PROBABILISTIC PROOFS OF PASCAL'S RECURSION FORMULAS FOR SUMS OF POWERS

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Abstract

This note provides probabilistic proofs of the original Pascal's recurrence formula and the version involving the alternating sum.

1. Introduction

Let $n \geq 1$ and $k \geq 0$ be integers. Putting $S_k(n) = 1^k + 2^k + \cdots + n^k$, we see that $S_0(n) = n,\ S_1(n) = n(n+1)/2,\ S_2(n) = n(n+1)(2n+1)/6$, and $S_3(n) = \{n(n+1)/2\}^2$. The sum has been studied for a long time (see Edwards [2], Beardon [1], and references therein). It is known that Faulhaber's formula provides a concrete expression for $S_m(n)$ (see, e.g., Beardon [1, Equation (2.2)], Knuth [6]). While it involves Bernoulli numbers, we do not need to use it in recurrence formulas. We call the equation

$$S_m(n) = \frac{(n+1)^{m+1} - 1}{m+1} - \frac{1}{m+1} \sum_{r=0}^{m-1} \binom{m+1}{r} S_r(n) \quad \text{for } m \geq 1 \tag{1}$$

the *original Pascal's recursion formula* (see Farhadian [3, Equation (4)]). A modified one,

$$S_m(n) = \frac{n(n+1)^m}{m+1} - \frac{1}{m+1} \sum_{r=1}^{m-1} \binom{m}{r-1} S_r(n) \quad \text{for } m \geq 1, \tag{2}$$

was given by Farhadian [3, Equation (6)]. Moreover, Hu and Zhong [5, Equation (5)] and [3, Equation (5)] wrote the *recursion formula involving the alternating sum*

$$S_m(n) = \frac{n^{m+1}}{m+1} + \sum_{r=0}^{m-1} \binom{m}{r} \frac{(-1)^{m-r+1}}{m-r+1} S_r(n) \quad \text{for } m \geq 1. \tag{3}$$

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For example, for $m = 4$, calculating each of Equations (1), (2), and (3) using $S_0(n), \ldots, S_3(n)$ yields

$$S_4(n) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

We provide notes on Equations (1), (2), and (3) as follows.

Remark 1. We note Equation (1). As stated by Edwards [2, p. 23] and Beardon [1, Section 2, p. 202], Pascal's identity

$$(n+1)^{m+1} - 1 = \sum_{r=0}^{m} \binom{m+1}{r} S_r(n) \quad \text{for } m \geq 1 \tag{4}$$

was found by Pascal himself in 1654. Equation (4) is proved by summing over

$$(l+1)^{m+1} - l^{m+1} = \sum_{r=0}^{m} \binom{m+1}{r} l^r \quad \text{for } l = 1, \ldots, n.$$

Solving Equation (4) for $S_m(n)$ gives us Equation (1).

Remark 2. We note Equation (2). A simple calculation shows that Equations (1) and (2) are equivalent. In fact, the difference between the right-hand side of Equations (1) and (2) times $(m+1)$ is

$$\begin{aligned}(n+1)^{m+1} - 1 - n(n+1)^m - \sum_{r=1}^{m-1} \left\{ \binom{m+1}{r} - \binom{m}{r-1} \right\} S_r(n) - n \\ = (n+1)^m - 1 - \sum_{r=0}^{m-1} \binom{m}{r} S_r(n) = 0,\end{aligned}$$

where the last equality follows from Equation (4).

Remark 3. We note Equation (3). Pascal's identity involving the alternating sum is

$$n^{m+1} = \sum_{r=0}^{m} \binom{m+1}{r} (-1)^{m-r} S_r(n) \quad \text{for } m \geq 1, \tag{5}$$

which is proved by summing over

$$l^{m+1} - (l-1)^{m+1} = \sum_{r=0}^{m} \binom{m+1}{r} (-1)^{m-r} l^r \quad \text{for } l = 1, 2, \ldots, n.$$

Thus, since it follows that

$$n^{m+1} = \sum_{r=0}^{m-1} \frac{m+1}{m-r+1} \binom{m}{r} (-1)^{m-r} S_r(n) + (m+1) S_m(n),$$

solving for $S_m(n)$ gives us Equation (3).

