# EXPLICIT ESTIMATES FOR INCOMPLETE EXPONENTIAL SUMS

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**Abstract**

We obtain explicit estimates for the incomplete exponential sum

$$S_p(f,B) := \sum_{x=1}^{B} e_p(f(x)),$$

where $e_p(u) = e^{2\pi i u/p}$, $f(x)$ is a polynomial over $\mathbb{Z}/(p)$, and $1 \le B \le p$, such as the following. For any $f$ of degree at least 2,

$$|S_p(f,B)| \le \min\{\tfrac{2}{\pi^2}\log p + 3.29 + \tfrac{B}{p},\ \tfrac{4}{\pi^2}\log p + \tfrac{B}{p} + .35\}\,\Phi,$$

where $\Phi = \max_{0\le y\le p-1}\left|\sum_{x=1}^{p} e_p(f(x)+yx)\right|$. For any quadratic polynomial $f$ and $B$ with $\sqrt{p} < B < p/2$, we have $|S_p(f,B)| < \sqrt{\frac{2}{\pi} p \log(16.6B^2/p)}$. For any monic cubic polynomial, and $B$ with $4p^{\frac{1}{3}} < B < \sqrt{p/12}$, we have $|S_p(f,B)| < (Bp)^{\frac{1}{4}}\sqrt{\log p}$.

## 1. Introduction

Let $p$ be a prime, $\mathbb{Z}_p = \{0,1,2,\ldots,p-1\}$ denote the finite field $\mathbb{Z}/(p)$, and $e_p(\cdot) := e^{2\pi i \cdot/p}$. For any polynomial $f(x)$ over $\mathbb{Z}$ or $\mathbb{Z}_p$, and positive integer $B$ with $1 \le B \le p$, let $S_p(f,B)$ denote the *incomplete* exponential sum

$$S_p(f,B) := \sum_{x=1}^{B} e_p(f(x)),$$

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and put $S_p(f) := S_p(f,p)$ for the *complete* sum. Let $d_p = d_p(f)$ denote the degree of $f$ viewed as a polynomial over $\mathbb{Z}_p$. Define

$$\Phi = \Phi(f,p) := \max_{0\le y\le p-1} \Big| \sum_{x=1}^{p} e_p(f(x)+yx) \Big|. \tag{1.1}$$

A standard estimate for $S_p(f,B)$, dating back to the work of Davenport and Heilbronn [5, 6], is the bound

$$|S_p(f,B)| \ll \Phi \log p. \tag{1.2}$$

Thus by the Weil bound [28], $\Phi \le (d_p-1)\sqrt{p}$, we have $|S_p(f,B)| \ll d_p\sqrt{p}\log p$ for $d_p \ge 2$. The first objective of this paper is to make (1.2) explicit.

**Theorem 1.1.** *For any prime $p$, polynomial $f$ over $\mathbb{Z}$ or $\mathbb{Z}_p$ with $d_p(f) \ge 2$, and integer $B$ with $1 \le B \le p$, we have*

$$|S_p(f,B)| \le \min\{ \tfrac{2}{\pi^2}\log p + 3.29 + \tfrac{B}{p},\ \tfrac{4}{\pi^2}\log p + \tfrac{B}{p} + .35\}\, \Phi. \tag{1.3}$$

The second expression in (1.3) is less than the first for $p < 1.99 \cdot 10^6$, and implies the following clean bound, first established by Vinogradov [27] for $p > 60$.

**Corollary 1.1.** *For any prime $p \ge 5$, polynomial $f$ over $\mathbb{Z}$ or $\mathbb{Z}_p$ with $d_p(f) \ge 2$, and integer $B$ with $1 \le B \le p$, we have*

$$|S_p(f,B)| < \Phi \log p. \tag{1.4}$$

The estimate in Theorem 1.1 is trivial for $B < \frac{2}{\pi^2}(d_p-1)\sqrt{p}\log p$ if we employ the Weil bound for $\Phi$. The second objective of this paper is to obtain an explicit nontrivial bound on $S_p(f,B)$ for $B$ in this range using the method of Weyl. We restrict our attention here to second and third degree polynomials.

### 1.1. Quadratic Polynomials

Korolev [14], improving on earlier works of Hardy and Littlewood [8], Fielder, Jurkat and Köner [7], and Oskolkov [21], established that for $p \nmid a$, and $1 \le B \le p$, one has $|S_p(ax^2,B)| \le 3.5254\sqrt{p}+1$. For a general quadratic $f(x) = ax^2+bx$ with $p \nmid a$, by completing the square and expressing $S_p(f(x),B)$ as a sum or difference of two monomial quadratic sums, one deduces from the Korolev bound that

$$|S_p(ax^2+bx,B)| \le 7.0508\sqrt{p}+2, \tag{1.5}$$

for $1 \le B \le p$. The bound is nontrivial for $B > 7.06\sqrt{p}$, and yields the bound $|S_p(ax^2+bx,B)| < B/2$ for $B > 14.11\sqrt{p}$. Lehmer [16] studied in detail the case where $a = 1$, obtaining for $1 \le B \le p$,

$$\big|S_p(x^2,B)\big| \le \begin{cases} \sqrt{p}+O(1), & \text{if } p \equiv 3 \pmod 4; \\ 1.0625461\sqrt{p}+O(1), & \text{if } p \equiv 1 \pmod 4. \end{cases} \tag{1.6}$$

Moreover, the constants on the $\sqrt{p}$ in (1.6) are sharp. We prove the following.

**Theorem 1.2.** *For any quadratic polynomial $f(x) = ax^2 + bx + c$ with $p \nmid a$, and positive integer $B$ with $\sqrt{p} < B < p/2$, we have*

$$|S_p(f,B)| < \sqrt{\tfrac{2}{\pi} p \log(16.6B^2/p)}. \tag{1.7}$$

This improves on Theorem 1.1 for all $p$, and improves on (1.5) for all $p, B$ with $B < 2.22 \cdot 10^{16} \sqrt{p}$. The bound in (1.7) is nontrivial for $B > 1.53\sqrt{p}$. We derive from (1.7), for instance, that if $B > 3.73\sqrt{p}$, then $|S_p(f,B)| < B/2$. Such a bound is not possible for $B$ slightly smaller than $\sqrt{p}$; see Example 6.1. For $B > p/2$, see Corollary 6.1.

### 1.2. Cubic Polynomials

We are not aware of any explicit Weyl type estimates for cubic polynomials, but see [9] for related work. Here we establish the following. Put

$$\delta_p := \frac{1.0661}{\log\log p}.$$

**Theorem 1.3.** *For any cubic polynomial $f(x) = ax^3 + bx^2 + cx$ with $p \nmid a$, and positive integer $B$, we have*

$$|S_p(f,B)| \le \begin{cases} 1.1\, B^{\frac{1}{4}} p^{\frac{1}{4}+\frac{\delta_p}{4}} \log^{\frac{1}{4}} p, & \text{for } 1 \le B < \sqrt{p/2}; \\ 1.38(1+\frac{p}{B^2})^{\frac{1}{4}}\, B^{\frac{3}{4}} p^{\frac{\delta_p}{2}} \log^{\frac{1}{4}} p, & \text{for } \sqrt{p/2} < B \le p. \end{cases}$$

The estimate is nontrivial for $B > 1.14\, p^{\frac{1}{3}+\frac{\delta_p}{3}} \log^{\frac{1}{3}} p$ and improves on Theorem 1.1 for $B \ll_\varepsilon p^{\frac{2}{3}-\varepsilon}$.

