2,5)$: $\sqrt{221}/5 \in \mathcal{L}_p$ *if and only if* p *is a square modulo* 13 *and* 17*.*

(d) $(1,5,13)$: $\sqrt{1517}/13 \in \mathcal{L}_p$ *if and only if* p *is a square modulo* 37 *and* 41*.*

(e) $(2,5,29)$: $\sqrt{7565}/29 \in \mathcal{L}_p$ *if and only if* p *is a square modulo* 5*,* 17*, and* 89*.*

(f) $(1,13,34)$: $10\sqrt{26}/17 \in \mathcal{L}_p$ *if and only if* $p \equiv \pm 1 \mod 8$ *is a square modulo* 13*.*

The first four of these show up frequently as $\min \mathcal{L}_p$. Note that the last two points can never be $\min \mathcal{L}_p$, because the associated congruence conditions imply the presence of an even lower point in $\mathcal{L}_p$: $\sqrt{5} \in \mathcal{L}_p$ and $2\sqrt{2} \in \mathcal{L}_p$, respectively.

For the next few Markoff triples following the first six, the class group of $\mathcal{O}_D$ is not 2-torsion and, indeed, is expected to be large owing to the presence of a small unit

$$u = \frac{3z + \sqrt{9z^2 - 4}}{2} \in \mathcal{O}_D^\times.$$

For fixed z, by the Chebotarev density theorem, half of all primes split in $\mathcal{O}_D$ and their factors are equidistributed in the class group of $\mathcal{O}_D$. It is then perhaps not surprising that there are a good many p for which $\mathcal{L}_p$ has no point below 3, as we illustrate below.

8. Numerical Data

Algorithm 21 allows us to compute the minimal element of $\mathcal{L}_p$ for any p, assuming the algorithm terminates. In Table 8, we list the lowest value of $\mathcal{L}_p$ for primes $p < 2000$ not covered by Corollary 27, which covers a proportion $55/64 \approx 86\%$ of primes. In Figure 10, we show the same data in graphical format.

In each case, the continued fraction for ξ is shown (periodic part only). The continued fraction for $p\xi$ is derived by one of three symmetry relations, listed in the last column:

- The continued fraction for ξ and $p\xi$ are **S**ymmetric and alike (as occurs in all the Markoff cases).
- The continued fractions for ξ and $p\xi$ are **A**symmetric and alike.
- the continued fractions for ξ and $p\xi$ are asymmetric and mutual **R**eversals.

We notice a few patterns. All the values shown are greater than 3, some only slightly so; that is to say, none of the countably many Markoff numbers show up in $\mathcal{L}_p$. Markoff numbers beyond those covered by Corollary 27 *can* show up as $\min \mathcal{L}_p$,

p	$\min \mathcal{L}_p = \min \mathcal{M}_p$	Continued fraction	Symmetry
3	$3.46410161513775 = 2\sqrt{3}$	[21]	S
67	$3.67821975514076 = \sqrt{7157}/23$	[33211112]	S
163	$3.42607262955615 = 3\sqrt{4853}/61$	[22121211]	R
227	$3.03973683071413 = \sqrt{231}/5$	[22211111]	S
277	$3.04378880403255 = 13\sqrt{29}/23$	[2222111]	S
283	$3.04378880403255 = 13\sqrt{29}/23$	[2222111]	S
293	$3.33732764987912 = 2\sqrt{19182}/83$	[2212111211]	R
317	$3.20024999023514 = \sqrt{6401}/25$	[211211111]	S
347	$3.20578401169006 = \sqrt{8643}/29$	[22211211111211]	S
397	$3.11986602498262 = \sqrt{2813}/17$	[22221]	S
547	$3.11986602498262 = \sqrt{2813}/17$	[22221]	S
557	$3.11986602498262 = \sqrt{2813}/17$	[22221]	S
587	$3.17316332974547 = 2\sqrt{2117}/29$	[2211211]	S
643	$3.11986602498262 = \sqrt{2813}/17$	[22221]	S
653	$3.28110118710167 = \sqrt{689}/8$	[22121121]	S
683	$3.11986602498262 = \sqrt{2813}/17$	[22221]	S
773	$3.20471655999477 = \sqrt{4522693909}/20985$	[$221121^{8}2112211122111$]	S
827	$3.22597102376446 = 2\sqrt{18981183427599}/2701041$	[22221121111 211112211111211]	R
853	$3.11735557792563 = \sqrt{1625621}/409$	[2222122111]	R
907	$3.28110118710167 = \sqrt{689}/8$	[22121121]	S
947	$3.32456094061007 = 2\sqrt{224439}/285$	[22121111121111]	A
997	$3.04963643415425 = \sqrt{174557}/137$	[22211222111]	S
1013	$3.40942239592361 = 2\sqrt{64517}/149$	[221212211]	S
1093	$3.00461791388810 = \sqrt{29870597}/1819$	[2211221^{12}]	S
1123	$3.08622198700304 = \sqrt{574738353221}/245645$	[22222211122122111111]	R
1163	$3.19289664252119 = \sqrt{9797}/31$	[21121111]	S
1213	$3.03973683071413 = \sqrt{231}/5$	[22211111]	S
1237	$3.08977153564575 = \sqrt{1535117}/401$	[2212212211]	S
1493	$3.33732764987912 = 2\sqrt{19182}/83$	[2212111211]	A
1523	$3.32267143334034 = \sqrt{10538139}/977$	[$121^{7}212211111$]	R
1597	$3.16759838060491 = \sqrt{11527532430881}/1071860$	[$2^{16}1122112112112211$]	S
1627	$3.28696105669264 = \sqrt{354606557}/5729$	[22222111211212]	R
1637	$3.04378880403255 = 13\sqrt{29}/23$	[2222111]	S
1693	$3.19289664252119 = \sqrt{9797}/31$	[21121111]	S
1747	$3.08627411116922 = \sqrt{1720469}/425$	[221221^{9}]	S
1787	$3.00487903864412 = \sqrt{60713}/82$	[2211221^{7}]	S
1867	$3.19289664252119 = \sqrt{9797}/31$	[21121111]	S
1907	$3.04378880403255 = 13\sqrt{29}/23$	[2222111]	S
1933	$3.25087478718599 = \sqrt{5834157}/743$	[211121^{13}]	S
1987	$3.00022037758055 = \sqrt{27229}/55$	[221^{7}]	S
1997	$3.12428276843314 = 2\sqrt{69697}/169$	[222122111]	R