Farhadian [3] showed Equation (2) using the tail sum formula for the moment of a non-negative integer-valued random variable. Hu and Zhong [5] proved Equation (3) using a probabilistic method involving the convolution of independent random variables, which is somewhat technical.

In this short note, motivated by Farhadian [3] and Hu and Zhong [5], we also provide probabilistic proofs of both Equations (1) and (3). Due to Remark 2, our probabilistic proof may be considered alternative one of Equation (2). These proofs are carried out by calculating the expected values of random variables. Let Y be a random variable, and let $\{A_i : i \in I\}$ be a countable partition of the sample space. Then,

$$\mathrm{E}(Y) = \sum_{l \in I} \mathrm{E}(Y|A_l)\mathrm{P}(A_l),$$

a result sometimes referred to as the *partition theorem* (see, e.g., Grimmett and Stirzaker [4, Theorem 3.7.4, p. 72]). We apply this theorem to the restricted and unrestricted moments of the standard discrete uniform random variable.

2. Probabilistic Proofs

In this section, we use a random variable X, which is *discrete and uniformly* distributed over $\{1, \ldots, n\}$ with $\mathrm{P}(X = l) = 1/n$ for $l = 1, \ldots, n$. Let $\mathbb{I}\{A\}$ denote the indicator function of an event A (see [4, Example 2.1.9, p. 31]). We now provide proofs of Equations (1) and (3).

Proof of Equation (1): Let us assume $n \geq 2$. We firstly consider the $(m+1)$th moment of X as follows:

$$\mathrm{E}(X^{m+1}) = \sum_{l=1}^{n} l^{m+1}\mathrm{P}(X = l) = \frac{1}{n}\sum_{l=1}^{n} l^{m+1} = \frac{1}{n}S_{m+1}(n). \tag{6}$$

Let $\nu \geq 1$ be an integer and $p \in (0, 1)$. Let $\mathrm{Bin}(\nu, p)$ be a binomial random variable, independent of X, with parameters ν and p. Consider the event $A_r(\nu, p) = \{\mathrm{Bin}(\nu, p) = r\}$ for $r = 0, \ldots, \nu$. We then see $\mathbb{I}\{\cup_{r=0}^{m+1} A_r(\nu, p)\} = 1$. Use the partition theorem on the events $\{X = l\}_{l=1,2,\ldots,n}$ to obtain

$$\mathrm{E}(Y) = \sum_{l=1}^{n} \mathrm{E}(Y|X = l)\mathrm{P}(X = l),$$

where $Y = X^{m+1}\mathbb{I}\{\cup_{r=0}^{m+1} A_r(m+1, 1/X)\}$. We then have

$$\begin{aligned}
\mathrm{E}(X^{m+1}) &= \mathrm{E}(X^{m+1}\mathbb{I}\{\cup_{r=0}^{m+1} A_r(m+1,1/X)\}) \\
&= \sum_{l=1}^{n} \mathrm{E}(X^{m+1}\mathbb{I}\{\cup_{r=0}^{m+1} A_r(m+1,1/X)\}|X=l)\mathrm{P}(X=l) \\
&= \frac{1}{n}\sum_{l=1}^{n} l^{m+1}\mathrm{E}(\mathbb{I}\{\cup_{r=0}^{m+1} A_r(m+1,1/l)\}|X=l) \\
&= \frac{1}{n}\sum_{l=1}^{n}\sum_{r=0}^{m+1} l^{m+1}\mathrm{P}\left(\mathrm{Bin}(m+1,1/l)=r\right).
\end{aligned}$$