For monic cubics we can eliminate the term $p^{\delta_p/4}$ for small $B$.

**Theorem 1.4.** *For any cubic polynomial $f(x) = x^3 + bx^2 + cx$ and positive integer $B$ with $4p^{\frac{1}{3}} < B < \sqrt{p/12}$, we have*

$$|S_p(f,B)| < (Bp)^{\frac{1}{4}} \log^{\frac{1}{4}}(B^3/p) \log^{\frac{1}{4}}(Bp).$$

In particular, $|S_p(f,B)| < (Bp)^{\frac{1}{4}} \log^{\frac{1}{2}} p$ for $B < \sqrt{p/12}$. The theorem yields a nontrivial bound for $|S_p(f,B)|$ for $p > 5\cdot 10^8$, $B > p^{\frac{1}{3}}(\log^{\frac{1}{3}} p)\log\log p$, and improves on Theorem 1.1 whenever it applies. The theorem also yields bounds of the following type for sufficiently large $p$: For $B > 2p^{\frac{1}{3}}(\log^{\frac{1}{3}} p)\log\log p$, we have $|S_p(f,B)| < B/2$.

## 2. Proof of Theorem 1.1, Part I

To obtain the second bound in (1.3) we use the *method of completing a sum*, a method dates back at least to the 1918 work of Polya [23], Vinogradov [26], Landau

[15] and Schur [24], who used it for estimating the multiplicative character sum $\sum_{x=1}^{B} \chi(x)$, yielding the well known Polya-Vinogradov inequality. For the sum at hand, the first appearance we know of is the 1936 work of Davenport and Heilbronn [5, 6], who treated the case of monomials, followed by Hua [10] who treated general $f(x)$. Let $T_p(B)$ denote the trigonometric sum

$$T_p(B) := \frac{1}{p} \sum_{y=1}^{p-1} \left| \frac{\sin(\pi B y/p)}{\sin(\pi y/p)} \right|.$$

**Lemma 2.1.** *For any prime $p$, polynomial $f(x)$ over $\mathbb{Z}$ or $\mathbb{Z}_p$, and positive integer $B$ with $1 \le B \le p$, we have*

$$S_p(f,B) = \frac{B}{p} \sum_{x=1}^{p} e_p(f(x)) + \theta\, T_p(B) \max_{1 \le y \le p-1} \Big| \sum_{x=1}^{p} e_p(f(x) + yx) \Big|, \tag{2.1}$$

*for some $\theta$ with $|\theta| \le 1$,*

*Proof.* Let $I = \{1, 2, \ldots, B\} \subseteq \mathbb{Z}_p$, and $1_I$ be the characteristic function of $I$ with Fourier expansion

$$1_I(x) = \sum_{y=0}^{p-1} a(y) e_p(yx),$$

where $a(0) = B/p$, and for $1 \le y \le p-1$,

$$a(y) = \frac{1}{p} \sum_{x=1}^{B} e_p(-yx) = \frac{1}{p} e_p\Big(\frac{-B-1}{2}\, y\Big) \frac{\sin(\pi B y/p)}{\sin(\pi y/p)}.$$

Then

$$\begin{aligned} \sum_{x=1}^{B} e_p(f(x)) &= \sum_{x=0}^{p-1} 1_I(x) e_p(f(x)) = \sum_{x=0}^{p-1} \sum_{y=0}^{p-1} a(y) e_p(yx) e_p(f(x)) \\ &= a(0) \sum_{x=0}^{p-1} e_p(f(x)) + \sum_{y=1}^{p-1} a(y) \sum_{x=0}^{p-1} e_p(f(x) + yx), \end{aligned}$$

and so

$$\left| \sum_{x=1}^{B} e_p(f(x)) - \frac{B}{p} \sum_{x=0}^{p-1} e_p(f(x)) \right| \le \Phi' \sum_{y=1}^{p-1} |a(y)| = \Phi' T_p(B),$$

where $\Phi' := \max_{1 \le y \le p-1} \left| \sum_{x=0}^{p-1} e_p(f(x) + yx) \right|$. □

It is elementary to show $T_p(B) < \log p$; see, e.g., [11, Theorem 7.3]. Improvements were made by Vinogradov [27], Lidl and Neiderreiter [17], Cochrane [3], Kongting [13], Peral [22], and Cochrane and Peral [4]. Presently, the best bound is that of Bourgain, Cochrane, Paulhus and Pinner [2, Lemma 11.1], a slight improvement on [3, Theorem 1]: For any prime $p \geq 5$ and $1 \leq B \leq p$,

$$T_p(B) \leq \tfrac{4}{\pi^2} \log p + .35. \tag{2.2}$$

The constant $4/\pi^2$ on the $\log p$ term is best possible.

Let $\Phi = \Phi(f, p)$ be as defined in (1.1).

**Lemma 2.2.** *For any prime $p$, and polynomial $f$ over $\mathbb{Z}$, we have*

$$\Phi(f, p) \geq \sqrt{p}.$$

*Proof.* This is immediate from the identity

$$\sum_{y=0}^{p-1} \Big| \sum_{x=0}^{p-1} e_p(f(x) + yx) \Big|^2 = \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} e_p(f(u) - f(v)) \sum_{y=0}^{p-1} e_p(y(u-v)) = p^2.$$

□

*Proof of Theorem 1.1, Part I.* From (2.2) and (2.1) we get for $p \geq 5$,

$$|S_p(f, B)| \leq \tfrac{B}{p}\Phi + |T_p(B)|\Phi \leq \tfrac{B}{p}\Phi + (\tfrac{4}{\pi^2} \log p + .35)\Phi,$$

yielding the second bound in (1.3). For $p = 2, 3$, we note that using $|S_p(f, B)| \leq B$ and the bound $\Phi \geq \sqrt{p}$ from Lemma 2.2, it suffices to show that

$$B \leq \big(\tfrac{4}{\pi^2} \log p + \tfrac{B}{p} + .35\big)\sqrt{p},$$

that is,

$$B\big(1 - \tfrac{1}{\sqrt{p}}\big) \leq \big(\tfrac{4}{\pi^2} \log p + .35\big)\sqrt{p}.$$

Computation confirms this for $p = 2, 3$, and $B \leq p$. □

To prove the first bound in (1.3), we need a number of lemmas on trigonometric sums.

