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ENUMERATION OF CATALAN AND SMOOTH WORDS ACCORDING TO CAPACITY

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Abstract

A *bargraph* is a sequence of rectangles lying in the first quadrant each of unit width and positive integral length whose widths are flush with the x-axis starting from the origin. By the *capacity* of a bargraph, we mean the amount of a liquid that would be retained when poured over the bargraph from above by virtue of its shape. In this paper, we consider the capacity statistic on two classes of words satisfying certain growth restrictions, represented geometrically as bargraphs. A *Catalan* word $w = w_1 \cdots w_n$ is one with positive integer entries such that $w_{i+1} - w_i \leq 1$ for $1 \leq i \leq n-1$, with $w_1 = 1$. Let $\mathcal{A}_n$ denote the set of Catalan words of length n, which are enumerated by the n-th Catalan number C_n for all $n \geq 1$. We derive an explicit formula for the generating function of the capacity distribution on $\mathcal{A}_n$ and study further properties of this distribution such as the number of members of $\mathcal{A}_n$ achieving the maximum and minimum capacity. A similar treatment is provided for the set of *smooth* words, that is, those satisfying the condition $|w_{i+1} - w_i| \leq 1$ for $1 \leq i \leq n-1$ instead, and also study the distribution of capacity on a restricted class of smooth words. As special cases of our results in the various cases, we obtain infinite series identities involving the reciprocals of Chebyshev polynomials.

1. Introduction

Given a word $v = v_1 \cdots v_n$ on the alphabet of positive integers, we represent v as a bargraph consisting of n columns flush with the x-axis and comprised of unit squares such that the height of the i-th column is given by v_i for $1 \leq i \leq n$. We will frequently identify a word with its bargraph representation and use the terms *word* and *bargraph* interchangeably when speaking of a statistic on either. By a *water*

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cell of column i in v, where $2 \leq i \leq n-1$, we mean a unit square in this column at height p such that $v_i < p < \min\{v_j, v_\ell\}$ for some $j < i < \ell$. Let k_i for $2 \leq i \leq n-1$ denote the number of water cells of column i in v. Then define the *capacity* of v by $\sum_{i=2}^{n-1} k_i$, which will be denoted by $\text{cap}(v)$. For example, if $v = 122341123212323$, then $\text{cap}(v) = 10$ as illustrated below in Figure 1, where the individual water cells are shaded. We assume any word of length less than three to have capacity zero, the sum in question being vacuous in this case. It is easy to see that all (weakly) monotonic words have capacity zero as well.

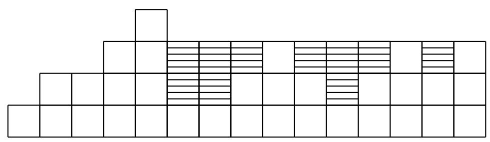


Figure 1: The bargraph v with cap(v)=10, where the water cells are shaded.

Informally, the capacity of a bargraph B corresponds to the total number of cells that would be immersed when a large amount of a liquid is poured over the top of B from above. Here, it is understood that liquid in which there is no basin for it to collect would flow off of B altogether (with no restriction at the left or right endpoints of B to stop the flow). Then a water cell corresponds to a square s that would be immersed in the liquid by virtue of their being a column of strictly greater height both to the left and to the right of the column of B above which s lies, thereby forming a basin containing s where water would collect. Thus, in addition to addressing a new class of restrictions on certain classes of k-ary words, our results below could be of potential interest from a physical standpoint as bargraphs are used to model various kinds of surfaces, particularly those satisfying certain growth restrictions.

The capacity parameter has been considered on a variety of discrete structures represented sequentially as bargraphs, among them, compositions [7, 15], geometrically distributed words [3], k-ary words [6], permutations [9] and finite set partitions [15]. In [8], a comparable study was made on the capacity of Dyck paths, represented graphically as a sequence of diagonal up and down steps. This research continues work on the subject of bargraph enumeration with respect to various statistics initiated in [21] and [12], where a formula was found for the bivariate generating function tracking the number of horizontal and up steps in a bargraph, viewed as a first quadrant lattice path. Further, there has been some interest in generating function formulas related to bargraphs in statistical physics [19, 20], where they are used to model certain kinds of polymers. See also the review paper [14] on bargraph enumeration and references contained therein.

By a *Catalan* word $w = w_1 \cdots w_n$, we mean one with positive integer entries satisfying $w_1 = 1$ with $w_{i+1} - w_i \leq 1$ for all $i \in [n-1]$. Catalan words have been studied recently in the context of pattern avoidance [4, 5] and the exhaustive generation of Gray codes for growth-restricted words [17]. The distribution of various parameters has also been considered on Catalan words, among them, area [10, 16], semi-perimeter [10, 16] and the number of interior lattice points [13]. Let $\mathcal{A}_n$ denote the set of Catalan words of length n. For example, if $n = 4$, we have

$$\mathcal{A}_4 = \{1111, 1121, 1211, 1221, 1231, 1112, 1122,\\ 1212, 1222, 1232, 1123, 1223, 1233, 1234\}.$$

The cardinality of $\mathcal{A}_n$ is given by the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geq 1$; see, for example, [24, Exercise 80].

By a *smooth* word $w = w_1 \cdots w_n$, we mean one where $w_i \geq 1$ for all i with $w_1 = 1$ such that $|w_{i+1} - w_i| \leq 1$ for each $i \in [n-1]$. Smooth words were briefly considered in [16], where the generating function of the joint distribution for the statistics recording the sums of the ascent tops and level values was found. Let $\mathcal{R}_n$ denote the set of smooth words of length n. For example, we have $\mathcal{R}_4 = \mathcal{A}_4 - \{1231\}$, where $\mathcal{A}_4$ is as given above. Then $|\mathcal{R}_n|$ for all $n \geq 1$ is given by sequence A005773[n] in the OEIS [18], which we will denote here by L_n. Note that L_n for $n \geq 1$ enumerates the set $\mathcal{L}_n$ of lattice paths from $(0, 0)$ to the line $x = n - 1$ that never go below the x-axis and use $u = (1, 1)$, $d = (1, -1)$ and $h = (1, 0)$ steps, the members of which are known as *Motzkin left factors* (see, for example, [2, p. 111]). Upon encoding each difference $w_{i+1} - w_i$ for $i \in [n-1]$ by u, d or h according to if it equals 1, -1 or 0, respectively, one may identify each $w \in \mathcal{R}_n$ with a member of $\mathcal{L}_n$, which will be denoted by $\jmath(w)$. For example, if $w = 12322122 \in \mathcal{R}_8$, then $\jmath(w) = uudhduh \in \mathcal{L}_8$. Note that the final letter of w corresponds to one more than the final height of $\jmath(w)$ for all w. At times, we will identify members of $\mathcal{R}_n$ with their corresponding lattice paths in $\mathcal{L}_n$ under $\jmath$.

In this paper, we study the distribution of the capacity statistic on $\mathcal{A}_n$ and $\mathcal{R}_n$. This extends not only recent work concerning the capacity distribution on other discrete structures but also work pertaining to the distribution of various statistics on $\mathcal{A}_n$ or $\mathcal{R}_n$. Further, one obtains apparently new polynomial generalizations of the underlying counting sequences C_n and L_n as a result. Indeed, we consider the joint distribution of capacity with the final letter statistic on the respective structures leading to a bivariate generalization. Note that by restricting the distribution polynomial for capacity on $\mathcal{R}_n$ for $n \geq 1$ to those members that end in 1, one obtains the corresponding distribution on a subset of the smooth words that are equinumerous with the Motzkin paths of length $n - 1$. In this way, one obtains a q-generalization of the Motzkin number sequence as well.

The organization of this paper is as follows. In the next section, we consider the capacity distribution on Catalan words and compute a formula for its generating function. To do so, we make use of a system of recurrences satisfied by the distribution on $\mathcal{A}_n$ and two auxiliary arrays, which leads to a linear system in the associated generating functions that can be solved explicitly using Cramer's rule. We also determine the degree of the distribution polynomial on $\mathcal{A}_n$ and find that it depends upon n mod 3. A simple formula in terms of Fibonacci numbers is given for the number of members of $\mathcal{A}_n$ with zero capacity and also for the number of such members ending in a fixed letter. Algebraic and combinatorial proofs are provided for this result. In the third section, a comparable treatment is presented for the set of smooth words of length n. We remark that a similar system of linear recurrences can be found for the associated arrays, though a somewhat different technique–one involving direct iteration–is needed to ascertain the formula for the generating function of the distribution in this case. As a consequence of our results, we obtain a pair of infinite series identities involving the reciprocals of Chebyshev polynomials (see Corollary 5 below).

In the final section, we consider the capacity distribution on restricted smooth words wherein no two adjacent 1's are allowed. Note that the cardinality of the set of such smooth words of length n is given by the grand Motzkin number G_{n-1} for all $n \geq 1$, whereas the subset of these words ending in 1 is enumerated by the Riordan number R_{n-1}. Thus, one obtains q-generalizations for these sequences in terms of the capacity distribution. Further, as a consequence of our results, we obtain, in the case $q = 1$, new infinite series expansions for the generating functions of G_n and R_n in terms of Chebyshev polynomials (Theorem 8). Finally, an explicit bijection is found between the set of Motzkin left factors with no horizontal steps on the x-axis and the set of grand Motzkin paths of the same length (Theorem 11), which establishes the cardinality of the restricted class of smooth words under consideration.

2. Distribution of Capacity on Catalan Words

Given $n \geq 1$ and $1 \leq i \leq n$, let $\mathcal{A}_{n,i}$ denote the set of Catalan words of length n ending in i and hence $\mathcal{A}_n = \cup_{i=1}^{n} \mathcal{A}_{n,i}$. Let $a(n, i; q)$ denote the distribution (polynomial) of the capacity statistic on $\mathcal{A}_{n,i}$, that is,

$$a(n, i; q) = \sum_{\pi \in \mathcal{A}_{n,i}} q^{\mathrm{cap}(\pi)},$$

and let $a(n; q) = \sum_{i=1}^{n} a(n, i; q)$ be the corresponding distribution on $\mathcal{A}_n$. The q argument in the counting functions $a(n, i; q)$ and $a(n; q)$ (and also in others that follow) will often be suppressed in cases where the usual meaning is understood. We wish to determine information on $a(n)$ and $a(n, i)$. The problem of finding a recurrence for $a(n, i)$ by itself seems difficult, if not intractable. Alternatively, it is possible to write a system of recurrences satisfied by $a(n,i)$ and two other arrays as follows by considering the restriction of the capacity distribution to a certain subset of $\mathcal{A}_{n,i}$.

Given $n \geq 1$ and $1 \leq i \leq n$, let $\mathcal{B}_{n,i}$ denote the subset of $\mathcal{A}_{n,i}$ containing those members whose largest letter is also i and let $\mathcal{B}_n = \cup_{i=1}^{n}\mathcal{B}_{n,i}$. Let $b(n,i;q)$ and $b(n;q)$ denote the respective distributions of the capacity statistic on $\mathcal{B}_{n,i}$ and $\mathcal{B}_n$. In order to write a system of recurrences for $a(n,i)$ and $b(n,i)$, we need to consider a further array enumerating a certain class of words that satisfy the Catalan growth restriction. Given $m,j \geq 1$, let $\mathcal{C}_{m,j}$ denote the set of words $w = w_1 \cdots w_m$ with positive integer letters that both end in and have greatest letter j such that $w_{i+1} \leq w_i + 1$ for each $i \in [m-1]$. Given $\ell \in [j]$, let $\mathcal{C}_{m,j,\ell}$ denote the subset of $\mathcal{C}_{m,j}$ consisting of those members starting with ℓ. Let $c(m,j,\ell;q)$ denote the distribution of the area statistic on $\mathcal{C}_{m,j,\ell}$, where the area is that of the corresponding bargraphs, and let $c(m,j;q) = \sum_{\ell=1}^{j} c(m,j,\ell;q)$ be the area distribution on $\mathcal{C}_{m,j}$.

Considering cases based on the second letter within a member of $\mathcal{C}_{m,j,\ell}$ where $m \geq 2$ yields the following recurrence for $c(m,j,\ell)$, where it is assumed $c(m,j,\ell) = 0$ if $\ell > j$.

Lemma 1. *If $m \geq 2$, $j \geq 1$ and $\ell \in [j]$, then*

$$c(m,j,\ell) = q^{\ell} \sum_{i=1}^{\ell+1} c(m-1,j,i), \tag{1}$$

with $c(1,j,\ell) = q^{\ell} \cdot \delta_{j,\ell}$.

Then $a(n,i)$ and $b(n,i)$ satisfy the following system of recurrences in terms of $c(m,j)$.

Lemma 2. *The arrays $a(n,i)$ and $b(n,i)$ are given recursively by*

$$\begin{aligned} a(n,i) &= b(n-1,i-1) + \sum_{j=i}^{n-1} a(n-1,j) \\ &\quad + \sum_{m=1}^{n-i-1} \sum_{j=i}^{n-m-1} q^{mi} a(n-m-1,j) c(m,i-1;1/q), \qquad 1 < i < n, \end{aligned} \tag{2}$$

and

$$\begin{aligned} b(n,i) &= b(n-1,i) + b(n-1,i-1) \\ &\quad + \sum_{m=1}^{n-i-1} q^{mi} b(n-m-1,i) c(m,i-1;1/q), \qquad 1 < i < n, \end{aligned} \tag{3}$$

where $c(m,j)$ is as given in Lemma 1, $a(n,1) = a(n-1)$ for all $n \geq 2$ and $a(n,n) = b(n,1) = b(n,n) = 1$ for all $n \geq 1$.

