A SHORT SIMPLE PROBABILISTIC PROOF OF A WELL KNOWN IDENTITY AND THE DERIVATION OF RELATED NEW IDENTITIES INVOLVING THE BERNOULLI NUMBERS AND THE EULER NUMBERS

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Abstract

An alternative short, elementary probabilistic and combinatorial proof is presented to derive a closed form expression for $I_k := (1/\pi)\int_0^\infty (\sin^k x)/x^k dx$, a well known identity in the literature. An alternative simpler and more elegant form for the value of I_k is also obtained for $k = 2, 4, 6, 8$, generalizing a result derived by R. Butler in 1960. Furthermore, our results show new relations between I_k, the Dirichlet beta functions, the Bernoulli numbers, and the Euler numbers in a way not realized before.

1. Introduction

The integral I_k appears in a number of diverse problems (see, e.g., [2, 4, 7, 8, 9, 11, 17, 19, 21]).

In closed form, I_k is given (see, e.g., [8]) for all integers $k \geq 1$ by

$$I_k = \frac{1}{2^k(k-1)!}\sum_{i=0}^{\gamma_{k/2}}(-1)^i\binom{k}{i}(k-2i)^{k-1}, \tag{1}$$

where $\gamma_s := \lfloor s \rfloor = \text{Floor}(s)$. The values of the first ten values of I_k are:

$$\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{1}{3}, \frac{115}{384}, \frac{11}{40}, \frac{5887}{23040}, \frac{151}{630}, \frac{259723}{1146880}, \frac{15619}{72576}.$$

The numerators and denominators of these ratios are the sequences OEIS A049330 and OEIS A049331, respectively.

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Several rather tedious mathematical methods have been suggested in the literature to prove the identity in Equation (1). For example, Goddard's approach in his version in [7] is based on a relation which is given by

$$\log\left(\frac{\sin x}{x}\right) = -\sum_{i=1}^{\infty} \frac{B_i}{2i(2i)!}(2x)^{2i},$$

where the B_i's are the Bernoulli numbers. The values of these numbers are $B_0 = 1, B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30, B_{10} = 5/66$, etc. (see OEIS A000367 and OEIS A002445). All odd-indexed Bernoulli numbers are zero, except for the convention that $B_1 = -1/2$ is used, and the signs of B_{2n} alternate. Butler applies in [2] a formula due to Poisson (see, e.g., [20, p.443]) to derive Equation (1) by the trapezoidal rule using certain ranges of intervals, dependent on k. He also presented the following alternative and simpler form for the value of I_k in the cases $1 \leq k \leq 4$:

$$I_k = \frac{1}{4}\left\{\frac{E_{k-1}}{(k-1)!} + 1\right\}, \tag{2}$$

where E_{k-1} represents the appropriate Eulerian numbers, which are those used in [10], namely, $E_0 = 1$, $E_1 = 1$, $E_2 = 1$, $E_3 = 2$, $E_4 = 5$, $E_5 = 16$, $E_6 = 61$, $E_7 = 272$, $E_8 = 1385$, $E_9 = 7936$, $E_{10} = 50521$, etc.

Below we provide an alternative elementary probabilistic and combinatorial proof of the identity in Equation (1). Also, a simpler form for I_k is derived for $k = 2, 4, 6, 8$. Furthermore, some interesting relations between I_k, the Dirichlet beta functions, the Bernoulli numbers, and the Euler numbers are discussed.

2. Alternative Proof

We first introduce some notation. Suppose that $U_1, U_2, \ldots, U_k$ are independent and identically distributed uniform random variables on $(0, 1)$. Let $V_i = 2U_i - 1, i = 1, \ldots, k$. Then clearly $V_1, V_2, \ldots, V_k$ are independent and identically distributed uniform variables on $(-1, +1)$, with probability density function

$$g(y) = \begin{cases} 1/2, & \text{for } -1 < y < +1; \\ 0, & \text{elsewhere.} \end{cases} \tag{3}$$

Define $S_k := \sum_{i=1}^{k} U_i$, $T_k := \sum_{i=1}^{k} V_i = 2S_k - k$, and denote the cumulative distribution function of S_k and that of T_k by F_k and G_k, respectively, with corresponding probability density functions f_k and g_k, which exist almost everywhere since F_k and G_k are absolutely continuous with respect to Lebesgue measure.

Theorem 1. *The identity in Equation (1) holds for all integers $k \geq 1$.*

Proof. Note that

$$\begin{aligned} G_k(t) &= P(T_k \le t) = P(2S_k - k \le t) \\ &= P(S_k \le (t+k)/2) = F_k((t+k)/2), \end{aligned}$$

so that, since $0 < S_k < k$ and $-k < T_k < k$,

$$g_k(t) := \frac{d}{dt} G_k(t) = \begin{cases} (1/2) f_k((t+k)/2), & \text{for} - k < t < k; \\ 0, & \text{elsewhere.} \end{cases} \tag{4}$$