## 3. Lemmas on Trigonometric Sums

**Lemma 3.1.** *([3, Lemma 2.2]). For $0 < x < 1$, we have*

$$\frac{1}{\sin \pi x} < \frac{1}{\pi} \left( \frac{1}{x} + \frac{2x}{1 - x^2} \right).$$

**Lemma 3.2.** *If $p$ is a prime, $1 \le B < p/2$, and $p \nmid a$, we have*

$$\sum_{h=1}^{B} \frac{1}{|\sin(\pi a h/p)|} < \frac{p}{\pi}\Big(\log B + \gamma + \frac{1}{2B}\Big) + \frac{pB(B+1)}{\pi(p^2 - B^2)}.$$

*Proof.* We first show that the maximum value of the sum occurs when $a = 1$. Since $B < p/2$, we have $0 < \sin(\pi/p) < \sin(2\pi/p) < \cdots < \sin(B\pi/p)$. If there is a value of $a$ where the sum exceeds $\sum_{h=1}^{B} \frac{1}{|\sin(\pi h/p)|}$, then there must exist an integer $b$ and distinct values $h, h'$ with $1 \le h, h' \le B$, such that $ah \equiv \pm b \bmod p$ and $ah' \equiv \pm b \bmod p$. Then either $ah \equiv ah' \bmod p$, or $ah \equiv -ah' \bmod p$. In the first case we have $h = h'$, a contradiction. In the second case, $p|a(h+h')$. Since $2 \le h + h' \le 2B < p$, we again have a contradiction. Thus, the maximum value is attained when $a = \pm 1$.

By Lemma 3.1 and the Euler-Maclaurin estimate $\sum_{1 \le n \le x} \frac{1}{n} < \log x + \gamma + \frac{1}{2x}$,

$$\begin{aligned}
\sum_{h=1}^{B} \frac{1}{|\sin(\pi h/p)|} &< \frac{p}{\pi} \sum_{h=1}^{B} \left(\frac{1}{h} + \frac{2h}{p^2 - h^2}\right) \\
&< \frac{p}{\pi}(\log B + \gamma + \frac{1}{2B}) + \frac{2p}{\pi(p^2 - B^2)} \sum_{h=1}^{B} h \\
&= \frac{p}{\pi}(\log B + \gamma + \frac{1}{2B}) + \frac{pB(B+1)}{\pi(p^2 - B^2)}.
\end{aligned}$$

□

**Lemma 3.3.** *If $p$ is a prime and $\sqrt{p/\pi} < B < p/2$, then for any $a$ with $p \nmid a$, we have*

$$\sum_{h=1}^{B} \min\left\{B,\ \tfrac{1}{|\sin(\pi a h/p)|}\right\} < \tfrac{p}{\pi}\big(\log(\pi B^2/p) + 1 + \tfrac{\pi B}{p}\big) + \tfrac{pB(B+1)}{\pi(p^2 - B^2)}.$$

*Proof.* As in the preceding proof, we may assume that $a = 1$. By Lemma 3.1, the estimate $\sum_{u < h \le v} \frac{1}{h} < \log(v/u) + \frac{1}{u}$ for real $u, v$ with $v \ge u \ge 1$, and noting that $p/(\pi B) < B$ by assumption, we have

$$\begin{aligned}
\sum_{h=1}^{B} \min\left\{B,\ \frac{1}{|\sin(\pi h/p)|}\right\} &\le \sum_{1 \le h \le p/(\pi B)} B + \sum_{p/(\pi B) < h \le B} \frac{p}{\pi}\left(\frac{1}{h} + \frac{2h}{p^2 - h^2}\right) \\
&\le \frac{p}{\pi} + \frac{p}{\pi} \sum_{p/(\pi B) < h \le B} \frac{1}{h} + \frac{2p}{\pi(p^2 - B^2)} \sum_{h=1}^{B} h \\
&\le \frac{p}{\pi} + \frac{p}{\pi}\left(\log(\pi B^2/p) + \frac{\pi B}{p}\right) + \frac{pB(B+1)}{\pi(p^2 - B^2)}.
\end{aligned}$$

□

**Lemma 3.4.** *([3, Lemma 2.1]). For any positive integer $K$ and real number $\xi$,*

$$\sum_{y=1}^{K} \tfrac{|\sin(y\xi)|}{y} < \tfrac{2}{\pi}\Big(\log(K+1)+\gamma+\log 2\Big) + \tfrac{3}{\pi(K+1)}.$$

In particular, since $\log(K+1) < \log K + \frac{1}{K}$,

$$\sum_{y=1}^{K} \tfrac{|\sin(y\xi)|}{y} < \tfrac{2}{\pi}\log K + \tfrac{2}{\pi}(\gamma+\log 2) + \tfrac{5}{\pi K}. \tag{3.1}$$

**Lemma 3.5.** *For any positive integers $q, K, B$, with $K < q$, we have*

$$\sum_{k=1}^{K} \tfrac{|\sin(\pi Bk/q)|}{|\sin(\pi k/q)|} < \tfrac{2}{\pi^2} q\log K + \tfrac{2q}{\pi^2}(\gamma+\log 2) + \tfrac{q}{\pi}\tfrac{K}{q-K} + \tfrac{5q}{\pi^2 K}.$$

*Proof.* By Lemma 3.1, and then (3.1),

$$\begin{aligned}\sum_{k=1}^{K} \tfrac{|\sin(\pi Bk/q)|}{|\sin(\pi k/q)|} &< \tfrac{q}{\pi}\sum_{k=1}^{K}\tfrac{|\sin(\pi Bk/q)|}{k} + \tfrac{q}{\pi}\sum_{k=1}^{K}\tfrac{|\sin \pi Bk/q|}{q-k}\\ &\le \tfrac{q}{\pi}\left(\tfrac{2}{\pi}\log K + \tfrac{2}{\pi}(\gamma+\log 2) + \tfrac{5}{\pi K}\right) + \tfrac{q}{\pi}\tfrac{K}{q-K}.\end{aligned}$$

□

## 4. Proof of Theorem 1.1, Part II

We make use of the Beurling–Selberg majorizing and minorizing functions. The following comes from the work of Vaaler [25, Theorem 19], but is stated here in the form appearing in [19].

**Lemma 4.1.** *For any interval $J = [a, b]$ in $\mathbb{T}$ with length $b - a < 1$, and for any positive integer $K$, there are trigonometric polynomials*

$$S_\pm(x) = \sum_{k=-K}^{K} \alpha_\pm(k) e^{2\pi i k x},$$

*such that $\alpha_\pm(0) = b - a \pm \frac{1}{K+1}$ and*

$$S_-(x) \le 1_J(x) \le S_+(x), \quad \textit{for all } x \in \mathbb{T}\ .$$

Let $I = \{1, 2, \ldots, B\} \subseteq \mathbb{Z}_p$. We apply Lemma 4.1 with $J = [\frac{1}{2p}, \frac{B}{p} + \frac{1}{2p}]$, so that, $1_J(\frac{n}{p}) = 1_I(n)$ for $n \in \mathbb{Z}$, and $\alpha_\pm(0) = \frac{B}{p} \pm \frac{1}{K+1}$. Set

$$T_\pm(x) := S_\pm(\tfrac{x}{p}) = \sum_{k=-K}^{K} \alpha_\pm(k) e_p(kx). \tag{4.1}$$