Proof. The boundary conditions when $i = 1$ or $i = n$ follow from observing that each of the sets $\mathcal{A}_{n,n}$, $\mathcal{B}_{n,1}$ and $\mathcal{B}_{n,n}$ are singletons whose sole member has capacity zero, whereas the members of $\mathcal{A}_{n,1}$ arise by appending 1 to members of $\mathcal{A}_{n-1}$ with this operation leaving the capacity unchanged. To show (2), let $\pi \in \mathcal{A}_{n,i}$ where $1 < i < n$ and we consider cases based on the penultimate letter j of π. If $j \geq i$, then deleting the terminal i results in an arbitrary member of $\mathcal{A}_{n-1,j}$ of equal capacity, which accounts for the first summation on the right side of (2). So assume $j = i-1$, the only other option. If no other letter $\geq i$ exists in π, then we have $\pi = \pi' i$, where $\pi' \in \mathcal{B}_{n-1,i-1}$ with $\mathrm{cap}(\pi) = \mathrm{cap}(\pi')$. Hence, there are $b(n-1, i-1)$ possibilities for π in this case. Otherwise, there exists some letter $\geq i$ other than the last, the rightmost of which must occur in position $n-m-1$ for some $m \in [n-i-1]$.

That is, we have $\pi = \pi' j \pi'' i$, where $|\pi''| = m$, $j \in [i, n-m-1]$ and $\max(\pi'') = i-1$ with the final letter of π'' also equal to $i-1$. Hence, we have $\pi' j \in \mathcal{A}_{n-m-1,j}$ and $\pi'' \in \mathcal{C}_{m,i-1}$, with $\mathrm{cap}(\pi) = \mathrm{cap}(\pi' j) + \mathrm{cap}(j\pi'' i)$. Note that since $\max(\pi'') < i \leq j$, we have $\mathrm{cap}(j\pi'' i) = \mathrm{cap}(i\pi'' i) = mi - \mathrm{area}(\pi'')$. Hence, the weight of all members $\pi \in \mathcal{A}_{n,i}$ of the stated form for a fixed m and j is given by the product $a(n-m-1, j) \cdot q^{mi} c(m, i-1; 1/q)$. Considering all possible m and j then accounts for the second summation on the right side and completes the proof of (2). A similar argument applies to (3), the principle differences being that the first summation on the right is replaced by $b(n-1, i)$ and $j = i$ is required in the decomposition $\pi = \pi' j \pi'' i$ above with $\pi' i$ belonging to $\mathcal{B}_{n-m-1,i}$. Figure 2 below illustrates a particular case of the decomposition $\pi = \pi' j \pi'' i \in \mathcal{A}_{n,i}$ used above, where $n = 14$, $m = 4$, $i = 3$ and $j = 4$. $\square$

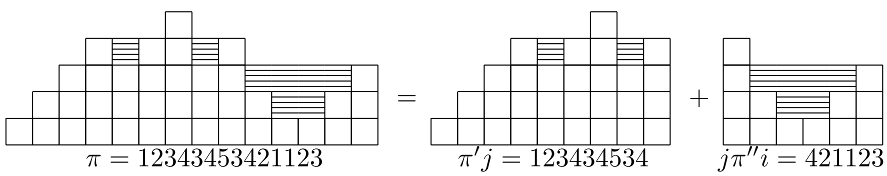


Figure 2: The Catalan word $\pi = 12343453421123 \in \mathcal{A}_{14,3}$, with $\mathrm{cap}(\pi)$ given by $\mathrm{cap}(\pi' j) + \mathrm{cap}(j\pi'' i) = 2 + 6 = 8$.

We have the following expression for the generating functions

$$C_{j,i}(x; q) = \sum_{n\geq 1} c(n, j, i; q)x^n \text{ and } C_j(x; q) = \sum_{n\geq 1} \sum_{i=1}^{j} c(n, j, i; q)x^n,$$

where $j \geq 1$ and $i \in [j]$.

Lemma 3. *Let $d_j = d_j(x; q)$ be defined recursively by*

$$d_j = (1 - xq^j)d_{j-1} - x^2 q^{2j-1} d_{j-3}, \qquad j \geq 3, \tag{4}$$

with initial values $d_0 = 1$, $d_1 = 1 - xq$ *and* $d_2 = 1 - xq - xq^2$. *Then we have*

$$C_{j,i}(x;q) = \frac{x^{j-i+1} q^{\binom{j+1}{2} - \binom{i}{2}} d_{i-1}(x;q)}{d_j(x;q)}, \qquad 1 \le i \le j, \tag{5}$$

and hence

$$C_j(x;q) = \frac{1}{d_j(x;q)} \sum_{i=1}^{j} x^{j-i+1} q^{\binom{j+1}{2} - \binom{i}{2}} d_{i-1}(x;q), \qquad j \ge 1. \tag{6}$$

Proof. Formula (6) follows from (5) and the definitions, and hence we need only prove (5). Multiplying both sides of (1) by x^m and summing over all $m \ge 2$, we obtain

$$C_{j,\ell}(x;q) = xq^\ell \cdot \delta_{j,\ell} + xq^\ell \sum_{i=1}^{\ell+1-\delta_{j,\ell}} C_{j,i}(x;q), \tag{7}$$

for $\ell = 1, \ldots, j$. Define the matrix $\mathbf{A} = (\mathbf{A}_{ab})_{1 \le a,b \le j}$, where

$$\mathbf{A}_{ab} = \begin{cases} 1 - xq^a, & a = b; \\ -xq^a, & b = 1, 2, \ldots, a-1, a+1, \text{ with } a < j, \text{ or if } b < a = j; \\ 0, & \text{otherwise.} \end{cases}$$

Let $e_j = e_j(x;q)$ denote the determinant of the matrix $\mathbf{A}$. By applying Cramer's rule to the linear $j \times j$ system (7) in the quantities $C_{j,i}(x;q)$ for a fixed $j \ge 1$, one has

$$C_{j,i}(x;q) = \frac{x^{j+1-i} q^{\binom{j+1}{2} - \binom{i}{2}} e_{i-1}}{e_j}, \qquad 1 \le i \le j,$$

where $e_0 = 1$. Thus, to complete the proof of (5), we need to show $e_j = d_j$ for all $j \ge 1$.

One may assume $j \ge 4$, since the equality is readily verified for $1 \le j \le 3$. For $j \ge 4$, define the matrices $\mathbf{B} = (\mathbf{B}_{ab})_{1 \le a,b \le j-1}$ and $\mathbf{C} = (\mathbf{C}_{ab})_{1 \le a,b \le j-2}$ by

$$\mathbf{B}_{ab} = \begin{cases} 1 - xq^a, & a = b < j-1; \\ -xq^a, & b = 1, 2, \ldots, a-1, a+1, \text{ with } a < j-1; \\ -xq^j, & a = j-1; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{C}_{ab} = \begin{cases} 1 - xq^a, & a = b < j-2; \\ -xq^a, & b = 1, 2, \ldots, a-1, a+1, \text{ with } a < j-2; \\ -xq^j, & a = j-2; \\ 0, & \text{otherwise.} \end{cases}$$

Expanding along the final column of $\mathbf{A}$ and $\mathbf{B}$ gives

$$e_j = (1 - xq^j) e_{j-1} + xq^{j-1} \det(\mathbf{B}) \text{ and } \det(\mathbf{B}) = -xq^j e_{j-2} + xq^{j-2} \det(\mathbf{C}).$$

By the linearity of the determinant in the last row, we have

$$\frac{1}{q^2}\det(\mathbf{C})+\det(\mathbf{D})=e_{j-2},$$

where $\mathbf{D}=(\mathbf{D}_{ab})_{1\le a,b\le j-2}$ is obtained from $\mathbf{C}$ by replacing the final row with the vector $(0,\dots,0,1)$ of size $j-2$. From the definitions, we have $\det(\mathbf{D})=e_{j-3}$ and thus

$$\det(\mathbf{B})=-xq^je_{j-2}+xq^j(e_{j-2}-e_{j-3})=-xq^je_{j-3},\qquad j\ge 4.$$

Hence, e_j satisfies the same recurrence as d_j for $j\ge 4$, which implies $e_j=d_j$ for all $j\ge 1$, as desired. □

Let $U_n=U_n(x)$ denote the n-th Chebyshev polynomial of the second kind (see, for example, [22]) given by $U_n=2xU_{n-1}-U_{n-2}$ for $n\ge 2$, with $U_0=1$ and $U_1=2x$. We have the following explicit formulas for $C_{j,i}(x;q)$ and $C_j(x;q)$ at $q=1$ in terms of Chebyshev polynomials.

Corollary 1. *For all $j\ge 1$ and $i\in[j]$,*

$$C_{j,i}(x;1)=\frac{x^{(j-i+1)/2}U_i(t)}{U_{j+1}(t)}\quad \text{and}\quad C_j(x;1)=\frac{U_{j-1}(t)}{U_{j+1}(t)},\tag{8}$$

where $t:=\frac{1}{2\sqrt{x}}$.

Proof. Note $d_j(x;1)=(1-x)d_{j-1}(x;1)-x^2d_{j-3}(x;1)$ for $j\ge 3$, with $d_0(x;1)=1$, $d_1(x;1)=1-x$ and $d_2(x;1)=1-2x$, by (4). By induction on j, we have $d_j(x;1)=x^{(j+1)/2}U_{j+1}(t)$ for all $j\ge 0$. Hence, by Lemma 3, we get

$$C_{j,i}(x;1)=\frac{x^{(j-i+1)/2}U_i(t)}{U_{j+1}(t)}$$

and

$$C_j(x;1)=\frac{1}{U_{j+1}(t)}\sum_{i=1}^{j}x^{(j-i+1)/2}U_i(t)=\frac{U_{j-1}(t)}{U_{j+1}(t)},$$

where in the last equality we used $\sum_{i=1}^{j}x^{(j-i+1)/2}U_i(t)=U_{j-1}(t)$, which can be shown by induction on j. □

Define the generating function $B_i(x;q)=\sum_{n\ge i}b(n,i;q)x^n$ for $i\ge 1$.

Lemma 4. *If $i\ge 1$, then*

$$B_i(x;q)=\frac{x^i}{\prod_{j=1}^{i}\left(1-x-\frac{x}{d_{j-1}(q^jx;1/q)}\sum_{m=1}^{j-1}x^{j-m}q^{j(j-m)+\binom{m}{2}-\binom{j}{2}}d_{m-1}(q^jx;1/q)\right)},\tag{9}$$

where $d_j(x;q)$ is given by (4).

Proof. Multiplying both sides of (3) by x^n and summing over $n \geq i+1$, we obtain

$$\begin{aligned} B_i(x;q) &= xB_i(x;q) + xB_{i-1}(x;q) + \sum_{m\geq 1}\sum_{n\geq i} q^{mi} b(n,i)c(m,i-1;1/q)x^{n+m+1} \\ &= xB_i(x;q) + xB_{i-1}(x;q) + xB_i(x;q)\sum_{m\geq 1} q^{mi}c(m,i-1;1/q)x^m, \end{aligned}$$

which leads to

$$B_i(x;q) = \frac{xB_{i-1}(x;q)}{1-x-xC_{i-1}(q^i x;1/q)}, \qquad i \geq 2, \tag{10}$$

with $B_1(x;q) = \frac{x}{1-x}$. Iteration of (10), together with use of (6), completes the proof. $\square$

Corollary 2. *If $i \geq 1$, then*

$$B_i(x;1) = \frac{x^{(i-1)/2}}{U_{i+1}(t)}, \tag{11}$$

where $t := \frac{1}{2\sqrt{x}}$.

Proof. Let us assume $C_0(x;q) = U_{-1}(x) = 0$. By (10) at $q=1$ and (8), we then get

$$\begin{aligned} B_i(x;1) &= \prod_{j=0}^{i-1}\frac{x}{1-x-xC_j(x;1)} = \prod_{j=0}^{i-1}\frac{xU_{j+1}(t)}{(1-x)U_{j+1}(t)-xU_{j-1}(t)} \\ &= \prod_{j=0}^{i-1}\frac{xU_{j+1}(t)}{U_{j+1}(t)-x^{1/2}U_j(t)} = \prod_{j=0}^{i-1}\frac{x^{1/2}U_{j+1}(t)}{U_{j+2}(t)} = \frac{x^{(i-1)/2}}{U_{i+1}(t)}, \end{aligned}$$

where we have used the fact $U_{j+1}(t) = x^{-1/2}U_j(t) - U_{j-1}(t)$. $\square$

We can now establish explicit formulas for the generating functions of the distribution of the capacity statistic on $\mathcal{A}_n$. Let $A_i(x;q) = \sum_{n\geq i} a(n,i;q)x^n$ and $A(x;q) = \sum_{n\geq 1}\sum_{i=1}^n a(n,i;q)x^n$. For convenience, we will take $B_0(x;q) = 1$ and $C_0(x;q) = 0$.