The last equality holds from the independence of X and $\mathrm{Bin}(m+1, 1/l)$. Now write the term corresponding to $l = 1$ separately from the rest to obtain

$$\begin{aligned}
\mathrm{E}(X^{m+1}) &= \frac{1}{n}\left\{\sum_{r=0}^{m+1}\sum_{l=2}^{n} l^{m+1}\mathrm{P}\left(\mathrm{Bin}(m+1,1/l)=r\right) + \sum_{r=0}^{m+1}\mathrm{P}\left(\mathrm{Bin}(m+1,1)=r\right)\right\} \\
&= \frac{1}{n}\left\{\sum_{r=0}^{m+1}\binom{m+1}{r}\sum_{l=2}^{n}(l-1)^r + 1\right\} = \frac{1}{n}\left\{\sum_{r=0}^{m+1}\binom{m+1}{r}S_r(n-1)+1\right\}.
\end{aligned} \tag{7}$$

Using Equations (6), (7), and

$$\sum_{r=0}^{m+1}\binom{m+1}{r}S_r(n-1) = \sum_{r=0}^{m-1}\binom{m+1}{r}S_r(n-1)+(m+1)S_m(n-1)+S_{m+1}(n)-n^{m+1},$$

we have

$$\sum_{r=0}^{m-1}\binom{m+1}{r}S_r(n-1)+(m+1)S_m(n-1)-n^{m+1}+1=0.$$

Solving for $S_m(n-1)$ yields

$$S_m(n-1) = \frac{n^{m+1}-1}{m+1} - \frac{\sum_{r=0}^{m-1}\binom{m+1}{r}S_r(n-1)}{m+1},$$

which completes the proof of Equation (1). □

Proof of Equation (3): Consider continuous independent random variable $U_1, \ldots, U_{m+1}$, uniformly distributed over $[0, 1]$ (see, e.g., [4, Section 4.4.1, p. 106]) and also assume these are independent of X. Let $V_{m+1} = \min_{1\le i\le m+1} U_i$. Therefore,

$$\mathrm{P}\left(V_{m+1}\le t\right) = 1-\mathrm{P}\left(V_{m+1}>t\right) = 1-(1-t)^{m+1} \quad \text{for } t\in(0,1), \tag{8}$$

(see also [7, Equation (2)]). Expand the right-hand side of Equation (8) gives

$$\mathrm{P}\left(V_{m+1} \leq t\right) = \sum_{r=0}^{m} \binom{m+1}{r} (-1)^{m-r} t^{m-r+1}. \tag{9}$$

It follows from the partition theorem for the events $\{X = l\}_{l=1,2,\ldots,n}$ that

$$\begin{aligned}
&\mathrm{E}(X^{m+1}\mathbb{I}\{XV_{m+1} \leq 1\}) \\
&\quad = \sum_{l=1}^{n} \mathrm{E}(X^{m+1}\mathbb{I}\{XV_{m+1} \leq 1\} | X = l)\mathrm{P}(X = l) \\
&\quad = \frac{1}{n}\sum_{l=1}^{n} l^{m+1}\mathrm{P}(V_{m+1} \leq l^{-1}).
\end{aligned} \tag{10}$$

Substituting Equation (8) into Equation (10) gives

$$\mathrm{E}(X^{m+1}\mathbb{I}\{XV_{m+1} \leq 1\}) = \frac{1}{n}\sum_{l=1}^{n}\left\{l^{m+1} - (l-1)^{m+1}\right\} = n^m.$$

Similarly, substituting Equation (9) into Equation (10) gives

$$\mathrm{E}(X^{m+1}\mathbb{I}\{XV_{m+1} \leq 1\}) = \frac{1}{n}\sum_{r=0}^{m} \binom{m+1}{r} (-1)^{m-r} S_r(n).$$

Hence we have Equation (5), which completes the proof of Equation (3). $\square$

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