If $K < \frac{p}{2}$, then we can view $T_\pm$ as functions on $\mathbb{Z}_p$ with Fourier expansions as given in (4.1). Let $1_I(x) = \sum_{k=0}^{p-1} a(k) e_p(kx)$ be the Fourier expansion for $1_I(x)$. Then for $|k| \le K$,

$$\alpha_\pm(k) = \frac{1}{p}\sum_{x=0}^{p-1} T_\pm(x) e_p(-kx), \qquad a(k) = \frac{1}{p}\sum_{x=0}^{p-1} 1_I(x) e_p(-kx).$$

Also, by design, for any $x \in \mathbb{Z}_p$,

$$T_-(x) \le 1_I(x) \le T_+(x).$$

Thus for $|k| \le K$, we have

$$\begin{aligned} |a(k) - \alpha_-(k)| &\le \tfrac{1}{p}\sum_{x=0}^{p-1} |1_I(x) - T_-(x)| = \tfrac{1}{p}\sum_{x=0}^{p-1}\Big(1_I(x) - T_-(x)\Big) \\ &= \tfrac{1}{p}\sum_{x=0}^{p-1} 1_I(x) - \tfrac{1}{p}\sum_{x=0}^{p-1} T_-(x) = a(0) - \alpha(0) = \tfrac{1}{K+1}, \end{aligned} \tag{4.2}$$

and the same for $|a(k) - \alpha_+(k)|$.

Set $T = T_\pm$, $\alpha = \alpha_\pm$ with a fixed $\pm$ choice. We have

$$\begin{aligned} \sum_{x=1}^{B} e_p(f(x)) &= \sum_{x=0}^{p-1} 1_I(x) e_p(f(x)) = \sum_{x=0}^{p-1} T(x) e_p(f(x)) + \sum_{x=0}^{p-1} (1_I(x) - T(x)) e_p(f(x)) \\ &= \Sigma_1 + \Sigma_2, \end{aligned}$$

say. Now, as seen in (4.2), $|\Sigma_2| \le \frac{p}{K+1}$ and for $|k| \le K$, we have $|\alpha(k)| \le |a(k)| + \frac{1}{K+1}$. Using the Fourier expansion for $T$, we have

$$\Sigma_1 = \sum_{k=-K}^{K} \alpha(k) \sum_{x=0}^{p-1} e_p(f(x) + kx),$$

and so with $\Phi$ denoting a uniform upper bound on the exponential sum over $x$,

$$|\Sigma_1| \le \Phi \sum_{k=-K}^{K} |\alpha(k)| \le \Phi\Big(|\alpha(0)| + \sum_{1 \le |k| \le K}\Big(|a(k)| + \tfrac{1}{K+1}\Big)\Big).$$

Now $\alpha(0) = \frac{B}{p} \pm \frac{1}{K+1}$ and by Lemma 3.5,

$$\begin{aligned} \sum_{1 \le |k| \le K} |a(k)| &= \frac{2}{p}\sum_{k=1}^{K} \frac{|\sin(\pi k B/p)|}{|\sin(\pi k/p)|} \\ &\le \frac{4}{\pi^2}\log K + \frac{4}{\pi^2}(\gamma + \log 2) + \frac{2}{\pi}\frac{K}{p-K} + \frac{10}{\pi^2 K}. \end{aligned}$$

Thus, for $T = T_-$,

$$|\Sigma_1| \le \Phi\Big(\tfrac{B}{p} - \tfrac{1}{K+1}\Big) + \Phi\Big(\tfrac{4}{\pi^2}\log K + \tfrac{4}{\pi^2}(\gamma + \log 2) + \tfrac{2}{\pi}\tfrac{K}{p-K} + \tfrac{10}{\pi^2 K} + \tfrac{2K}{K+1}\Big)$$
$$= \Phi\left(\tfrac{4}{\pi^2}\log K + C_1 - \tfrac{3}{K+1} + \tfrac{2}{\pi}\tfrac{K}{p-K} + \tfrac{10}{\pi^2 K}\right),$$

where $C_1 = 2 + \frac{4}{\pi^2}(\gamma + \log 2) + \frac{B}{p} < 2.51486 + \frac{B}{p}$, and

$$|\Sigma_1| + |\Sigma_2| \le \tfrac{p}{K+1} + \Phi\left(\tfrac{4}{\pi^2}\log K + C_1 + \tfrac{2}{\pi}\tfrac{K}{p-K} + \tfrac{10}{\pi^2 K} - \tfrac{3}{K+1}\right).$$

Let $K := \lfloor \frac{\pi^2}{4}\sqrt{p} \rfloor$. Noting that the coefficient on $\Phi$ is increasing with $K$, we have

$$|\Sigma_1| + |\Sigma_2| \le \tfrac{4}{\pi^2}\sqrt{p} + \Phi\Big(\tfrac{4}{\pi^2}\log\tfrac{\pi^2}{4}\sqrt{p} + C_1 + \tfrac{2\pi}{4\sqrt{p}-\pi^2} + \tfrac{40}{\pi^4\sqrt{p}} - \tfrac{12}{\pi^2\sqrt{p}}\Big)$$
$$\le \tfrac{2}{\pi^2}\Phi\log p + \tfrac{4}{\pi^2}\sqrt{p} + \Phi\Big(\tfrac{4}{\pi^2}\log\tfrac{\pi^2}{4} + C_1 + \tfrac{2\pi}{4\sqrt{p}-\pi^2} - \tfrac{.805}{\sqrt{p}}\Big).$$

Using $\sqrt{p} \le \Phi$ we get

$$|\Sigma_1| + |\Sigma_2| \le \tfrac{2}{\pi^2}\Phi\log p + \Phi\Big(C_2 + \tfrac{2\pi}{4\sqrt{p}-\pi^2} - \tfrac{.805}{\sqrt{p}}\Big),$$

with

$$C_2 = \tfrac{4}{\pi^2} + \tfrac{4}{\pi^2}\log\tfrac{\pi^2}{4} + C_1 \le 3.28619 + \tfrac{B}{p}.$$

Thus for $p > 43000$ we have

$$|S_p(f,B)| \le |\Sigma_1| + |\Sigma_2| \le \tfrac{2}{\pi^2}\Phi\log p + \Phi(3.29 + \tfrac{B}{p}).$$

For $p < 43000$, the same holds by the second bound in (1.3), already established.

## 5. Proof of Corollary 1.1

In this section we establish Corollary 1.1.

*Proof of Corollary 1.1.* For $p \ge 11$, the corollary follows immediately from Theorem 1.1, that is, $\frac{4}{\pi^2}\log p + 1.35 < \log p$. For $p = 7$, we use Lemma 2.2 to get $\Phi\log p \ge \sqrt{7}\log 7 > 5.148$, and so (1.4) is immediate for $B \le 5$, that is, $|S_p(f,B)| \le B < 5.148 < \Phi\log p$. For $B = 7$, the sum is a complete sum, and so the result again is immediate. For $B = 6$, we have

$$|S_p(f,6)| \le |S_p(f)| + 1 \le \Phi + 1 < \Phi\log 7.$$

For $p = 5$ the same argument holds, that is, for $B \le 3$ or $B = 5$, the result is immediate, while for $B = 4$, we can use $|S_p(f,4)| \le \Phi + 1$ to get the result. $\square$

**Remark 5.1.** For $p = 3$, inequality (1.4) is immediate for $B = 1$ or 3, while for $B = 2$, if $f$ is not constant mod 3 on $[1,2]$, we have $|S_p(f,B)| = |1 + e^{2\pi i/3}| = 1$, yielding the inequality. For $p = 2$, (1.4) is immediate for $B = 1$, while for $B = 2$, if $f$ is not constant mod 2, then $S_p(f,B) = S_p(f) = 0$.