Theorem 1. *We have*

$$\begin{aligned} A_i(x;q) = x\sum_{m=1}^{i}(-x)^{m-1}\sum_{1\leq i_1<\cdots<i_{m-1}<i_m=i}\Big(&(1+C_{i_1-1}(q^{i_1}x;1/q))A(x;q) \\ &+ B_{i_1-1}(x;q)\Big)\prod_{j=2}^{m}(1+C_{i_j-1}(q^{i_j}x;1/q)) \end{aligned} \tag{12}$$

and

$$A(x;q) = \frac{x\sum_{m\geq 1}(-x)^{m-1}\sum_{1\leq i_1<\cdots<i_m} B_{i_1-1}(x;q)\prod_{j=2}^{m}(1+C_{i_j-1}(q^{i_j}x;1/q))}{1-x\sum_{m\geq 1}(-x)^{m-1}\sum_{1\leq i_1<\cdots<i_m}\prod_{j=1}^{m}(1+C_{i_j-1}(q^{i_j}x;1/q))}, \tag{13}$$

where $B_i(x;q)$ *and* $C_i(x,q)$ *are as given in Lemmas 3 and 4.*

Proof. Multiplying both sides of (2) by x^n and summing over $n \geq i+1$, we obtain

$$\begin{aligned}
A_i(x;q) &= \sum_{n\geq i+1}\sum_{j=i}^{n-1} a(n-1,j)x^n + xB_{i-1}(x;q) \\
&\quad + \sum_{n\geq i+2}\sum_{m=1}^{n-i-1}\sum_{j=i}^{n-m-1} q^{mi}a(n-m-1,j)c(m,i-1;1/q)x^n \\
&= x\sum_{j\geq i}A_j(x;q) + xB_{i-1}(x;q) \\
&\quad + x\sum_{n\geq 1}\sum_{m=1}^{n}\sum_{j=i}^{n+i-m} q^{mi}a(n+i-m,j)c(m,i-1;1/q)x^{n+i} \\
&= x\sum_{j\geq i}A_j(x;q) + xB_{i-1}(x;q) \\
&\quad + x\sum_{m\geq 1}\sum_{n\geq i}\sum_{j=i}^{n} q^{mi}a(n,j)c(m,i-1;1/q)x^{n+m} \\
&= x\sum_{j\geq i}A_j(x;q) + xB_{i-1}(x;q) \\
&\quad + x\sum_{j\geq i}\sum_{m\geq 1} q^{mi}A_j(x;q)c(m,i-1;1/q)x^m,
\end{aligned}$$

which leads to

$$A_i(x;q) = x(1+C_{i-1}(q^ix;1/q))\sum_{j\geq i}A_j(x;q) + xB_{i-1}(x;q), \qquad i\geq 2. \tag{14}$$

Since $B_0(x;q) = 1$ and $C_0(x;q) = 0$, we have that (14) is seen to hold also for $i = 1$, as $a(n,1) = a(n-1)$ for $n \geq 2$ with $a(1,1) = 1$ implies $A_1(x;q) = x + xA(x;q)$. Since $A(x;q) = \sum_{i\geq 1} A_i(x;q)$, one may rewrite (14) as

$$\begin{aligned}
A_i(x;q) &= x(1+C_{i-1}(q^ix;1/q))A(x;q) \\
&\quad + xB_{i-1}(x;q) - x(1+C_{i-1}(q^ix;1/q))\sum_{j=1}^{i-1}A_j(x;q), \qquad i\geq 1.
\end{aligned} \tag{15}$$

To determine $A_i(x;q)$, consider, more generally, a recurrence of the form

$$u_i = k_i + \ell_i \sum_{j=1}^{i-1} u_j, \qquad i \geq 1,$$

where k_i and ℓ_i denote arbitrary sequences. By induction on i, one can show

$$u_i = \sum_{m=1}^{i} \sum_{1 \leq i_1 < \cdots < i_{m-1} < i_m = i} k_{i_1} \prod_{j=2}^{m} \ell_{i_j}, \qquad i \geq 1. \tag{16}$$

Applying formula (16) to (15) yields (12). Summing (12) over $i \geq 1$ implies

$$\begin{aligned} A(x;q) = x \sum_{m \geq 1} (-x)^{m-1} \sum_{1 \leq i_1 < \cdots < i_m} \Big(&(1 + C_{i_1-1}(q^{i_1}x; 1/q))A(x;q) \\ &+ B_{i_1-1}(x;q) \Big) \prod_{j=2}^{m} (1 + C_{i_j-1}(q^{i_j}x; 1/q)), \end{aligned}$$

and solving for $A(x;q)$ in the last equality gives (13) and completes the proof. □

Let $C = C(x)$ denote the Catalan number generating function $\sum_{n \geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$.

Corollary 3. *For all $i \geq 1$, we have $A_i(x;1) = x^i C^i(x)$.*

Proof. We provide two proofs, the first of which will show how the equality follows from the preceding results. By (14), we have that $v_i = A_i(x;1)$ satisfies

$$v_i = x(1 + C_{i-1}(x;1)) \sum_{j \geq i} v_j + xB_{i-1}(x;1), \qquad i \geq 1, \tag{17}$$

where $\sum_{j \geq 1} v_j = C - 1$. We demonstrate that $(xC)^i$ also satisfies (17), whence the result follows. By Corollaries 1 and 2, in order to do so, we must show

$$(xC)^i = x \left(1 + \frac{U_{i-2}(t)}{U_i(t)}\right) \sum_{j \geq i} (xC)^j + \frac{x^{i/2}}{U_i(t)},$$

which is equivalent to

$$(xC)^i U_i(t) = (xC)^{i+1}(U_i(t) + U_{i-2}(t)) + x^{i/2}, \qquad i \geq 1. \tag{18}$$

We prove (18) by induction on i, the $i = 1$ case reading $xC/\sqrt{x} = (xC)^2/\sqrt{x} + x^{1/2}$, which is true by $C = 1 + xC^2$. Let us assume the result for some $i \geq 1$ and prove it for $i + 1$:

$$(xC)^{i+1} U_{i+1}(t) = (xC)^{i+2}(U_{i+1}(t) + U_{i-1}(t)) + x^{(i+1)/2}.$$

Applying $U_{i+1}(t) = \frac{1}{\sqrt{x}}U_i(t) - U_{i-1}(t)$, we have

$$(xC)^{i+1}\frac{1}{\sqrt{x}}U_i(t)(1-xC) = (xC)^{i+1}U_{i-1}(t) + x^{(i+1)/2},$$

which, by the fact $1 - xC = \frac{1}{C}$, is equivalent to

$$(xC)^i U_i(t) = (xC)^{i+1}\frac{1}{\sqrt{x}}U_{i-1}(t) + x^{i/2}.$$

Thus, by $U_i(t) = \frac{1}{\sqrt{x}}U_{i-1}(t) - U_{i-2}(t)$, we obtain

$$(xC)^i U_i(t) = (xC)^{i+1}(U_i(t) + U_{i-2}(t)) + x^{i/2},$$

which holds by the induction hypothesis.

Alternatively, this formula may also be explained combinatorially as follows. Decompose $\pi \in \mathcal{A}_{n,i}$ according to the last occurrence of each letter $j \in [i]$; that is, write $\pi = \pi^{(1)}1\pi^{(2)}2\cdots\pi^{(i)}i$, where the section $\pi^{(j)}$ for $1 \leq j \leq i$ contains only letters in $\{j, j+1, \ldots\}$. Note that for each $\pi^{(j)}$, the sequence $\pi^{(j)} - (j-1)$ is a possibly empty Catalan word. By the definitions and the fact that $|\mathcal{A}_n| = C_n$ for all $n \geq 0$, we have $A_i(x;1) = \sum_{n\geq i}|\mathcal{A}_{n,i}|x^n = (xC)^i$, as desired. □

Remark 1. The $q = 1$ cases of the array $a(n,i)$ and of the row sum $b(n) = \sum_{i=1}^{n} b(n,i)$ correspond to the respective entries A033184 and A287709 of the OEIS [18], which may be realized bijectively.

We have the following result for the maximum capacity achieved by a member of $\mathcal{A}_n$.

Theorem 2. *If $n \geq 1$, then*

$$deg(a(n;q)) = \begin{cases} 3\binom{m}{2}, & \text{if } n = 3m; \\ \frac{m(3m-1)}{2}, & \text{if } n = 3m+1; \\ \frac{m(3m+1)}{2}, & \text{if } n = 3m+2, \end{cases} \tag{19}$$

where $deg(f(q))$ denotes the degree of a polynomial $f(q)$.

Proof. We first define some terms related to the capacity of $w = w_1\cdots w_n \in \mathcal{A}_n$ before proceeding with the proof of (19). Consider a string $w_x w_{x+1}\cdots w_y$ within w such that $y \geq x+2$ and $w_x, w_y > \max\{w_{x+1}, \ldots, w_{y-1}\}$. Note that w a Catalan word implies $w_x \geq w_y$, for otherwise $w_{y-1} \geq w_y - 1 \geq w_x$, contrary to the assumption on w_x. This prompts the following definition. By a *special string* of w, we mean a subsequence of consecutive letters $w_a w_{a+1}\cdots w_b$ such that $b \geq a+2$ and $w_a \geq w_b > \max\{w_{a+1}, \ldots, w_{b-1}\}$. Note that special strings correspond to sections of w capable of holding water within the bargraph representation and hence contribute positively towards the capacity. Let us refer to a special string that is not strictly contained within any other special strings (that is, is maximal with respect to containment) as a *reservoir*.

Note that a reservoir corresponds to a maximal connected region where water would collect when poured over the bargraph of w from above. Further, it is seen that two distinct reservoirs of w consist of disjoint strings of letters except for possibly the last letter of one coinciding with the first of the other. Suppose a reservoir of w is given by the string $\alpha = w_a \cdots w_b$. By an *i-reservoir*, we mean one in which $w_b = i$. Define the depth of an entry w_r, where $a+1 \leq r \leq b-1$, within the i-reservoir α as $i - w_r$, that is, the number of water cells of column r. Note that the capacity of w is given by the sum of the capacities of its various reservoirs and that the capacity of a reservoir equals the sum of the depths of its intermediate entries. Let us say that $w = w_1 \cdots w_n \in \mathcal{A}_n$ has *height* j if the greatest i for which there exists an i-reservoir in w is j. For example, the word v in Figure 1, which is a member of $\mathcal{A}_{15}$, has three reservoirs and is of height three. Let $\mathcal{A}_n^{(j)}$ denote the subset of $\mathcal{A}_n$ consisting of its height j members. Note that $\mathcal{A}_n^{(j)}$ is nonempty only when $n \geq 4$ and $j \in [2, n-2]$.

We now wish to maximize the capacity over all members of $\mathcal{A}_n^{(j)}$. Note first that for any $\pi \in \mathcal{A}_n$, the columns in the bargraph of π corresponding to the initial occurrence of each $i \in [\ell]$, where $\ell = \max(\pi)$, all fail to contain any water cells. Further, if π contains a reservoir, at least one other column will not contain any water cells as well, namely, one forming the right boundary of a reservoir. Thus, if $\pi \in \mathcal{A}_n^{(j)}$, there are at least $j+1$ columns that do not contain a water cell. Of the remaining entries of π, there must be one of depth d for each $d \in [j-2]$, by the growth restriction on members of $\mathcal{A}_n$. This leaves at most $n-(2j-1)$ entries of π that can have the greatest possible depth of $j-1$. From the preceding observations, it follows that if $2 \leq j \leq \lfloor n/2 \rfloor$, then the maximum capacity of a member of $\mathcal{A}_n^{(j)}$ is achieved (only) by $\pi = 12\cdots j1^{n-2j+1}23\cdots j$. On the other hand, if $\lfloor n/2 \rfloor < j \leq n-2$, then it is seen that the maximal π is given (only) by $\pi = 12\cdots j(2j-n+1)(2j-n+2)\cdots j$ in this case. Let us denote the uniquely determined π for which the capacity statistic is maximized over $\mathcal{A}_n^{(j)}$ by $\pi^{(j)}$. Then, to ascertain the degree of the polynomial $a(n;q)$, it suffices to find $\max\{\mathrm{cap}(\pi^{(j)}) : 2 \leq j \leq n-2\}$, where we may assume $n \geq 4$. Since $\mathrm{cap}(\pi^{(t)}) < \mathrm{cap}(\pi^{\lfloor n/2 \rfloor})$ for $t = \lfloor n/2 \rfloor + 1, \ldots, n-2$, as one may verify, we need only find $M := \max\{\mathrm{cap}(\pi^{(j)}) : 2 \leq j \leq \lfloor n/2 \rfloor\}$.

Note that $\mathrm{cap}(\pi^{(j)}) = \binom{j}{2} + (j-1)(n-2j)$ for $2 \leq j \leq \lfloor n/2 \rfloor$. So to determine M, let us consider the function $f(x)$ of a real variable given by $f(x) = \frac{x^2-x}{2} + (x-1)(n-2x)$. Then we have that $f(x)$ achieves its maximum value at $x_0 = \frac{n}{3} + \frac{1}{2}$. We now consider cases on n mod 3. If $n = 3m$ for some $m \geq 2$, then we have that M is achieved both when $j = m$ and $j = m+1$, since $x_0 = m + \frac{1}{2}$ and hence the graph of $f(x)$ is symmetric with respect to the vertical line $x = m + \frac{1}{2}$ in this case.

This implies $M = 3\binom{m}{2}$, which yields the first formula in (19). If $n = 3m+1$ for some $m \geq 1$, then $x_0 = m + \frac{5}{6}$ in this case and M is achieved only by $j = m+1$, as it is the value of j that is closest to the x-coordinate of the vertex in the graph of $f(x)$. This gives $M = \binom{m+1}{2} + m(m-1) = \frac{m(3m-1)}{2}$, and hence the second formula of (19). Finally, if $n = 3m+2$, then $x_0 = m + \frac{7}{6}$ and again M is achieved only when $j = m+1$. One then gets $M = \frac{m(3m+1)}{2}$ in this case, which yields the third formula in (19) and completes the proof. □

It is also possible to deduce from the preceding argument the leading two coefficients in the polynomial $a(n;q)$.