Table 1: Minimum point of the p-Lagrange spectrum $\mathcal{L}_p$ for primes $p < 2000$ not covered by Corollary 27. For brevity, the terms 1, 2, and 3 of the continued fractions are written without separators, and the notation a^b means that the term a is repeated b times.

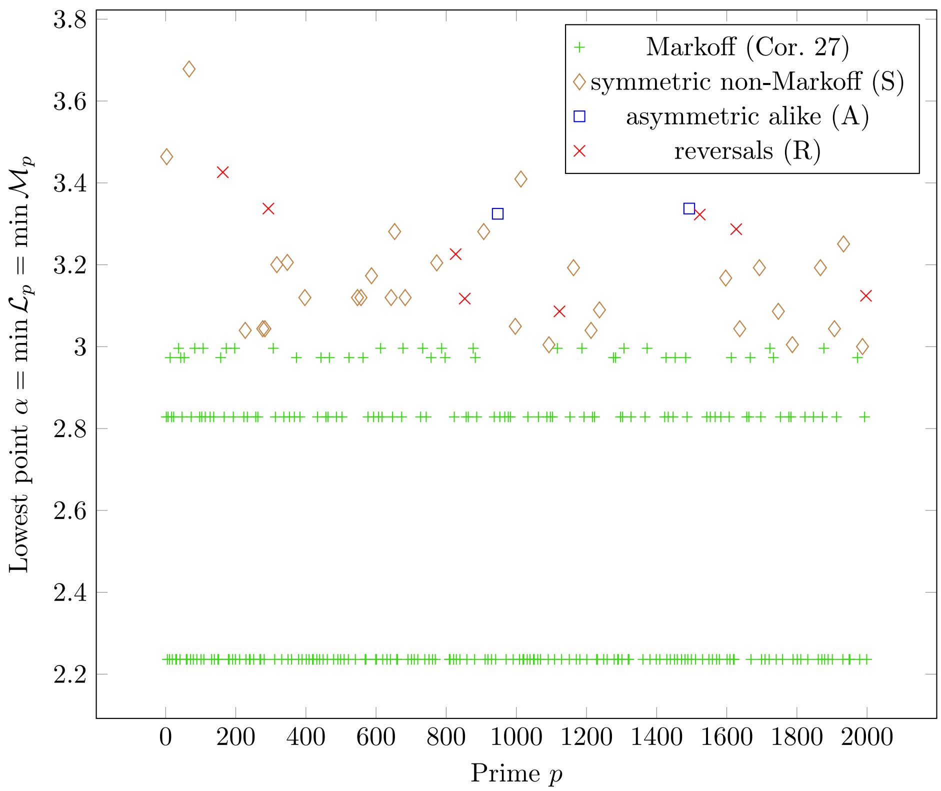


Figure 10: Minimum point of the p-Lagrange spectrum $\mathcal{L}_p$ for primes p

but rarely; for instance, the Markoff triple $(5, 29, 433)$ gives the minimum point for one out of every 512 primes, such as

$$\min \mathcal{L}_{3907} = 2.99999644423373236\ldots = \sqrt{9 - \frac{4}{433^2}}.$$

The three largest observed values of $\min \mathcal{L}_p$ occur for $p = 3, 67, 163$, which are (absolute) discriminants of imaginary quadratic fields of class number 1 (also known as Heegner numbers). This does not seem to be entirely a coincidence, although its full significance remains unclear. In both problems the prime p must satisfy the condition that certain small primes are non-squares modulo p. The most striking behavior occurs for $p = 67$, which is the unique p found for which $\min \mathcal{L}_p$ lies to the right of the gap $(2\sqrt{3}, \sqrt{13})$ in $\mathcal{L}$, separating continued fractions built of 1's and 2's from those containing a term at least 3. In view of the data, and by analogy with the finitude of the Heegner numbers, we conjecture the following.

Conjecture 28. Over all primes p, $\min \mathcal{L}_p$ attains a maximum at $p = 67$, where $\min \mathcal{L}_{67} = \sqrt{7157}/23$.

The symmetry relations for ξ and $n\xi$ are striking. Indeed, the S type occurs in a large majority of cases, even for some very long continued fractions where the symmetry cannot be attributed to mere chance. When the continued fraction for ξ is asymmetric, the R relation is apparently more frequent than the A relation, but both occur. Note that the continued fraction $[2, 2, 1, 2, 1, 1, 1, 2, 1, 1]$, with approximability value $\lambda = 2\sqrt{19182}/83$, occurs twice, once for $p = 293$ with the R relation, and once for $p = 1493$ with the A relation. We make the following conjecture (which may be hard, perhaps as hard as the Markoff uniqueness conjecture).

Conjecture 29. For each prime p, the periodic part of the continued fraction of ξ attaining the minimal $\lambda_p(\xi) = \alpha$ is unique up to reversal. We have $\lambda(\xi) = \lambda(p\xi) = \alpha$, and the periodic parts of ξ and $p\xi$ are either equal or mutual reversals, depending only on p.

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