## 6. Proof of Theorem 1.2

Let $\gamma$ denote Euler's constant, $\gamma := .577215664...$ , and

$$R := \frac{2pB(B+1)}{\pi(p^2-B^2)}.$$

**Proposition 6.1.** *For any quadratic polynomial $f(x) = ax^2+bx+c$ with $p \nmid a$, and positive integer $B$ with $\sqrt{p} < B < p/2$, we have*

$$|S_p(f,B)|^2 \le \min\left\{ \frac{2p}{\pi}\log(B^2/p) + 1.366\,p + 3B + R, \right.$$
$$\left. \frac{2p}{\pi}\log B + .368\,p + B + \frac{p}{\pi B} + R \right\}. \tag{6.1}$$

*Proof.* Let $f(x) = ax^2+bx+c$ with $p \nmid a$, and $B$ be an integer with $1 \le B < p/2$. Following the method of Weyl, letting $y = x+h$, we have

$$\begin{aligned} |S_p(f,B)|^2 &= \sum_{x=1}^{B}\sum_{y=1}^{B} e_p(f(y)-f(x)) = B + 2\mathcal{R}\sum_{1\le x<y\le B} e_p(f(y)-f(x)) \\ &= B + 2\mathcal{R}\sum_{h=1}^{B-1}\sum_{x=1}^{B-h} e_p(2hax + ah^2 + bh) \le B + 2\sum_{h=1}^{B-1}\left|\sum_{x=1}^{B-h} e_p(2hax)\right|, \end{aligned} \tag{6.2}$$

and so

$$|S_p(f,B)|^2 \le B + 2\sum_{h=1}^{B-1}\min\left\{B-h,\ \frac{1}{|\sin(2\pi ha/p)|}\right\}. \tag{6.3}$$

We deduce from Lemma 3.3 that for $\sqrt{p} < B < p/2$, we have

$$\begin{aligned} |S_p(f,B)|^2 &\le B + \frac{2p}{\pi}\left(\log(\pi B^2/p) + 1 + \frac{\pi B}{p}\right) + R \\ &= \frac{2p}{\pi}\log(B^2/p) + \frac{2p}{\pi}\log(\pi e) + 3B + R \\ &< \frac{2p}{\pi}\log(B^2/p) + 1.366\,p + 3B + R. \end{aligned}$$

Similarly, by Lemma 3.2,

$$\begin{aligned} |S_p(f,B)|^2 &\le B + \frac{2p}{\pi}\left(\log B + \gamma + \frac{1}{2B}\right) + R \\ &= \frac{2p}{\pi}\log B + \frac{2\gamma}{\pi}p + B + \frac{p}{\pi B} + R \\ &< \frac{2p}{\pi}\log B + .368\,p + B + \frac{p}{\pi B} + R. \end{aligned}$$

$\square$

We can now establish Theorem 1.2.

*Proof of Theorem 1.2.* If $B < 1.52\sqrt{p}$, we have trivially

$$|S_p(f,B)|^2 \le B^2 < \frac{2p}{\pi}\log(16.6B^2/p),$$

since letting $u := \frac{B^2}{p}$, we have $u \le \frac{2}{\pi}\log(16.6u)$ for $u < (1.52)^2$. Thus, we may assume that $B \ge 1.52\sqrt{p}$. Using this restriction on $B$ and computer computation, we can readily verify the theorem for $p < 2000$ using (6.3). Suppose now that $p > 2000$. If $B < .136\,p$ so that $R < .0121\,p$, we use the first bound in (6.1) to get the result provided that

$$\tfrac{2p}{\pi}\log(B^2/p) + 1.366p + 3B + .0121p \le \tfrac{2p}{\pi}\log(16.6B^2/p),$$

that is,

$$1.3781p + 3B \le \tfrac{2p}{\pi}\log(16.6).$$

Since $B < .136p$, the latter holds.

Suppose now that $B = \lambda p$ with $.136 \le \lambda < .5$. The second bound in (6.1) gives the result provided that

$$\tfrac{2p}{\pi}\log B + .368\,p + B + \tfrac{p}{\pi B} + R < \tfrac{2p}{\pi}\log(16.6B^2/p),$$

that is,

$$.368 + \lambda + \tfrac{1}{\pi\lambda p} + \tfrac{2\lambda}{\pi}\tfrac{\lambda+1/p}{1-\lambda^2} \le \tfrac{2}{\pi}\log(16.6\lambda),$$

which is the case for $p > 2000$ and $\lambda \ge .136$. $\square$

For intervals of length $1 \le B \le p$, we have the following uniform bound.

**Corollary 6.1.** *For any quadratic polynomial $f(x) = ax^2 + bx + c$ with $p \nmid a$, and positive integer $B$ with $1 \le B \le p$, we have*

$$|S_p(f,B)| < \sqrt{p} + \sqrt{\tfrac{2}{\pi}p\log p + \tfrac{1}{2}\,p}.$$

This improves on (1.5) for $p < 4.31 \cdot 10^{24}$.

*Proof.* Inserting $B = \frac{p-1}{2}$ into the second expression in (6.1), we get for $p \ge 31$,

$$|S_p(f,\tfrac{p-1}{2})|^2 < \tfrac{2p}{\pi}\log\tfrac{p}{2} + .368p + \tfrac{p}{2} + \tfrac{2}{\pi} + \tfrac{p}{2\pi}\tfrac{p-1}{3p-1} < \tfrac{2}{\pi}p\log p + .5\,p.$$

The same can be verified by computer computation for $p < 31$. Now, it is easy to see that the second expression in (6.1) is increasing with $B$, and so $|S_p(f,B)| \le \sqrt{\frac{2}{\pi}p\log p + .5\,p}$, for any $B \le \frac{p-1}{2}$. If $B > p/2$, we can write $S_p(f(x),B) = S_p(f) - S_p(f(-x),p-B) = \chi_2(a)\mathcal{G}_p - S_p(f(-x),p-B)$ with $\mathcal{G}_p = \sqrt{p}$ or $\sqrt{p}\,i$, and apply the bound from the preceding sentence to the latter sum. $\square$

**Example 6.1.** A nontrivial bound of the sort $|S_p(f,B)| < B/2$ is not possible in general for $B$ slightly smaller than $\sqrt{p}$, as the following example illustrates. For $B < p$ and $B, p$ sufficiently large, we claim that

$$\Big|\sum_{x=1}^{B} e_p(x^2)\Big| > B\Big(1 - \tfrac{4B^4}{p^2}\Big). \tag{6.4}$$