Corollary 4. *Let $\alpha = deg(a(n;q))$ and c_n and d_n denote the coefficients of q^α and $q^{\alpha-1}$ in $a(n;q)$, respectively. If $n \geq 4$, then $c_n = 2$ if n is divisible by 3, with $c_n = 1$, otherwise. If $n \geq 7$, then*

$$d_n = \begin{cases} 2m, & \text{if } n = 3m; \\ m+1, & \text{if } n = 3m+1; \\ m+2, & \text{if } n = 3m+2, \end{cases} \tag{20}$$

with $d_4 = 13$, $d_5 = 7$ and $d_6 = 8$.

Proof. We will draw upon the notation and terminology used in the proof of Theorem 2. The first statement is apparent from the preceding proof since we saw that both $\pi^{(m)}$ and $\pi^{(m+1)}$ and no others achieved the maximum capacity α when $n = 3m$, with only $\pi^{(m+1)}$ doing so otherwise. For the second statement, let us refer to $\pi \in \mathcal{A}_n$ for which $\text{cap}(\pi) = \alpha - 1$ as being *near maximal*. If $n = 3m$ where $m \geq 3$, then a near maximal member of $\mathcal{A}_{3m}$ cannot occur in $\mathcal{A}_{3m}^{(j)}$ for $j \neq m, m+1$ as

$$\text{cap}(\pi^{(m-1)}) = \binom{m-1}{2} + (m-2)(n-2m+2) = \frac{3(m+1)(m-2)}{2} < 3\binom{m}{2} - 1$$

and since $\text{cap}(\pi^{(j)}) \leq \text{cap}(\pi^{(m-1)})$ for all $j \in [2, 3m-2] - \{m, m+1\}$, by the symmetry of the graph of $y = f(x)$ with respect to the line $x = m + \frac{1}{2}$ in this case. Further, $m \geq 3$ implies a near maximal member of $\mathcal{A}_{3m}$ belonging to $\mathcal{A}_{3m}^{(m)} \cup \mathcal{A}_{3m}^{(m+1)}$ cannot contain two or more reservoirs. To realize this, note that by observations made in the third paragraph of the proof of Theorem 2, if $\rho \in \mathcal{A}_{3m}^{(m)} \cup \mathcal{A}_{3m}^{(m+1)}$ where $m \geq 3$ were to contain two or more reservoirs, then the sum of the depths of ρ, that is, $\text{cap}(\rho)$, would be at least $m-1$ less than $\text{cap}(\pi^{(m)})$ if $\rho \in \mathcal{A}_{3m}^{(m)}$ and at least m less than $\text{cap}(\pi^{(m+1)})$ if $\rho \in \mathcal{A}_{3m}^{(m+1)}$. In either case, ρ would fail to be near maximal.

Thus, a near maximal $\pi \in \mathcal{A}_{3m}^{(m)} \cup \mathcal{A}_{3m}^{(m+1)}$ where $m \geq 3$ must contain a single reservoir. It is then seen that the only way in which one can achieve $\text{cap}(\pi) = \alpha - 1$

is for one of the 1's within the interior string of 1's in $\pi^{(j)} = 12\cdots j1^{n-2j+1}23\cdots j$ to be changed to a 2, with all other entries remaining the same. This yields $m+1$ near maximal π in $\mathcal{A}_{3m}^{(m)}$, and $m-1$ in $\mathcal{A}_{3m}^{(m+1)}$, for a total of $2m$ altogether, which implies the first case of (20). If $n = 3m+1$ with $m \geq 2$, then similar reasoning applies and a near maximal member belonging to $\mathcal{A}_{3m+1}^{(m+1)}$ is one obtained from $\pi^{(m+1)} = 12\cdots(m+1)1^m 23\cdots(m+1)$ by again changing an intermediate 1 to a 2, yielding m possibilities. In addition, one must also check if $\pi^{(m)}$ is near maximal as $\mathrm{cap}(\pi^{(j)}) < \mathrm{cap}(\pi^{(m)}) < \mathrm{cap}(\pi^{(m+1)})$ for all $j \in [2, 3m-1] - \{m, m+1\}$, by the symmetry of $f(x)$ with respect to the line $x = m + \frac{5}{6}$ in this case. Note

$$\mathrm{cap}(\pi^{(m)}) = \binom{m}{2} + (m-1)(n-2m) = \frac{(m-1)(3m+2)}{2} = \frac{m(3m-1)}{2} - 1,$$

and indeed $\pi^{(m)}$ is near maximal. Since there are no other possibilities for a near maximal member of $\mathcal{A}_{3m+1}$, the second case of (20) is established. Finally, a similar argument applies if $n = 3m+2$ with $m \geq 2$, the near maximal π in this case consisting of the $m+1$ members of $\mathcal{A}_{3m+2}^{(m+1)}$ obtained by changing an intermediate 1 within $\pi^{(m+1)} = 12\cdots(m+1)1^{m+1}23\cdots(m+1)$ to a 2, together with $\pi^{(m+2)}$. This yields the third formula in (20) and completes the proof. $\square$

We now consider the case when $q = 0$ in $a(n,i)$ and $a(n)$. Let F_n denote the Fibonacci number defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with $F_0 = 0$, $F_1 = 1$. We will need the following pair of Fibonacci identities, where it is assumed $F_{-1} = 1$.

Lemma 5. *If $n \geq 1$ and $2 \leq i \leq n+1$, then*

$$\sum_{m=0}^{n-i+1} F_{2m-1}\binom{n-m-1}{i-2} = \binom{n-1}{i-2} + \sum_{j=i}^{n}\sum_{m=0}^{n-j} F_{2m-1}\binom{n-m-2}{j-2}, \tag{21}$$

and for all $n \geq 1$, we have

$$F_{2n} = \sum_{j=1}^{n}\sum_{m=0}^{n-j} F_{2m-1}\binom{n-m-1}{j-1}. \tag{22}$$

Proof. To show (21), we compute the generating function of both sides over $n \geq i-1$ for a fixed $i \geq 2$. Now

$$\begin{aligned}\sum_{n\geq i-1}\sum_{m=0}^{n-i+1} F_{2m-1}\binom{n-m-1}{i-2}x^n &= \sum_{m\geq 0} F_{2m-1}\sum_{n\geq i+m-1}\binom{n-m-1}{i-2}x^n \\ &= \sum_{m\geq 0} F_{2m-1}x^{m+1}\sum_{n\geq i-2}\binom{n}{i-2}x^n = \frac{1-2x}{1-3x+x^2}\cdot\left(\frac{x}{1-x}\right)^{i-1},\end{aligned}$$

upon recalling $\sum_{m\geq 0} F_{2m-1}x^m = \frac{1-2x}{1-3x+x^2}$. As for the right side of (21), we have

$$\sum_{n\geq i-1}\binom{n-1}{i-2}x^n + \sum_{n\geq i}\sum_{j=i}^{n}\sum_{m=0}^{n-j} F_{2m-1}\binom{n-m-2}{j-2}x^n$$
$$= \left(\frac{x}{1-x}\right)^{i-1} + \sum_{j\geq i}\sum_{n\geq 0}\sum_{m=0}^{n} F_{2m-1}\binom{n-m+j-2}{j-2}x^{n+j}.$$

Thus, to complete the proof of (21), we need to show

$$\sum_{j\geq i}\sum_{n\geq 0}\sum_{m=0}^{n} F_{2m-1}\binom{n-m+j-2}{j-2}x^{n+j}$$
$$= \left(\frac{x}{1-x}\right)^{i-1}\left(\frac{1-2x}{1-3x+x^2}-1\right) = \frac{x^i}{(1-x)^{i-2}(1-3x+x^2)}, \qquad i\geq 2.$$

Now

$$\sum_{j\geq i}\sum_{n\geq 0}\sum_{m=0}^{n} F_{2m-1}\binom{n-m+j-2}{j-2}x^{n+j} = \sum_{j\geq i}\sum_{m\geq 0} F_{2m-1}\sum_{n\geq j-2}\binom{n}{j-2}x^{n+m+2}$$
$$= \sum_{j\geq i}\frac{x^j}{(1-x)^{j-1}}\sum_{m\geq 0}F_{2m-1}x^m = \frac{\frac{x^i}{(1-x)^{i-1}}}{1-\frac{x}{1-x}}\cdot\frac{1-2x}{1-3x+x^2}$$
$$= \frac{x^i}{(1-x)^{i-2}(1-3x+x^2)},$$

as desired. A similar proof can be given for (22). □

Using (5), one can show $C_{j,i}(q^{j+1}x;1/q)\mid_{q=0}= 0$ for $1\leq i\leq j$ and hence $C_j(q^{j+1}x;1/q)\mid_{q=0}= 0$ for all $j\geq 1$. By (10) at $q=0$, we then have $B_i(x;0) = \frac{x^i}{(1-x)^i}$, whence $b(n,i;0)=\binom{n-1}{i-1}$ for $1\leq i\leq n$. Thus, taking $q=0$ in (2) gives

$$a(n+1,i;0) = \binom{n-1}{i-2} + \sum_{j=i}^{n} a(n,j;0), \qquad 2\leq i\leq n+1, \tag{23}$$

with $a(n+1,1;0)=a(n;0)=\sum_{j=1}^{n}a(n,j;0)$ for $n\geq 1$ and $a(1,1;0)=1$. One can now establish the following formulas for $a(n,i;0)$ by induction on n using (23) and Lemma 5, upon considering separately the cases when $i=1$ and $i>1$.

Theorem 3. *If $2\leq i\leq n$, then*

$$a(n,i;0) = \sum_{m=0}^{n-i} F_{2m-1}\binom{n-m-2}{i-2}, \tag{24}$$

with $a(n,1;0)=F_{2n-3}$ for $n\geq 1$. Moreover, we have $a(n;0)=\sum_{j=1}^{n}a(n,j;0) = F_{2n-1}$ for all $n\geq 1$.

It is instructive to demonstrate the last result also by a combinatorial argument.

Combinatorial Proof of Theorem 3. We first show $a(n;0) = F_{2n-1}$ for $n \geq 1$, from which the formula for $a(n,1;0)$ follows as an immediate consequence. Let $\mathcal{A}'_n$ denote the subset of $\mathcal{A}_n$ whose members have zero capacity. It is seen that $\lambda \in \mathcal{A}_n$ belongs to $\mathcal{A}'_n$ if and only if λ can be decomposed (uniquely) as $\lambda = \lambda'\lambda''$, where λ' is weakly increasing, λ'' is weakly decreasing with λ'' possibly empty, and $\max(\lambda'') < \max(\lambda')$ if λ'' is nonempty. Let $k_n = |\mathcal{A}'_n|$ for $n \geq 1$ and note $k_1 = 1 = F_1$ and $k_2 = 2 = F_3$. To show $k_n = F_{2n-1}$ for all $n \geq 1$, we then argue $k_n = 3k_{n-1} - k_{n-2}$ for $n \geq 3$. First note that there are clearly k_{n-1} members of $\mathcal{A}'_n$ ending in 1, upon appending 1 to an arbitrary member of $\mathcal{A}'_{n-1}$. There are also k_{n-1} members of $\mathcal{A}'_n$ not ending in 1 in which the maximum letter occurs only once. To see this, suppose $\lambda = \lambda'\lambda'' \in \mathcal{A}'_{n-1}$, where λ' and λ'' are as described above, and let $u = \max(\lambda)$, which also must be the final letter of λ'. We then insert a single copy of $u+1$ between λ' and λ'' (or at the very end of λ if $\lambda'' = \varnothing$) and increase each letter of λ'' by one to obtain a member of $\mathcal{A}'_n$ not ending in 1 and whose largest letter occurs once. As this operation is reversible, it follows that there are k_{n-1} members of $\mathcal{A}'_n$ of the stated form.

Now suppose $\lambda = \lambda'\lambda'' \in \mathcal{A}'_{n-1}$ is as before, with the further assumption that λ does not end in 1. Consider in this case inserting an additional copy of u between λ' and λ'' (or at the end of λ if $\lambda'' = \varnothing$). This operation is seen to yield a bijection between the subset of $\mathcal{A}'_{n-1}$ whose members do not end in 1 and the subset of $\mathcal{A}'_n$ whose members have their maximum letter occurring at least twice and do not end in 1. By subtraction, there are $k_{n-1} - k_{n-2}$ members of the former set, and hence also of the latter. Combining with the previous cases then implies $k_n = 3k_{n-1} - k_{n-2}$ for $n \geq 3$, as desired.

The preceding argument can now be extended to show (24) as well. Let $\mathcal{A}'_{n,i} = \mathcal{A}'_n \cap \mathcal{A}_{n,i}$, and we wish to enumerate the members of $\mathcal{A}'_{n,i}$ for $2 \leq i \leq n$. Suppose first that the letter i within $\lambda = \lambda_1 \cdots \lambda_n \in \mathcal{A}'_{n,i}$ occurs only once (at the end). Then $\lambda_1 \cdots \lambda_{n-1}$ is a weakly increasing sequence in $[i-1]$ where each letter occurs at least once. By [23, p. 14], the number of possible λ is given by $\binom{n-2}{i-2}$, which accounts for the $m = 0$ term of the summation in (24). So assume i within $\pi \in \mathcal{A}'_{n,i}$ occurs at least twice. We decompose π as $\pi = \pi'\pi''$, where π' contains no i and π'' starts with (and ends in) i. Note that π' is weakly increasing (for otherwise, π would have nonzero capacity) and contains each letter in $[i-1]$ at least once. Further, π'' can only contain letters in $\{i, i+1, \ldots\}$, or membership in $\mathcal{A}'_n$ would be violated, since π'' both begins and ends with i. Let $|\pi''| = m+1$ for some $m \geq 1$; note π' containing every letter in $[i-1]$ implies $m \leq n-i$. Subtraction of $i-1$ from each letter in π'' is seen to result in a member of $\mathcal{A}'_{m+1,1}$, as π'' has zero capacity. By the formula for $a(n,1;0)$ already established, there are thus F_{2m-1} possibilities for π'' for a given m. Then $|\pi'| = n-m-1$ implies there are $\binom{n-m-2}{i-2}$ possibilities for π' and allowing m to vary gives $\sum_{m=1}^{n-i} F_{2m-1}\binom{n-m-2}{i-2}$ members of $\mathcal{A}'_{n,i}$ wherein i

occurs at least twice. Combining this case with the prior yields (24) and completes the proof. □

3. Capacity of Smooth Words

Given $n \geq 1$ and $1 \leq i \leq n$, let $\mathcal{R}_{n,i}$ denote the set of smooth words of length n ending in i and hence $\mathcal{R}_n = \cup_{i=1}^{n} \mathcal{R}_{n,i}$. Let $r(n, i; q)$ denote the distribution of the capacity statistic on $\mathcal{R}_{n,i}$ (whose q argument will often be suppressed and likewise for subsequent distributions). Let $r(n; q) = \sum_{i=1}^{n} r(n, i; q)$ for $n \geq 1$ be the corresponding distribution on $\mathcal{R}_n$. As it was with the array $a(n, i)$ studied in the prior section, it seems difficult to write a recursion for the array $r(n, i)$ by itself. So again we consider a system of recurrences with two other arrays.