Thus in order to have $|S_p(f,B)| < B/2$, we need $\frac{4B^4}{p^2} \geq \frac{1}{2}$, that is $B > \frac{1}{2^{3/4}}\sqrt{p}$. To establish (6.4), note that $\cos(2x) \geq 1 - 2x^2$, and so the real part of the sum is

$$\begin{aligned}\sum_{x=1}^{B} \cos(2\pi x^2/p) &\geq B - \tfrac{2\pi^2}{p^2}\sum_{x=1}^{B} x^4 = B - \tfrac{2\pi^2}{p^2}\tfrac{B(B+1)(2B+1)(3B^2+3B-1)}{30}\\ &\geq B - \tfrac{2\pi^2}{5}\tfrac{B^5}{p^2}\Big(1 + O(\tfrac{1}{B})\Big).\end{aligned}$$

**Remark 6.1.** One can also make use of van der Corput's Lemma, see, e.g., [18, pg. 18], to prove a result similar to Proposition 6.1. In this case, the analog of the sum over $h$ in (6.3) can be restricted to an interval $1 \leq h \leq H$ with $H < B$, yielding a bound for $|S_p(f,B)|^2$ with dominant term $\frac{2p}{\pi}\frac{B+H}{H+1}\log(BH/p)$. Unfortunately, the slight increase in the factor in front of the logarithm (in comparison to $\frac{2p}{\pi}$) prevents any numeric improvement in our results.

## 7. Lemmas for the Divisor Function

To prove Theorems 1.3 and 1.4, we need the following results about the divisor function $\tau(n)$.

### 7.1. Upper Bound for $\tau(n)$

Nicolas and Robin [20] established the uniform upper bound

$$\tau(n) \leq n^{\frac{1.066018\ldots}{\log\log n}},$$

for $n \geq 2$ with equality at $n = 6983776800 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

### 7.2. Average Value of $\tau(n)$

Next, for any positive real $x$, write

$$\sum_{1 \leq n \leq x} \tau(n) = x\log x + (2\gamma - 1)x + \Delta(x).$$

Dirichlet established that $\Delta(x) = O(\sqrt{x})$ and many improvements in the order of magnitude of $\Delta(x)$ have been given since. It is widely believed that $\Delta(x) =$

$O_\varepsilon(x^{\frac{1}{4}+\varepsilon})$ with the fraction 1/4 being best possible. Little work has been done on obtaining explicit upper bounds for $\Delta(x)$. The standard textbook proof of $\Delta(x) = O(\sqrt{x})$, gives with a little extra care, $|\Delta(x)| \leq \sqrt{x}$ for $x \geq 1$. This was improved to

$$|\Delta(x)| \leq .961\sqrt{x}, \tag{7.1}$$

for $x \geq 1$ by Berkane, Bordellès and Ramaré [1]. They also established the explicit estimate [1, Theorem 1.2],

$$\Delta(x) < .764\, x^{1/3} \log x, \qquad \text{for } x \geq 9995. \tag{7.2}$$

### 7.3. Average Value of $\tau(n)/n$

Finally, consider the sum $\sum_{n=1}^{\infty} \frac{\tau(n)}{n}$. The Laurent expansion

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \zeta^2(s) = \tfrac{1}{(s-1)^2} + \tfrac{2\gamma}{(s-1)} + C + \cdots,$$

with $C = \gamma^2 - 2\gamma_1 = .478809...$, where $\gamma = .577215...$ is Euler's constant and $\gamma_1 = -.0728158...$ is the first Stieltjes constant, gives the asymptotic $\sum_{n=1}^{x} \frac{\tau(n)}{n} \sim \frac{1}{2}\log^2 x + 2\gamma \log x + C$ as $x \to \infty$. This was made explicit in [1, Corollary 2.2]. In the next lemma, we correct and refine their result.

**Lemma 7.1.** *For real $x \geq 2$, we have*

$$\sum_{n \leq x} \tfrac{\tau(n)}{n} = \tfrac{1}{2}\log^2 x + 2\gamma \log x + C + O^*\Big(\tfrac{2.1 \log x}{x^{2/3}}\Big).$$

As noted earlier, the $O^*$ means the constant in the big-oh can be taken to be 1. In [1], the constant $C$ was stated as $\gamma^2 - \gamma_1$ and the big-oh term was $O^*(\frac{1.16}{x^{1/3}})$.

*Proof.* Let $x$ be a positive real number and $c$ a constant such that for $t \geq x$, $\Delta(t) \leq ct^{1/3} \log t$. We have

$$\begin{aligned}
\sum_{n \leq x} \tfrac{\tau(n)}{n} &= \sum_{n \leq x} \tau(n)\Big(\tfrac{1}{x} + \int_n^x \tfrac{dt}{t^2}\Big) \\
&= \tfrac{1}{x}\sum_{n \leq x} \tau(n) + \int_1^2 \tau(1)\tfrac{dt}{t^2} + \int_2^3 (\tau(1)+\tau(2))\tfrac{dt}{t^2} + \cdots \\
&= \tfrac{1}{x}\int_1^x \tau(n) + \int_1^x \Big(\sum_{n \leq t} \tau(n)\Big)\tfrac{dt}{t^2} \\
&= \tfrac{1}{x}\sum_{n \leq x} \tau(n) + \int_1^x \tfrac{1}{t^2}\Big(t\log t + (2\gamma-1)t + \Delta(t)\Big)\, dt \\
&= \tfrac{1}{x}\sum_{n \leq x} \tau(n) + \tfrac{1}{2}\log^2 x + (2\gamma-1)\log x + \int_1^\infty \tfrac{\Delta(t)}{t^2}\, dt - \int_x^\infty \tfrac{\Delta(t)}{t^2}\, dt.
\end{aligned}$$

Now,

$$\int_1^\infty \tfrac{\Delta(t)}{t^2} = \gamma^2 - 2\gamma - 2\gamma_1 + 1;$$

see, e.g., [12, (33)]. Using $|\Delta(x)| \le cx^{1/3}\log x$, we get

$$\begin{aligned}\tfrac{1}{x}\sum_{n\le x}\tau(n) &= \tfrac{1}{x}\Big(x\log x + (2\gamma-1)x + O^*(cx^{1/3}\log x)\Big)\\ &= \log x + 2\gamma - 1 + O^*\Big(\tfrac{c\log x}{x^{2/3}}\Big).\end{aligned}$$

Again, using $|\Delta(t)| \le ct^{1/3}\log t$ for $t \ge x$ and integrating by parts,

$$\Big|\int_x^\infty \tfrac{\Delta(t)}{t^2}\,dt\Big| \le c\int_x^\infty \tfrac{\log t}{t^{5/3}}\,dt = \tfrac{3c\log x}{2x^{2/3}} + \tfrac{3c}{2}\int_x^\infty \tfrac{dt}{t^{5/3}} = \tfrac{3c\log x}{2x^{2/3}} + \tfrac{9c}{4x^{2/3}}.$$

Putting together the above, we get

$$\sum_{n\le x}\tfrac{\tau(n)}{n} = \tfrac{1}{2}\log^2 x + 2\gamma\log x + \gamma^2 - 2\gamma_1 + O^*\Big(\tfrac{5c\log x}{2x^{2/3}} + \tfrac{9c}{4x^{2/3}}\Big).$$

By (7.2) we can take $c = .764$ for $x > 9995$, and thus the error term is bounded by $2.1\log x/x^{2/3}$ for $x > 9995$. Computer computation shows that the same holds for $2 \le x \le 9995$. □

## 8. Proof of Theorem 1.3

Let $C(x) = ax^3 + bx^2 + cx$. Following the method of Weyl as in (6.2), we have

$$\begin{aligned}|S_p(f,B)|^2 &= B + 2\mathcal{R}\sum_{h=1}^{B-1}\sum_{x=1}^{B-h} e_p(f(x+h) - f(x))\\ &\le B + 2\sum_{h=1}^{B-1}\Big|\sum_{x=1}^{B-h} e_p(3ahx^2 + (3ah^2+2bh)x)\Big| = B + 2C, \qquad (8.1)\end{aligned}$$

say, with

$$C := \sum_{h=1}^{B-1}\Big|\sum_{x=1}^{B-h} e_p(3ahx^2 + (3ah^2+2bh)x)\Big|. \qquad (8.2)$$

i) Suppose that $B < \sqrt{p/2}$ and put $\delta = \delta_p = \frac{1.0661}{\log\log p}$. We may assume that $B + 2C > 1.21B^{\frac12}p^{\frac12+\frac{\delta}{2}}\log^{\frac12} p$, else we are already done. Then

$$\frac{B}{2C} \le \frac{B}{1.21B^{\frac12}p^{\frac12+\frac{\delta}{2}}\log^{\frac12} p - B} \le \frac{1}{1.43p^{\frac{\delta}{2}}p^{\frac14}\log^{\frac12} p - 1} := \lambda_p,$$

and

$$|S_p(f,B)|^2 \le B + 2C \le 2\lambda_p C + 2C = 2(1+\lambda_p)C.$$

Now, by Cauchy's inequality,

$$C^2 \le (B-1)\sum_{h=1}^{B-1}\left|\sum_{x=1}^{B-h} e_p(3ahx^2 + (3ah^2+2bh)x)\right|^2.$$

Applying (6.3) to the quadratic sum over $x$ gives

$$\begin{aligned}
C^2 &\le (B-1)\sum_{h=1}^{B-1}\left(B-h+2\sum_{j=1}^{B-h}\min\left\{B-h,\ \frac{1}{|\sin(6ajh\pi/p)|}\right\}\right)\\
&\le (B-1)\left(\frac{B(B-1)}{2}+2\sum_{h=1}^{B-1}\sum_{j=1}^{B-1}\min\left\{B-1,\ \frac{1}{|\sin(6ajh\pi/p)|}\right\}\right)\\
&= \tfrac{1}{2}B(B-1)^2 + 2(B-1)\sum_{1\le n<p} N(n)\min\left\{B-1,\ \frac{1}{|\sin(6an\pi/p)|}\right\}, \qquad (8.3)
\end{aligned}$$

where

$$N(n) = \#\{(h,j)\in\mathbb{Z}^2 : hj\equiv n \pmod p,\ 1\le h,j\le B-1\}. \qquad (8.4)$$

Note that for $p/2 < n < p$, we have $N(n) = 0$ since $(B-1)^2 < p/2$, while for $n < p/2$, we have $N(n) < \tau(n) < p^{\frac{1.0661}{\log(\log p)}} = p^\delta$. Thus,

$$C^2 \le \tfrac{1}{2}B(B-1)^2 + 2(B-1)p^\delta \sum_{n=1}^{(p-1)/2}\min\left\{B-1,\ \frac{1}{|\sin(6an\pi/p)|}\right\}.$$

Now, as in the proof of Lemma 3.2, the sum over $n$ is maximized when $6a\equiv 1 \bmod p$, where it is bounded by

$$\begin{aligned}
\sum_{n=1}^{(p-1)/2}\min\left\{B-1,\ \frac{1}{|\sin(n\pi/p)|}\right\} &\le \sum_{n<p/(\pi B)} B + \sum_{p/(\pi B)<n<p/2}\frac{p}{\pi}\Big(\frac{1}{n}+\frac{2n}{p^2-n^2}\Big)\\
&< \frac{p}{\pi}+\frac{p}{\pi}\log\tfrac{\pi B}{2} + B + \frac{2p}{\pi(p^2-(p/2)^2)}\sum_{1\le n\le (p-1)/2} n\\
&= \frac{p}{\pi}\log B + \frac{p}{\pi}\left(1+\log(\pi/2)\right) + B + \frac{p}{3\pi}\frac{p^2-1}{p^2}\\
&\le \frac{p}{\pi}\log B + .5682\,p + B. \qquad (8.5)
\end{aligned}$$

Thus for $B < \sqrt{p/2}$,

$$\begin{aligned}
C^2 &\le \tfrac{1}{2}B(B-1)^2 + 2(B-1)p^\delta\left(\frac{p}{\pi}\log B + .5682\,p + B\right)\\
&\le \tfrac{1}{4}pB + 2Bp^\delta\big(\tfrac{p}{2\pi}\log p + .4579\,p + B\big),
\end{aligned}$$

and

$$|S_p(f,B)|^4 \le 4(1+\lambda_p)^2\left(\tfrac14 pB + 2Bp^\delta\left(\tfrac{p}{2\pi}\log p + .4579\,p + B\right)\right) \\ \le (1.1)^4\,Bp^{1+\delta}\log p,$$

for $p > 10^9$. For $p < 10^9$ and $B < \sqrt{p/2}$, we trivially have $|S_p(f,B)|^4 \le B^4 < (1.1)^4 Bp^{1+\delta}\log p$. Thus in all cases, if $B < \sqrt{p/2}$, then $|S_p(f,B)| \le 1.1\,(Bp)^{\frac14} p^{\delta/4}\log^{\frac14} p$.

ii) Suppose that $B > \sqrt{p/2}$. If $B > p^{2/3}$, then by (1.4),

$$|S_p(f,B)| < \Phi\log p \le 2\sqrt{p}\log p < 1.38 B^{\frac34} p^{\delta/2}\log^{\frac14} p.$$

Assume now that $B < p^{2/3}$. We have trivially

$$|S_p(f,B)| \le B \le 1.38\,B^{\frac34} p^{\frac{\delta}{2}}\log^{\frac14} p,$$

for $B < (1.38)^4 p^{2\delta}\log p$. Since $B < p^{\frac23}$, it suffices to have $p < 1.27\cdot 10^{20}$.