Given $n \geq 1$ and $1 \leq i \leq n$, let $\mathcal{S}_{n,i}$ denote the subset of $\mathcal{R}_{n,i}$ containing those members whose largest letter is also i. Let $s(n, i; q)$ be the capacity distribution on $\mathcal{S}_{n,i}$. In order to write a system of recurrences for $r(n, i)$ and $s(n, i)$, we need to consider a further array enumerating a certain class of words that satisfy the smooth word restriction. Given $m, j \geq 1$, let $\mathcal{T}_{m,j}$ denote the set of words $w = w_1 \cdots w_m$ on the alphabet of positive integers that begin, end *and* have greatest letter j such that $|w_{i+1} - w_i| \leq 1$ for each $i \in [m-1]$. Let $t(m, j; q)$ denote the distribution of the area statistic on $\mathcal{T}_{m,j}$. Assume $r(n, i) = 0$ if $i > n$ or $i \leq 0$ and likewise for the $s(n, i)$ array, with $t(m, j) = 0$ if $j \leq 0$.

Consider whether $\lambda \in \mathcal{T}_{m,j}$ ends in two or more j's or a single j, where $m, j \geq 2$. If the latter, then λ both beginning and ending in j implies it can be decomposed as $\lambda = \lambda'\lambda''j$, where $\lambda' \in \mathcal{T}_{m-\ell-1,j}$ and $\lambda'' \in \mathcal{T}_{\ell,j-1}$ for some $1 \leq \ell \leq m-2$. This leads to the following recursive formula for $t(m, j)$.

Lemma 6. *If $m, j \geq 2$, then*

$$t(m, j) = q^j t(m-1, j) + q^j \sum_{\ell=1}^{m-2} t(\ell, j-1) t(m-\ell-1, j), \tag{25}$$

with $t(m, 1) = t(1, m) = q^m$ for all $m \geq 1$.

Then $r(n, i)$ and $s(n, i)$ are given in terms of a system of linear recurrence involving $t(m, j)$.

Lemma 7. *The arrays $r(n, i)$ and $s(n, i)$ are given recursively by*

$$\begin{aligned} r(n, i) &= r(n-1, i) + r(n-1, i+1) + s(n-1, i-1) \\ &\quad + \sum_{m=1}^{n-i-1} q^{mi} r(n-m-1, i) t(m, i-1; 1/q), \quad 1 < i < n, \end{aligned} \tag{26}$$

with $r(n,n) = 1$ for all $n \geq 1$ and $r(n,1) = r(n-1,1) + r(n-1,2)$ for $n \geq 2$, and

$$s(n,i) = s(n-1,i) + s(n-1,i-1) + \sum_{m=1}^{n-i-1} q^{mi} s(n-m-1,i) t(m,i-1;1/q), \quad 1 < i < n, \tag{27}$$

with $s(n,n) = s(n,1) = 1$ for all $n \geq 1$, where $t(m,j)$ is as given in Lemma 6.

Proof. The boundary conditions when $i = 1$ or $i = n$ follow from the fact that $\mathcal{R}_{n,n}$, $\mathcal{S}_{n,1}$ and $\mathcal{S}_{n,n}$ each are singletons whose sole member has capacity zero, whereas $\rho \in \mathcal{R}_{n,1}$ for $n \geq 2$ implies $\rho = \rho'1$, with $\rho' \in \mathcal{R}_{n-1,1} \cup \mathcal{R}_{n-1,2}$. To show (26), let $\pi \in \mathcal{R}_{n,i}$ where $1 < i < n$ and consider the penultimate letter j of π. Note $\pi \in \mathcal{R}_{n,i}$ implies $j \in \{i-1, i, i+1\}$. If $j = i$ or $i+1$, then there are clearly $r(n-1,i)$ and $r(n-1,i+1)$ possibilities, respectively. If $j = i-1$ and no i occurs within π other than at the end, then there are $s(n-1,i-1)$ possibilities, by the definitions. Otherwise, $j = i-1$, with π decomposable as $\pi = \pi' i \pi''(i-1)i$, where π'' contains no i and is possibly empty. Then we must have $\pi' i \in \mathcal{R}_{n-m-1,i}$ and $\pi''(i-1) \in \mathcal{T}_{m,i-1}$ for some $m \in [n-i-1]$, since π'' must start with $i-1$, if nonempty. Employing now the same argument used to establish the comparable case in the proof of (2), the contribution towards the total weight of those π decomposed as above is seen to be $r(n-m-1,i) \cdot q^{mi} t(m,i-1;1/q)$ for each m. Summing over all possible m then accounts for the summation on the right-hand side and combining with the prior cases completes the proof of (26). A similar proof applies to (27). First note $\pi \in \mathcal{S}_{n,i}$ implies the penultimate letter of π must either be i or $i-1$, and if it is $i-1$, consider whether or not π contains more than one i. If it does, with $\pi = \pi' i \pi''(i-1)i$ as before, then the section $\pi' i$ must now belong to $\mathcal{S}_{n-m-1,i}$ for some $m \in [n-i-1]$, where $|\pi''| = m-1$. Combining the various cases then yields (27) and completes the proof. $\square$

We seek explicit formulas for $R_i(x;q) = \sum_{n \geq i} r(n,i;q)x^n$ where $i \geq 1$ is fixed and $R(x;q) = \sum_{i \geq 1} R_i(x;q)$, which are the generating functions for the distribution of the capacity statistic over $\mathcal{R}_{n,i}$ for $n \geq i$ and over $\mathcal{R}_n$ for $n \geq 1$, respectively. We will need expressions for the auxiliary generating functions $S_i(x;q) = \sum_{n \geq i} s(n,i;q)x^n$ and $T_j(x;q) = \sum_{m \geq 1} t(m,j;q)x^m$. By (25), we have

$$T_j(x;q) = q^j x + q^j x T_j(x;q) + q^j x T_j(x;q) T_{j-1}(x;q),$$

that is,

$$T_j(x;q) = \frac{q^j x}{1 - q^j x - q^j x T_{j-1}(x;q)}, \qquad j \geq 2,$$

with $T_1(x;q) = \frac{qx}{1-qx}$.

An induction on j then yields the following finite continued fraction expression for $T_i(x;q)$.

Lemma 8. *For all $j \geq 2$,*

$$T_j(x;q) = \cfrac{q^j x}{1 - q^j x - \cfrac{q^{2j-1}x^2}{1 - q^{j-1}x - \cfrac{q^{2j-3}x^2}{\ddots\ \cfrac{q^5x^2}{1-q^2x-\frac{q^3x^2}{1-qx}}}}},$$

with $T_1(x;q) = \frac{qx}{1-qx}$.

Multiplying (27) by x^n and summing over $n \geq i+1$, we obtain

$$S_i(x;q) = xS_i(x;q) + xS_{i-1}(x;q) + xS_i(x;q)T_{i-1}(q^i x; 1/q),$$

that is,

$$S_i(x;q) = \frac{xS_{i-1}(x;q)}{1 - x - xT_{i-1}(q^i x; 1/q)}, \qquad i \geq 2,$$

with $S_1(x;q) = \frac{x}{1-x}$. Iteration of the last formula leads to the following result.

Lemma 9. *For all $i \geq 1$,*

$$S_i(x;q) = \frac{x^i}{\prod_{j=0}^{i-1}(1 - x - xT_j(q^{j+1}x; 1/q))},$$

where $T_j(x;q)$ for $j \geq 1$ with $T_0(x;q) = 0$ is as given in Lemma 8.

From (26), we obtain

$$R_i(x;q) = xR_i(x;q) + xR_{i+1}(x;q) + xS_{i-1}(x;q) + xR_i(x;q)T_{i-1}(q^i x; 1/q), \quad i \geq 2,$$

with $R_1(x;q) = x + xR_1(x;q) + xR_2(x;q)$. Thus,

$$R_i(x;q) = \frac{xR_{i+1}(x;q)}{1 - x - xT_{i-1}(q^i x; 1/q)} + \frac{xS_{i-1}(x;q)}{1 - x - xT_{i-1}(q^i x; 1/q)}, \qquad i \geq 2,$$

with $R_1(x;q) = \frac{x}{1-x} + \frac{x}{1-x}R_2(x;q)$. Therefore, by iteration,

$$R_i(x;q) = \sum_{m \geq i-1} \frac{x^{m-i+2}S_m(x;q)}{\prod_{j=i-1}^{m}(1 - x - xT_j(q^{j+1}x; 1/q))}, \qquad i \geq 2, \tag{28}$$

and, in particular,

$$\begin{aligned} R_2(x;q) &= \sum_{m \geq 1} \frac{x^m S_m(x;q)}{\prod_{j=1}^{m}(1 - x - xT_j(q^{j+1}x; 1/q))} \\ &= \sum_{m \geq 1} \frac{x^{2m}}{(1-x)(1 - x - xT_m(q^{m+1}x; 1/q)) \prod_{j=1}^{m-1}(1 - x - xT_j(q^{j+1}x; 1/q))^2}. \end{aligned}$$

Hence, we get the following formulas for $R_1(x;q)$ and $R(x;q)$.

Theorem 4. *We have*

$$R_1(x;q) = \sum_{m\geq 0} \frac{x^{2m+1}}{(1-x-xT_m(q^{m+1}x;1/q))\prod_{j=0}^{m-1}(1-x-xT_j(q^{j+1}x;1/q))^2}, \tag{29}$$

where $T_j(x;q)$ for $j \geq 1$ with $T_0(x;q) = 0$ is as given in Lemma 8. Moreover, we have $R(x;q) = \sum_{i\geq 1} R_i(x;q)$, where $R_i(x;q)$ for $i \geq 2$ is given by (28).

Note that $R_1(x;q)$ is the generating function for the distribution of the capacity statistic on members of $\mathcal{R}_{n,1}$ for $n \geq 1$, which are synonymous with the Motzkin paths of length $n-1$ via the correspondence $\jmath$.

By the recurrence for $T_i(x;q)$ at $q = 1$ and induction on i,

$$T_i(x;1) = \frac{U_{i-1}(v)}{U_i(v)}, \qquad i \geq 1, \tag{30}$$

where $v := \frac{1-x}{2x}$. Thus, when $q = 1$ in Lemma 9, one obtains a telescoping product expression, giving

$$S_i(x;1) = \frac{x^i}{\prod_{j=0}^{i-1}(1-x-xT_j(x;1))} = \frac{1}{U_i(v)}, \qquad i \geq 1. \tag{31}$$

As a consequence of the preceding observations and Theorem 4, one can obtain the following pair of Chebyshev identities.

Corollary 5. *We have*

$$R_1(x;1) = \sum_{j\geq 0} \frac{1}{U_j(v)U_{j+1}(v)} = \frac{1-x-\sqrt{1-2x-3x^2}}{2x} \tag{32}$$

and

$$R(x;1) = \frac{x}{3x-1}\sum_{j\geq 0} \frac{1+U_j(v)-U_{j+1}(v)}{U_j(v)U_{j+1}(v)} = \frac{2x}{3x-1+\sqrt{1-2x-3x^2}} - 1, \tag{33}$$

where $v := \frac{1-x}{2x}$.

Proof. By (29) at $q = 1$, together with (30) and (31), we obtain

$$R_1(x;1) = \sum_{j\geq 0} \frac{1}{U_j(v)U_{j+1}(v)}.$$

On the other hand, by $\jmath$, we have $r(n,1;1) = |\mathcal{R}_{n,1}| = M_{n-1}$, and hence

$$R_1(x;1) = \sum_{n\geq 1} M_{n-1}x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x},$$

which implies (32), where M_n denotes the n-th Motzkin number (see, for example, [18, A001006]).

Note that by (28) at $q=1$ and (31), we have for all $m \geq 2$,

$$\begin{aligned} R_m(x;1) &= \sum_{j\geq m-1} \frac{x^{j-m+2}S_j(x;1)}{\prod_{i=m-1}^{j}(1-x-xT_i(x;1))} \\ &= \sum_{j\geq m-1} \frac{x^{j-m+2}S_j(x;1)\prod_{i=0}^{m-2}(1-x-xT_i(x;1))}{\prod_{i=0}^{j}(1-x-xT_i(x;1))} \\ &= \sum_{j\geq m-1} \frac{x^{j-m+2}\cdot\frac{1}{U_j(v)}\cdot x^{m-1}U_{m-1}(v)}{x^{j+1}U_{j+1}(v)} = \sum_{j\geq m-1}\frac{U_{m-1}(v)}{U_j(v)U_{j+1}(v)}. \end{aligned}$$

Thus, by Theorem 4 at $q=1$, we get

$$\begin{aligned} R(x;1) = \sum_{m\geq 1} R_m(x;1) &= \sum_{j\geq 0}\frac{1}{U_j(v)U_{j+1}(v)} + \sum_{m\geq 2}\sum_{j\geq m-1}\frac{U_{m-1}(v)}{U_j(v)U_{j+1}(v)} \\ &= \sum_{m\geq 1}\sum_{j\geq m-1}\frac{U_{m-1}(v)}{U_j(v)U_{j+1}(v)} = \sum_{j\geq 0}\frac{1}{U_j(v)U_{j+1}(v)}\sum_{m=1}^{j+1}U_{m-1}(v) \\ &= \frac{x}{3x-1}\sum_{j\geq 0}\frac{1+U_j(v)-U_{j+1}(v)}{U_j(v)U_{j+1}(v)}, \end{aligned}$$

where in the last equality, we used the fact $\sum_{i=0}^{n}U_i(x) = \frac{1+U_n(x)-U_{n+1}(x)}{2(1-x)}$. On the other hand, by $\jmath$, we have $r(n;1) = |\mathcal{R}_n| = L_n$, and hence

$$R(x;1) = \sum_{n\geq 1} L_n x^n = \frac{2x}{3x-1+\sqrt{1-2x-3x^2}} - 1,$$

which implies (33) and completes the proof. □

Remark 2. See, for example, [1, 11] and references contained therein for finite sum and infinite series identities involving reciprocals of Horadam numbers. Note that the combinatorial derivation here of the identities (32) and (33) (as well as those in Theorem 8 below) differs from the methods presented in [1, 11], which entail the creative use of telescoping.

We have the following result for the maximum capacity achieved by a member of $\mathcal{R}_n$.

Theorem 5. *If $m \geq 1$ and $0 \leq p \leq 3$, then deg$(r(4m+p;q)) = m(2m+p-1)$.*

Proof. We seek to maximize the capacity of a member of $\mathcal{R}_n$, where $n \geq 4$. Upon proceeding as in the proof of Theorem 2, and making the appropriate modifications, in order to maximize the capacity, it suffices to consider only

$$\rho^{(j)} = 12\cdots j(j-1)\cdots 21^{n-3j+3}23\cdots j, \qquad 2 \leq j \leq t,$$

where $t = \lfloor (n+2)/3 \rfloor$, or members of $\mathcal{R}_n$ of the form

$$\pi = 12\cdots j(j-1)\cdots dd(d+1)\cdots j \text{ or } \pi = 12\cdots j(j-1)\cdots d(d+1)\cdots j,$$

where $2 \le d < j$ and $t < j \le n-2$. Upon changing the first j in π of either of the two stated forms to $j-2$, and then reducing each of the letters to the right of the first j by one, we obtain a member of $\mathcal{R}_n$ with a strictly larger capacity. Thus, no such π can be optimal, whence $\deg(r(n;q)) = \max\{\mathrm{cap}(\rho^{(j)}) : 2 \le j \le t\}$.

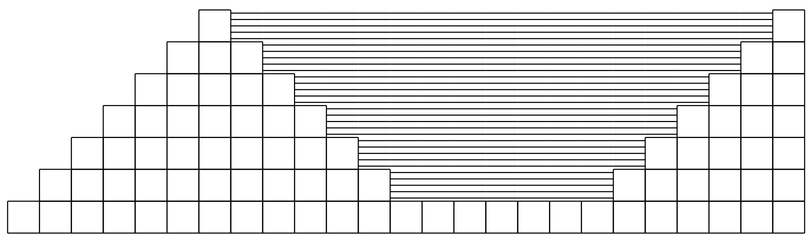


Figure 3: The maximum capacity over $\mathcal{R}_{25}$ of 72 achieved by $\rho^{(7)}$.

Note that $\mathrm{cap}(\rho^{(j)}) = 2\binom{j-1}{2} + (j-1)(n-3j+3) = (j-1)(n-2j+1)$, and we seek to maximize the function $g(x) = (x-1)(n-2x+1)$ of a real variable x. The function $g(x)$ achieves its maximum value at $x_0 = \frac{n+3}{4}$, and we consider cases on n mod 4. First suppose $n = 4m$ for some $m \ge 1$. Then $x_0 = m + \frac{3}{4}$ in this case and the value of $j \in [2,t]$ closest to x_0, that is, $m+1$, will yield the member of $\mathcal{R}_{4m}$ with maximum capacity, by the graph of $y = g(x)$. Thus, $\deg(r(4m;q)) = g(m+1) = m(n-2(m+1)+1) = m(2m-1)$, which establishes the $p = 0$ case of our formula. A similar argument applies in the other cases. Note that the maximum capacity of a member of $\mathcal{R}_{4m+p}$ is seen to be achieved only by $\rho^{(m+1)}$ if $0 \le p \le 2$, and by both $\rho^{(m+1)}$ and $\rho^{(m+2)}$ if $p = 3$. Illustrated in Figure 3 is the optimal member π of $\mathcal{R}_{25}$, namely, $\pi = \rho^{(7)}$, for which the maximum capacity of 72 is achieved. $\square$

We now consider the case when $q = 0$ in $r(n,i)$ and $r(n)$. Note first that $T_j(q^{j+1}x, 1/q)\mid_{q=0} = 0$ for each j and hence $S_i(x;0) = \left(\frac{x}{1-x}\right)^i$ for all $i \ge 1$. Therefore, by (28) at $q = 0$, we get for each $i \ge 2$,

$$\begin{aligned} R_i(x;0) &= \sum_{m \ge i-1} \left(\frac{x}{1-x}\right)^{2m-i+2} = \frac{x^i}{(1-2x)(1-x)^{i-2}} \\ &= \sum_{j \ge i-2} \frac{x^{j+2}}{(1-x)^{j+1}} = \sum_{n \ge i} x^n \sum_{j=i-2}^{n-2} \binom{n-2}{j}, \end{aligned}$$

with the second expression for $R_i(x;0)$ in the last line seen to hold for $i=1$ as well. Thus, we have

$$R(x;0)=\sum_{i\geq 1}R_i(x;0)=\sum_{i\geq 1}\frac{x^i}{(1-2x)(1-x)^{i-2}}=x\left(\frac{1-x}{1-2x}\right)^2.$$

Extracting the coefficient of x^n in the preceding formulas then gives the following result.

Theorem 6. *If $2\leq i\leq n$, then*

$$r(n,i;0)=\sum_{j=i-2}^{n-2}\binom{n-2}{j}, \tag{34}$$

with $r(n,1;0)=2^{n-2}$ for $n\geq 2$ and $r(1,1;0)=1$. Moreover, we have $r(n;0)=\sum_{i=1}^{n}r(n,i;0)=(n+2)2^{n-3}$ for $n\geq 2$, with $r(1;0)=1$.

It is also possible to provide a direct combinatorial explanation of the prior result.

Combinatorial Proof of Theorem 6.

Let $\mathcal{R}'_n$ denote the subset of $\mathcal{R}_n$ whose members have capacity zero and let $\mathcal{R}'_{n,i}=\mathcal{R}'_n\cap\mathcal{R}_{n,i}$ for $1\leq i\leq n$. Let $\pi\in\mathcal{R}'_{n,i}$, where $n\geq 2$ and $2\leq i\leq n$. Note that there are $p+i-1$ u's and p d's for some $0\leq p\leq\lfloor(n-i)/2\rfloor$ among the $n-1$ steps in the lattice path $\jmath(\pi)$. Further, the positions of the u and d steps within $\jmath(\pi)$ may be chosen arbitrarily, with the d's all occurring to the right of the last u, since π having zero capacity contains no ascents beyond the first descent (if it exists). Thus, there are $\binom{n-1}{i+2p-1}$ u and d steps combined for each p. This yields

$$r(n,i;0)=|\mathcal{R}'_{n,i}|=\binom{n-1}{i-1}+\binom{n-1}{i+1}+\cdots=\sum_{j=i-2}^{n-2}\binom{n-2}{j},$$

which gives (34) and also shows $r(n,1;0)=2^{n-2}$ for $n\geq 2$. From (34), we have

$$r(n,i;0)+r(n,n+3-i;0)=2^{n-2},\qquad 1\leq i\leq n+2,$$

assuming $r(n,m;0)=0$ if $m>n$. Summing both sides of the last equality over $1\leq i\leq n+2$ implies

$$r(n;0)=\sum_{i=1}^{n}r(n,i;0)=(n+2)2^{n-3},\qquad n\geq 2,$$

which completes the proof. $\square$

4. Restricted Smooth Words

Let $\mathcal{R}_n^*$ denote the subset of $\mathcal{R}_n$ whose members do not contain two adjacent 1's and $\mathcal{R}_{n,i}^* = \mathcal{R}_n^* \cap \mathcal{R}_{n,i}$ for $1 \leq i \leq n$. Note that members of $\mathcal{R}_n^*$ correspond under $\jmath$ to the subset of $\mathcal{L}_n$ consisting of those paths that have no horizontal steps at height zero (termed *low* h steps). Further, $\mathcal{R}_{n,1}^*$ is seen to correspond to the subset of the Motzkin paths of length $n-1$ whose members contain no low h's, which are called *Riordan* paths. Thus, $|\mathcal{R}_{n,1}^*| = R_{n-1}$ for all $n \geq 1$, where R_n denotes the n-th Riordan number (see A005043 in [18]). Define $\mathcal{S}_{n,i}^*$ where $1 \leq i \leq n$ and $\mathcal{T}_{m,j}^*$ where $m, j \geq 1$ analogously, where we require members of $\mathcal{S}_{n,i}$ or $\mathcal{T}_{m,j}$ to not contain any adjacent entries equal to 1. Let $r^*(n)$, $r^*(n,i)$ and $s^*(n,i)$ denote the capacity distribution on $\mathcal{R}_n^*$, $\mathcal{R}_{n,i}^*$ and $\mathcal{S}_{n,i}^*$, respectively, and $t^*(m,j)$ the area distribution on $\mathcal{T}_{m,j}^*$, where we have omitted here the q argument in each function.

Reasoning as in the proofs for Lemmas 6 and 7, we have that the arrays $t^*(m,j)$, $r^*(n,i)$ and $s^*(n,i)$ satisfy recurrences (25), (26) and (27), respectively, with the only difference being in the initial conditions as follows: $t^*(m,1) = q\cdot\delta_{m,1}$, $r^*(n,1) = r^*(n-1,2)$ for $n \geq 2$ with $r^*(1,1) = 1$, and $s^*(n,1) = \delta_{n,1}$.

Define $R^*(x;q) = \sum_{n\geq 1} r^*(n;q)x^n$, $R_i^*(x;q) = \sum_{n\geq i} r^*(n,i;q)x^n$, $S_i^*(x;q) = \sum_{n\geq i} s^*(n,i;q)x^n$ and $T_j^*(x;q) = \sum_{m\geq 1} t^*(m,j;q)x^m$. Reasoning as before leads to the following explicit formulas for $T_j^*(x;q)$ and $S_i^*(x;q)$.

Lemma 10. *For all $j \geq 2$,*

$$T_j^*(x;q) = \cfrac{q^j x}{1 - q^j x - \cfrac{q^{2j-1}x^2}{1 - q^{j-1}x - \cfrac{q^{2j-3}x^2}{\ddots \quad \cfrac{q^5x^2}{1-q^2x-\cfrac{q^3x^2}{1}}}}},$$

with $T_1^*(x;q) = qx$.

Lemma 11. *For all $i \geq 1$,*

$$S_i^*(x;q) = \frac{x^i}{\prod_{j=1}^{i-1}(1 - x - xT_j^*(q^{j+1}x;1/q))},$$

where $T_j^*(x;q)$ for $j \geq 1$ is as given in Lemma 10.

From the recurrence for $r^*(n,i)$, we get

$$R_i^*(x;q) = xR_i^*(x;q) + xR_{i+1}^*(x;q) + xS_{i-1}^*(x;q) + xR_i^*(x;q)T_{i-1}^*(q^ix;1/q),$$

and hence

$$R_i^*(x;q) = \frac{xR_{i+1}^*(x;q)}{1 - x - xT_{i-1}^*(q^ix;1/q)} + \frac{xS_{i-1}^*(x;q)}{1 - x - xT_{i-1}^*(q^ix;1/q)}, \quad i \geq 2,$$

with $R_1^*(x;q) = x + xR_2^*(x;q)$. Iterating the last equation, we have

$$R_i^*(x;q) = \sum_{m\geq i-1} \frac{x^{m-i+2}S_m^*(x;q)}{\prod_{j=i-1}^{m}(1-x-xT_j^*(q^{j+1}x;1/q))}, \qquad i \geq 2, \tag{35}$$

and, in particular,

$$R_2^*(x;q) = \sum_{m\geq 1} \frac{x^m S_m^*(x;q)}{\prod_{j=1}^{m}(1-x-xT_j^*(q^{j+1}x;1/q))}.$$

Thus, we can state the following result.

Theorem 7. *We have*

$$R_1^*(x;q) = x + \sum_{m\geq 1} \frac{x^{2m+1}}{(1-x-xT_m^*(q^{m+1}x;1/q))\prod_{j=1}^{m-1}(1-x-xT_j^*(q^{j+1}x;1/q))^2}, \tag{36}$$

where $T_j^*(x;q)$ for $j \geq 1$ is as given in Lemma 10. Moreover, we have $R^*(x;q) = \sum_{i\geq 1} R_i^*(x;q)$, where $R_i^*(x;q)$ for $i \geq 2$ is given by (35).