Suppose now that $p > 1.27\cdot 10^{20}$. Using the notation in (8.1), we may assume that $B + 2C > (1.38)^2 B^{\frac32} p^\delta \log^{\frac12} p$, and so

$$|S_p(f,B)|^2 \le B + 2C \le 2(1+\lambda'_p)C,$$

with

$$\lambda'_p := \frac{1}{1.6\,p^{\frac14+\delta}\log^{\frac12} p - 1}.$$

For any $n \le p$ there are at most $\frac{B^2}{p}+1$ integers $m$ with $1 \le m \le B^2$ and $m \equiv n \bmod p$, and thus with $N(n)$ as defined in (8.4),

$$N(n) \le (\tfrac{B^2}{p}+1)\max_{m\le B^2}\tau(m) \le (\tfrac{B^2}{p}+1)p^{2\delta}.$$

Thus from the analog of (8.3), allowing $n > p/2$, we have

$$C^2 \le \tfrac12 B(B-1)^2 + 2(B-1)\sum_{1\le n<p} N(n)\min\left\{B-1,\ \frac{1}{|\sin(6an\pi/p)|}\right\} \\ \le \tfrac12 B(B-1)^2 + 2(B-1)(\tfrac{B^2}{p}+1)p^{2\delta}\sum_{1\le n<p}\min\left\{B-1,\ \frac{1}{|\sin(6an\pi/p)|}\right\} \\ \le \tfrac12 B(B-1)^2 + 4(B-1)(\tfrac{B^2}{p}+1)p^{2\delta}\sum_{1\le n<p/2}\min\left\{B-1,\ \frac{1}{|\sin(n\pi/p)|}\right\}.$$

By (8.5), it follows that

$$C^2 \le \tfrac12 B^3 + 4B(\tfrac{B^2}{p}+1)p^{2\delta}\left(\tfrac{p}{\pi}\log B + .5682\,p + B\right),$$

and so for $p > 1.27 \cdot 10^{20}$, noting that $\lambda'_p < 10^{-10}$ and $B < p^{2/3}$, we have

$$\begin{aligned}
|S_p(f,B)|^4 &\le 4(1+\lambda'_p)^2\Big(\tfrac{1}{2}B^3 + 4B(\tfrac{B^2}{p}+1)p^{2\delta}\big(\tfrac{p}{\pi}\log B + .5682\,p + B\big)\Big) \\
&\le 4.0001\left(\tfrac{4}{\pi}\tfrac{2}{3}p^{2\delta}B^3\log p\,(1+\tfrac{p}{B^2})\Big(1+\tfrac{3\pi(.5682)}{2\log p}+\tfrac{3\pi}{2p^{\frac{1}{3}}\log p}+\tfrac{3\pi}{16p^{2\delta}\log p(1+1/p^{1/3})}\Big)\right) \\
&\le (1.38)^4 B^3\,\big(1+\tfrac{p}{B^2}\big)\,p^{2\delta}\log p,
\end{aligned}$$

as one can confirm by computation.

## 9. Proof of Theorem 1.4

Let $f(x) = x^3 + bx^2 + cx$ and suppose that $4p^{\frac{1}{3}} < B < \sqrt{p/12}$. In particular, $p > 7 \cdot 10^6$. Our goal is to show

$$|S_p(f,B)| < (Bp)^{\frac{1}{4}}\log^{\frac{1}{4}}(B^3/p)\log^{\frac{1}{4}}(Bp).$$

We have $|S_p(f,B)|^2 \le B + 2C$ with $C$ as given in (8.2), and so we are done if $B + 2C \le (Bp)^{\frac{1}{2}}$. Suppose now that $2C > (Bp)^{\frac{1}{2}} - B$, implying

$$\frac{B}{2C} \le \frac{1}{(p/B)^{\frac{1}{2}} - 1}.$$

From $B < \sqrt{p/12}$, we have $p/B > \sqrt{12p}$, and thus

$$\frac{B}{2C} \le \beta_p := \frac{1}{(12p)^{\frac{1}{4}} - 1} < .0106.$$

Then $B + 2C \le 2(1+\beta_p)C$ and

$$|S_p(f,B)|^4 \le (B+2C)^2 \le 4(1+\beta_p)^2C^2,$$

where by (8.3),

$$C^2 \le \tfrac{1}{2}B(B-1)^2 + 2(B-1)\sum_{1\le n<p} N(n)\min\Big\{B-1,\ \frac{1}{|\sin(6n\pi/p)|}\Big\}.$$

Noting that $N(n) = 0$ for $n > (B-1)^2$ and $N(n) \le \tau(n)$ for $n \le (B-1)^2$, the sum over $n$ is at most

$$(B-1)\sum_{1\le n\le \frac{p}{6B}}\tau(n) + \sum_{\frac{p}{6B}<n\le B^2}\frac{\tau(n)}{|\sin(6n\pi/p)|} = \Sigma_1 + \Sigma_2$$

say. By (7.1),

$$\begin{aligned}\Sigma_1 &\le (B-1)\left(\tfrac{p}{6B}\log\tfrac{p}{6B} + (2\gamma-1)\tfrac{p}{6B} + \sqrt{\tfrac{p}{6B}}\,\right)\\ &< \tfrac{p}{6}\log\tfrac{p}{6B} + \tfrac{p}{6}(2\gamma-1) + \sqrt{pB/6},\end{aligned}$$

and by Lemmas 3.1, 7.1, (7.2) and the fact that $\frac{6n}{p} < \frac{6B^2}{p} < \frac{1}{2}$, we have

$$\begin{aligned}\Sigma_2 &\le \sum_{\frac{p}{6B}<n\le B^2} \frac{\tau(n)}{\pi}\left(\frac{p}{6n} + \frac{2(6n)/p}{1-(6n/p)^2}\right) \le \sum_{\frac{p}{6B}<n\le B^2} \frac{\tau(n)}{\pi}\left(\frac{p}{6n}+\frac{4}{3}\right)\\ &= \frac{p}{6\pi}\sum_{\frac{p}{6B}<n\le B^2}\frac{\tau(n)}{n} + \frac{4}{3\pi}\sum_{\frac{p}{6B}<n\le B^2}\tau(n)\\ &\le \tfrac{p}{6\pi}\left(\tfrac{1}{2}\log^2 B^2 - \tfrac{1}{2}\log^2\tfrac{p}{6B} + 2\gamma\log\tfrac{6B^3}{p} + \tfrac{4.2\log(p/6B)}{(p/6B)^{2/3}}\,\right)\\ &\qquad + \tfrac{4}{3\pi}\Big(B^2\log B^2 - \tfrac{p}{6B}\log\tfrac{p}{6B} + (2\gamma-1)(B^2-\tfrac{p}{6B}) + 2B\Big).\end{aligned}$$

The dominant term in $\Sigma_1+\Sigma_2$ is

$$\tfrac{p}{12\pi}(\log^2 B^2 - \log^2\tfrac{p}{6B}) = \tfrac{p}{12\pi}\log\tfrac{6B^3}{p}\log\tfrac{Bp}{6}.$$

By computer computation, we find that for $4p^{\frac{1}{3}} < B < \sqrt{p/12}$, we have

$$\frac{\Sigma_1+\Sigma_2}{p\log(6B^3/p)\log(Bp/6)} < .0437.$$

Thus, for $B$ in this range,

$$\begin{aligned}|S_p(f,B)|^4 &\le 4(1+\beta_p)^2C^2 \le 4(1.0106)^2C^2\\ &\le 4.086\Big(\tfrac{1}{2}B(B-1)^2 + 2(B-1)(.0437)p\log(6B^3/p)\log(Bp/6)\Big)\\ &\le 4.086\Big(\tfrac{1}{24}Bp + (.0874)Bp\log(6B^3/p)\log(Bp/6)\Big)\\ &< .358\,Bp\log(6B^3/p)\log(Bp/6),\end{aligned}$$

yielding the desired bound.

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