As a consequence of the prior results, one can obtain the following pair of identities.

Theorem 8. *We have*

$$\begin{aligned} R_1^*(x;1) &= \sum_{j\geq 0} \frac{(1-x)^2}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))} \\ &= \frac{1+x-\sqrt{1-2x-3x^2}}{2(1+x)} \end{aligned} \tag{37}$$

and

$$\begin{aligned} R^*(x;1) &= \frac{1-x}{3x-1}\sum_{j\geq 0} \frac{2x(1-x)+(x^2+2x-1)U_j(v)+2x^2U_{j-2}(v)}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))} \\ &= \frac{x}{\sqrt{1-2x-3x^2}}, \end{aligned} \tag{38}$$

where $v := \frac{1-x}{2x}$.

Proof. Using the recurrence for $t^*(m,j)$, one can show

$$T_i^*(x;q) = \frac{q^i x}{1-q^i x - q^i x T_{i-1}^*(x;q)}, \qquad i \geq 2,$$

with $T_1^*(x;q) = qx$. By induction on i, and use of the recurrence

$$U_i(v) = \frac{1-x}{x}U_{i-1}(v) - U_{i-2}(v),$$

one can show

$$T_i^*(x;1) = \frac{U_{i-1}(v) + xU_{i-3}(v)}{U_i(v) + xU_{i-2}(v)}, \qquad i \geq 1, \tag{39}$$

where we assume $U_{-1}(v) = 0$ and $U_{-2}(v) = -1$. By (39) and the recurrence for $U_j(v)$, we have for $j \geq 1$,

$$\begin{aligned}
1 - x - xT_j^*(x;1) &= \frac{(1-x)(U_j(v) + xU_{j-2}(v)) - xU_{j-1}(v) - x^2U_{j-3}(v)}{U_j(v) + xU_{j-2}(v)} \\
&= \frac{x(U_{j+1}(v) + xU_{j-1}(v))}{U_j(v) + xU_{j-2}(v)}.
\end{aligned}$$

Thus, by Lemma 11 and (39), we get

$$S_i^*(x;1) = \frac{x^i}{\prod_{j=1}^{i-1}(1 - x - xT_j^*(x;1))} = \frac{1-x}{U_i(v) + xU_{i-2}(v)}, \qquad i \geq 1. \tag{40}$$

Hence, by (36) and (40), we have

$$\begin{aligned}
R_1^*(x;1) &= x + \sum_{m \geq 1} \frac{x^{2m+1}}{(1 - x - xT_m^*(x;1))\prod_{j=1}^{m-1}(1 - x - xT_j^*(x;1))^2} \\
&= x + \sum_{m \geq 1} \left(\frac{x^m}{\prod_{j=1}^{m-1}(1 - x - xT_j^*(x;1))}\right) \cdot \left(\frac{x^{m+1}}{\prod_{j=1}^{m}(1 - x - xT_j^*(x;1))}\right) \\
&= \sum_{m \geq 0} \frac{(1-x)^2}{(U_m(v) + xU_{m-2}(v))(U_{m+1}(v) + xU_{m-1}(v))}.
\end{aligned}$$

On the other hand, $r^*(n,1;1) = |\mathcal{R}_{n,1}^*| = R_{n-1}$ for all $n \geq 1$, and hence

$$R_1^*(x;1) = \sum_{n \geq 1} R_{n-1}x^n = \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2(1+x)},$$

which implies (37).

Note that by (35) at $q = 1$ and (40), we have for all $m \geq 2$,

$$\begin{aligned}
R_m^*(x;1) &= \sum_{j \geq m-1} \frac{x^{j-m+2}S_j^*(x;1)}{\prod_{i=m-1}^{j}(1 - x - xT_i^*(x;1))} \\
&= \sum_{j \geq m-1} \frac{x^{j-m+2}S_j^*(x;1)\prod_{i=1}^{m-2}(1 - x - xT_i^*(x;1))}{\prod_{i=1}^{j}(1 - x - xT_i^*(x;1))} \\
&= \sum_{j \geq m-1} \frac{x^{j-m+2} \cdot \frac{1-x}{U_j(v)+xU_{j-2}(v)} \cdot \frac{x^{m-1}(U_{m-1}(v)+xU_{m-3}(v))}{1-x}}{\frac{x^{j+1}(U_{j+1}(v)+xU_{j-1}(v))}{1-x}}
\end{aligned}$$

$$= \sum_{j\geq m-1} \frac{(1-x)(U_{m-1}(v)+xU_{m-3}(v))}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))}.$$

Therefore, by Theorem 7 at $q=1$, we get

$$\begin{aligned}
R^*(x;1) &= \sum_{m\geq 1} R^*_m(x;1)\\
&= \sum_{j\geq 0} \frac{(1-x)^2}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))}\\
&\quad + \sum_{m\geq 2}\sum_{j\geq m-1} \frac{(1-x)(U_{m-1}(v)+xU_{m-3}(v))}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))}\\
&= \sum_{m\geq 1}\sum_{j\geq m-1} \frac{(1-x)(U_{m-1}(v)+xU_{m-3}(v))}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))}\\
&= \sum_{j\geq 0} \frac{1-x}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))} \sum_{m=1}^{j+1}(U_{m-1}(v)+xU_{m-3}(v))\\
&= x + \frac{1-x^2}{U_2(v)+x} + \sum_{j\geq 2} \frac{1-x}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))} \sum_{m=0}^{j} U_m(v)\\
&\quad + \sum_{j\geq 2} \frac{1-x}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))} \left(-x + x\sum_{m=0}^{j-2} U_m(v)\right)\\
&= x + \frac{x^2(1+x)}{1-x-x^2}\\
&\quad + \sum_{j\geq 2} \frac{(1-x)(1+x-2x(1-v)+U_j(v)+xU_{j-2}(v)-U_{j+1}(v)-xU_{j-1}(v))}{2(1-v)(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))}\\
&= x + \frac{x^2(1+x)}{1-x-x^2}\\
&\quad + \frac{x(1-x)}{3x-1} \sum_{j\geq 2} \frac{2(1-x)+U_j(v)+xU_{j-2}(v)-U_{j+1}(v)-xU_{j-1}(v)}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))}\\
&= \frac{x(1-x)}{3x-1} \sum_{j\geq 0} \frac{2(1-x)+U_j(v)+xU_{j-2}(v)-U_{j+1}(v)-xU_{j-1}(v)}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))}\\
&= \frac{1-x}{3x-1} \sum_{j\geq 0} \frac{2x(1-x)+(x^2+2x-1)U_j(v)+2x^2U_{j-2}(v)}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))},
\end{aligned}$$

where we have used the identity $\sum_{m=0}^{j} U_m(v) = \frac{1+U_j(v)-U_{j+1}(v)}{2(1-v)}$ and, in the last equality, the recurrence for $U_j(v)$.

Let G_n denote the n-th grand Motzkin number for $n \geq 0$; see A002426 in [18]. In the proof of Theorem 11 below, a bijection is given demonstrating $|\mathcal{R}^*_n| = G_{n-1}$ for all $n \geq 1$. Thus, we get

$$R^*(x;1) = \sum_{n\geq 1} r^*(n;1)x^n = \sum_{n\geq 1} |\mathcal{R}^*_n| x^n = \sum_{n\geq 1} G_{n-1}x^n = \frac{x}{\sqrt{1-2x-3x^2}},$$

which implies (38) and completes the proof. $\square$

Remark 3. We have $r(n,i;1) = A064189[n-1,i-1]$ and $r^*(n,i;1) = A089942[n-1,i-1]$ for all $1 \le i \le n$. The first equality follows from applying the correspondence $\jmath$, whereas the second follows from comparing the defining recurrences and apparently yields the first combinatorial interpretation of entry A089942 in [18]. That is, $A089942[n,i]$ for $0 \le i \le n$ gives the cardinality of the subset of smooth words of length $n+1$ that end in $i+1$ in which no two 1's are adjacent. Let $R(x) = \sum_{n\ge1} R_{n-1}x^n = \frac{1+x-\sqrt{1-2x-3x^2}}{2(1+x)}$ and $M(x) = \sum_{n\ge1} M_{n-1}x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x}$. Using (28) and (35) at $q=1$, one can extend identities (32) and (37) to

$$R_m(x;1) = \sum_{j\ge m-1} \frac{U_{m-1}(v)}{U_j(v)U_{j+1}(v)} = M(x)^m, \qquad m \ge 1,$$

and

$$\begin{aligned} R^*_m(x;1) &= \sum_{j\ge m-1} \frac{(1-x)(U_{m-1}(v)+xU_{m-3}(v))}{(U_j(v)+xU_{j-2}(v))(U_{j+1}(v)+xU_{j-1}(v))} \\ &= R(x)M(x)^{m-1}, \qquad m\ge 1. \end{aligned}$$

There is the following result concerning the degree and leading coefficient of the polynomial $r^*(n;q)$.

Theorem 9. *Let $n = 8m+p \ge 1$, where $0 \le p \le 7$. Then we have*

$$deg(r^*(8m+p;q)) = 8m^2 + (2p-3)m + \begin{cases} 0, & 0 \le p \le 3; \\ 1, & p = 4; \\ 2^{p-5}, & 5 \le p \le 7. \end{cases} \tag{41}$$

Let $\alpha = deg(r^*(n;q))$ and c_n denote the coefficient of q^α in $r^*(n;q)$. Then we have for $m \ge 1$,

$$c_{8m+p} = \begin{cases} m+1, & p = 0,3; \\ 1, & p = 1,2,4,7; \\ m+3, & p = 5; \\ m+2, & p = 6, \end{cases} \tag{42}$$

with c_n for $1 \le n \le 7$ given by $1,1,3,1,5,2,1$, respectively.

Proof. The values of α and c_n for $1 \le n \le 7$ may be verified directly, so assume $n \ge 8$. Let

$$\pi^{(j)} = 12\cdots j(j-1)\cdots 2\rho 23\cdots j, \qquad 2 \le j \le \lfloor (n+2)/3 \rfloor,$$

where ρ is of the form $12\cdots121$ if $|\rho|$ is odd and ρ is a sequence consisting of an equal number of 1's and 2's such that no two 1's are adjacent if $|\rho|$ is even. Note that ρ is nonempty as $j \leq \lfloor (n+2)/3 \rfloor$ implies $|\rho| = n-3j+3 \geq 1$. Proceeding in a manner comparable to the proof of Theorem 5, in order to maximize the capacity over all members of $\mathcal{R}_n^*$, we need only consider the $\pi^{(j)}$ having the stated form above. Note, in particular, that σ given by $\sigma = 12\cdots\ell(\ell-1)\cdots3223\cdots\ell$ where $\ell \geq 4$ is seen to have capacity strictly less than that of $\sigma' = 12\cdots(\ell-1)(\ell-2)\cdots3212123\cdots(\ell-1)$, with σ' of the form $\pi^{(j)}$.

We consider cases based on the parity of n and j in determining the value of $M := \max\{\mathrm{cap}(\pi^{(j)}) : 2 \leq j \leq \lfloor (n+2)/3 \rfloor\}$ for a fixed n. First suppose n and j are both even and let $n = 2a$ and $j = 2\ell$ for some $a \geq 4$ and $\ell \geq 1$. Then we have $|\rho| = n-3j+3$ odd in this case and thus

$$\begin{aligned}\mathrm{cap}(\pi^{(j)}) &= 2\binom{j-1}{2} + (j-1)\left(\frac{n-3j+4}{2}\right) + (j-2)\left(\frac{n-3j+2}{2}\right)\\ &= (2\ell-1)(2\ell-2) + (2\ell-1)(a-3\ell+2) + (2\ell-2)(a-3\ell+1)\\ &= (2\ell-1)^2 + (4\ell-3)(a-3\ell+1), \qquad 1 \leq \ell \leq \lfloor (a+1)/3 \rfloor.\end{aligned}$$

We seek the integer ℓ in the indicated interval that is closest to the x-coordinate of the vertex of the graph of $g(x) = (2x-1)^2 + (4x-3)(a-3x+1)$, namely, $x_0 = \frac{a}{4} + \frac{9}{16}$. We now need to consider cases on a mod 4. First suppose a is divisible by 4, with $a = 4m$ for some $m \geq 1$. Then $x_0 = m + \frac{9}{16}$ in this case, and we choose $\ell = m+1$. Let us assume for now $m \geq 2$, in which case $m+1 \leq \lfloor (4m+1)/3 \rfloor$. Then we have

$$\mathrm{cap}(\pi^{(m+1)}) = g(m+1) = (2m+1)^2 + (4m+1)(m-2) = 8m^2 - 3m - 1, \qquad m \geq 2,$$

which is the maximum capacity of the $\pi^{(j)}$ for j even where $n = 8m$. If $m = 1$, then use $\ell = 1$ and the corresponding maximum works out to 3.

To determine M, we must also compute the maximum capacity of the $\pi^{(j)}$ for which j is odd. Let $j = 2\ell+1$, where $\ell \geq 1$. Then we have $|\rho| = n-3j+3$ even in this case and thus

$$\begin{aligned}\mathrm{cap}(\pi^{(j)}) &= 2\binom{j-1}{2} + (2j-3)\left(\frac{n-3j+3}{2}\right)\\ &= 2\ell(2\ell-1) + (4\ell-1)(a-3\ell), \qquad 1 \leq \ell \leq \lfloor (a-1)/3 \rfloor.\end{aligned}$$

If $h(x) = 2x(2x-1) + (4x-1)(a-3x)$, then the maximum of $h(x)$ occurs at $x_0 = \frac{a}{4} + \frac{1}{16}$. If $a = 4m$, then we choose $\ell = m$ and note $m \geq 1$ implies $m \leq \lfloor (a-1)/3 \rfloor$. Thus, the maximum capacity of the $\pi^{(j)}$ for j odd where $n = 8m$ is given by $h(m) = 2m(2m-1) + (4m-1)(a-3m) = 8m^2 - 3m$. Comparing this maximum with that of $\pi^{(j)}$ for j even found above, we have $M = 8m^2 - 3m$, which implies the $p = 0$ case of formula (41). Similar arguments apply to the other cases when p is even in (41), upon writing $a = 4m + s$ where $1 \le s \le 3$ and treating separately each case of s.

A similar proof which we briefly describe applies in the cases of (41) when n is odd, upon writing $n = 2a + 1$ for some $a \ge 4$ and proceeding as before. Note that if $j = 2\ell + 1$ where $\ell \ge 1$, then $|\rho|$ odd implies

$$\begin{aligned}\mathrm{cap}(\pi^{(j)}) &= 2\binom{j-1}{2} + (j-1)\left(\frac{n-3j+4}{2}\right) + (j-2)\left(\frac{n-3j+2}{2}\right)\\ &= 2\ell(2\ell-1) + 2\ell(a-3\ell+1) + (2\ell-1)(a-3\ell)\\ &= 4\ell^2 + (4\ell-1)(a-3\ell), \qquad 1 \le \ell \le \lfloor a/3 \rfloor,\end{aligned}$$

and one would pick ℓ in the given range closest to $\frac{a}{4} + \frac{3}{16}$. On the other hand, if j is even with $j = 2\ell$, then $|\rho|$ even implies

$$\mathrm{cap}(\pi^{(j)}) = (2\ell-1)(2\ell-2) + (4\ell-3)(a-3\ell+2), \qquad 1 \le \ell \le \lfloor (a+1)/3 \rfloor,$$

and one would pick ℓ in the given range closest to $\frac{a}{4} + \frac{11}{16}$. Taking the larger of the two maximum capacities obtained in this way yields M. Once again, one would need to consider cases on a mod 4. Illustrated in Figure 4 is the optimal member π of $\mathcal{R}^*_{26}$, namely, $\pi = \pi^{(8)}$, for which the maximum capacity of 75 is achieved.

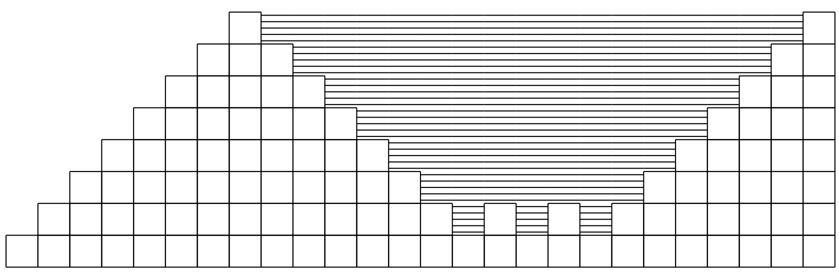


Figure 4: The maximum capacity over $\mathcal{R}^*_{26}$ of 75 achieved by $\pi^{(8)}$.

Formula (42) follows directly from the preceding proof when $p = 1, 2, 4, 7$, since the maximum value M arises in these cases (only) from instances in which $|\rho| = n - 3j + 3$ is odd, and hence there is only a single maximal member of $\mathcal{R}^*_{8m+p}$. If $p = 0$, we saw in the derivation above of this case that M arose when $|\rho|$ was even, with $|\rho| = 8m - 3(2m+1) + 3 = 2m$. The formula for c_{8m} then follows from the fact that there are $m + 1$ linear arrangements of m 1's and m 2's in which no two 1's are adjacent. To see this, consider inserting a single 2 between the i-th and $(i+1)$-st 1 for each $1 \le i \le m - 1$, which leaves a single 2 to insert in any one of $m + 1$ possible positions. The same explanation applies to c_{8m+3}. Finally, in the cases $p = 5, 6$, the maximum M is achieved by $\pi^{(j)}$ for both an even and an odd value of j. Since we have $|\rho| = 2m + 2$ when $p = 5$ and $|\rho| = 2m$ when $p = 6$ in the corresponding maximal cases for which $|\rho| = 8m + p - 3j + 3$ was even, one obtains the corresponding c_n values of $m + 3$ and $m + 2$, respectively. □

By modifying appropriately the algebraic or combinatorial argument given for Theorem 6 above (the details we omit), one obtains the following analogous results for $r^*(n, i)$ and $r^*(n)$ evaluated at $q = 0$.

Theorem 10. *If* $3 \leq i \leq n$*, then*

$$r^*(n, i; 0) = \sum_{j=i-3}^{n-3} \binom{n-3}{j}, \tag{43}$$

with

$$r^*(n, 1; 0) = \begin{cases} 1, & n = 1, 3; \\ 0, & n = 2; \\ 2^{n-4}, & n \geq 4, \end{cases} \quad \textit{and} \quad r^*(n, 2; 0) = \begin{cases} 1, & n = 2; \\ 2^{n-3}, & n \geq 3. \end{cases}$$

Moreover, we have $r^*(n; 0) = \sum_{i=1}^{n} r^*(n, i; 0) = (n + 2)2^{n-4}$ *for* $n \geq 4$*, with* $r^*(1; 0) = r^*(2; 0) = 1$ *and* $r^*(3; 0) = 3$*.*

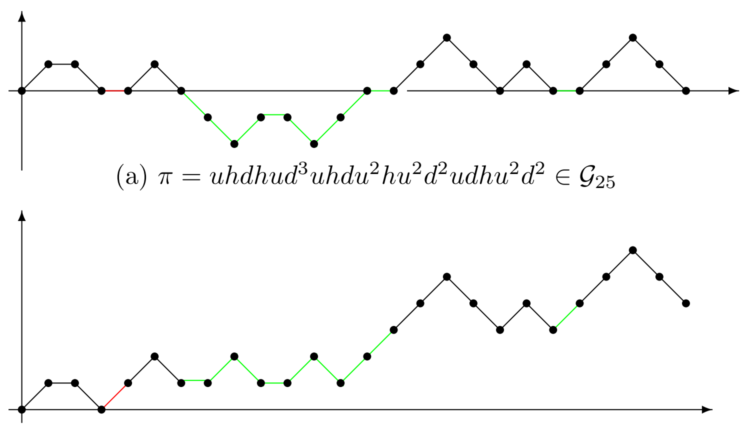
We conclude with the following enumerative result concerning the members of $\mathcal{R}_n^*$.

Theorem 11. *The cardinality of* $\mathcal{R}_n^*$ *is given by the grand Motzkin number* G_{n-1} *for all* $n \geq 1$*.*

Proof. Let us first recall some terminology and introduce a bit of notation. A *grand Motzkin* path of length n is any lattice path from the origin to $(n, 0)$ using u, d and h steps. Let $\mathcal{G}_n$ denote the set of all grand Motzkin paths of length n for $n \geq 1$, which are enumerated by G_n (see, for example, [18, A002426]). The subset of $\mathcal{G}_n$ consisting of those paths that do not dip below the x-axis at any point is denoted by $\mathcal{M}_n$. Members of $\mathcal{M}_n$ are referred to simply as *Motzkin* paths and are enumerated by the n-th Motzkin number M_n. By a *low* h step, it is meant one that lies on the x-axis. Let $\mathcal{K}_n$ denote the set of first quadrant lattice paths from the origin to the line $x = n - 1$ using u, d and h steps that do not contain any low h steps. Since $\mathcal{R}_n^*$ is equivalent to $\mathcal{K}_n$ via $\jmath$, to establish the result, it suffices to show $G_n = |\mathcal{K}_{n+1}|$ for all $n \geq 0$.

Let $\pi \in \mathcal{G}_n$, where $n \geq 1$. Recall that a *unit* ρ of π is a sequence of consecutive steps of the form $\rho = u\rho' d$, where ρ' is a (possibly empty) Motzkin path, or of the form $\rho = d\rho'' u$, where ρ'' is the reflection in the x-axis of some Motzkin path, such that the first step begins and the last step of ρ ends on the x-axis. We will refer to units of the two respective forms as being *positive* and *negative.* A *bad* unit will refer to either a low h step or negative unit. We define a bijection f between $\mathcal{G}_n$ and $\mathcal{K}_{n+1}$ as follows. If π consists exclusively of positive units, then let $f(\pi) = \pi$.

So assume π contains at least one bad unit. In this case, we write $\pi = \alpha\beta\gamma$, where α is a possibly empty sequence of positive units, β is a bad unit and γ comprises the remaining steps of π. We further decompose γ as $\gamma = u^{(1)}v^{(1)} \cdots u^{(k-1)}v^{(k-1)}u^{(k)}$ for some $k \geq 1$, where each $u^{(i)}$ if nonempty consists of positive units and each $v^{(i)}$ consists of a single bad unit. Note that if γ contains no bad units, then $k = 1$ with $\gamma = u^{(1)}$.



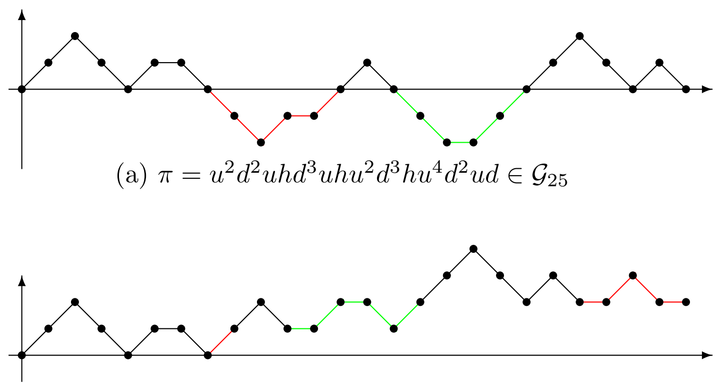
(a) $\pi = uhdhud^3uhdu^2hu^2d^2udhu^2d^2 \in \mathcal{G}_{25}$

(b) $f(\pi) = uhdu^2dhudhudu^4d^2udu^3d^2 \in \mathcal{K}_{26}$

Figure 5: The lattice paths $\pi \in \mathcal{G}_{25}$ and $f(\pi) \in \mathcal{K}_{26}$ in a case when β is a low h.

Given a lattice path τ, let $\text{ref}(\tau)$ be obtained from τ by changing each u to d and each d to u, leaving all h steps unchanged. Let $g(\gamma)$ be obtained from γ by leaving each $u^{(i)}$ section unchanged and replacing $v^{(i)}$ with u if $v^{(i)}$ is a low h and replacing $v^{(i)}$ with $h\text{ref}(\sigma_i)u$ if $v^{(i)}$ is a negative unit with $v_i = d\sigma_i u$. We can now define f by considering cases on β. If β is a low h within $\pi = \alpha\beta\gamma$, then let $f(\pi) = \alpha u g(\gamma)$. On the other hand, if $\beta = d\beta' u$ is a negative unit, then let $f(\pi) = \alpha u g(\gamma) h\text{ref}(\beta')$. Illustrated in Figures 5 and 6 are instances of f when β is a low h or a negative unit, respectively. The first bad unit of π (that is, β) is indicated in red in both figures, with the remaining bad units (that is, those contained in γ) in green. Further, the steps within $f(\pi)$ that correspond to those derived from the bad units of π after the transformation are colored accordingly.

One may verify that the mapping f provides the desired bijection between $\mathcal{G}_n$ and $\mathcal{K}_{n+1}$ by constructing its inverse. We outline how to do so as follows. Given $\lambda \in \mathcal{K}_{n+1}$, let s denote the final height of λ. If $s = 0$, then clearly $f^{-1}(\lambda) = \lambda$, so assume $s > 0$. Let λ^* denote the section of λ consisting of all steps beyond the rightmost step of λ ending at height $s - 1$. Note $\lambda^* = u\delta$, where δ is a possibly empty Motzkin path. Then consider cases based on whether or not δ contains a low h step (when δ is viewed as starting from the origin). If δ does not contain a low h, then δ would correspond to the section $u^{(k)}$ in $\pi = f^{-1}(\lambda)$, where β within π would be a low h in this case. If δ does contain a low h, then write $\delta = \delta^{(1)} h \delta^{(2)}$, where $\delta^{(1)}$ consists of only positive units if nonempty. Then $\delta^{(1)}$ in this case would correspond to $u^{(k)}$ in π and $\delta^{(2)}$ to $\text{ref}(\beta')$, in which case $\beta = d\,\text{ref}(\delta^{(2)})u$.



(a) $\pi = u^2d^2uhd^3uhu^2d^3hu^4d^2ud \in \mathcal{G}_{25}$

(b) $f(\pi) = u^2d^2uhdu^2dhuhdu^3d^2udhudh \in \mathcal{K}_{26}$

Figure 6: The lattice paths $\pi \in \mathcal{G}_{25}$ and $f(\pi) \in \mathcal{K}_{26}$ in a case when β is a negative unit.

The rest of π may be reconstructed successively in a similar fashion considering the remaining steps of λ to the left of λ^*. We make the following further observations. The parameter s of λ corresponds to the number of bad units of π. The units of λ when $s > 0$ are seen to correspond to the positive units of π occurring to the left of the first bad unit. Also, when $s > 0$, the rightmost u ending at a given height can only arise through the transformation of a bad unit. Finally, consider the h steps at a given (positive) level within λ, and among them those after which the path never reaches a lower level (if any). Then the leftmost of these h steps arises from the transformation of a negative unit (either of β or one contained within γ). $\square$

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