The distribution of S_k, the sum of independent and identically distributed uniform random variables on $(0,1)$, is well known in the statistical literature (see, e.g., [1,12,13,14,15,16]). From Remark 3 in [16, p.175] it follows for example that

$$f_k(s) := \frac{d}{ds} F_k(s) = \frac{1}{(k-1)!} \sum_{i=0}^{\gamma_s} (-1)^i \binom{k}{i} (s-i)^{k-1}, \tag{5}$$

for $0 < s < k$. Hence, applying Equations (4) and (5) we therefore have that

$$g_k(t) = \frac{1}{2^k (k-1)!} \sum_{i=0}^{\gamma_k(t)} (-1)^i \binom{k}{i} (t+k-2i)^{k-1}, \tag{6}$$

where γ_k(t):=$\lfloor (t+k)/2 \rfloor$. The following interesting identity can now be deduced from Equations (5) and (6):

$$g_k(0) = I_k. \tag{7}$$

Furthermore, since the V_j's, $j = 1, \ldots, k$, are identically distributed, they have a common characteristic function, that is (see Equation (3)),

$$\begin{aligned} \varphi_j(t) &:= \int_{-\infty}^{+\infty} e^{itv} g(v) dv, \quad (i^2 = -1) \\ &= \frac{\sin t}{t} \qquad\qquad (t \in \mathcal{R}). \end{aligned}$$

Using the fact that the V_j's are independent random variables, it follows that the characteristic function $C_k(t)$ of $T_k := \sum_{i=1}^k V_i$ is given by

$$C_k(t) := \int_{-\infty}^{+\infty} e^{itv} g_k(v) dv = \prod_{j=1}^{k} \varphi_j(t) = \left(\frac{\sin t}{t} \right)^k. \tag{8}$$

Applying the Fourier integral inversion theorem (see Equation (15.20) in [20, p.937]), from Equation (8) we obtain

$$g_k(v) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itv} C_k(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itv} \left(\frac{\sin t}{t} \right)^k dt,$$

which implies that

$$g_k(0) := \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\sin t}{t}\right)^k dt = \frac{1}{\pi}\int_{0}^{+\infty}\left(\frac{\sin t}{t}\right)^k dt. \tag{9}$$

Hence, Equation (1) follows from Equations (7) and (9). □

3. Simplified and New Identities for $I_k, k = 2, 4, 6, 8$

We need the following known results, which are stated as propositions.

Proposition 1 ([5]). *Let $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s, s > 1,$ be the Riemann zeta function, then*

$$\zeta(k)= \frac{(2\pi)^k}{2(k!)}|B_k|, \;\; (k = 2, 4, 6, \dots). \tag{10}$$

Proposition 2 ([20]). *We have*

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^k} = (1-2^{-k})\zeta(k), \quad k > 1. \tag{11}$$

To state the next proposition, we need the following definitions. Let $\beta(k) := \sum_{n=1}^{\infty}(-1)^{n-1}/(2n-1)^k$ be the well-known Dirichlet beta function, and suppose $\{E_k, k = 0, 2, 4, \dots\}$ is the set of Euler numbers (OEIS A028296) for the rest of the discussion below.

Proposition 3 ([3,6]). *For each even $k \geq 2$, we have*

$$\beta(k-1) = \left(\frac{\pi^{k-1}}{2^k}\right)\frac{|E_{k-2}|}{(k-2)!}\ , \tag{12}$$

which can be rewritten as $\beta(k-1) = r\pi^{k-1}$ for $k = 2, 4, 6, 8, \dots,$ where the multiples r are respectively (OEIS A046976 and OEIS A053005):

$$1/4,\ 1/32,\ 5/1536,\ 61/184320,\ \dots.$$

Proposition 4 ([18]). *We have*

$$\sum_{n=1}^{\infty}\frac{\sin^k(nt)}{n^k} = \pi t^{k-1}I_k - \frac{t^k}{2}, \tag{13}$$

for $0 < t < 2\pi$ if $k = 1$, and $0 \leq t \leq 2\pi/k$ if *$k \geq 2$.*

Note that in [18] the notation $\widetilde{S}_k$ is used instead of I_k.

3.1. Relation Between I_k and the Bernoulli Numbers for $k = 2, 4, 6, 8$

Let $t = \pi/4$. Straightforward calculations then show that, since k is even, Equation (13) becomes

$$\left(\frac{1}{\sqrt{2}}\right)^k \sum_{n=1}^{\infty} \frac{1}{(2n-1)^k} + \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^k} = \frac{\pi^k}{2^{2k-2}} \left(I_k - \frac{1}{8}\right). \tag{14}$$

Hence, from Equations (10), (11) and (14) we obtain the following interesting new identity for $k = 2, 4, 6, 8$:

$$I_k = \frac{1}{8} \left\{ \frac{2^k(2^k-1)(2^{k/2}+1)}{k!} |B_k| + 1 \right\}. \tag{15}$$

3.2. Relation Between I_k and the Euler Numbers for $k = 2, 4, 6, 8$

Differentiating with respect to t the left-hand side of Equation (13) term by term (which is permissible by virtue of the uniform convergence of the resulting sum and continuity of $\sin x$ and $\cos x$) and the right-hand side of Equation (13), we obtain

$$k \sum_{n=1}^{\infty} \frac{\sin^{k-1}(nt)\cos(nt)}{n^{k-1}} = (k-1)\pi t^{k-2} I_k - \frac{kt^{k-1}}{2}. \tag{16}$$

Choosing $t = \pi/4$ in Equation (16), it readily follows, since $k-1$ is odd, that

$$k \left(\frac{1}{\sqrt{2}}\right)^{k-1} \left(\frac{1}{\sqrt{2}}\right) \beta(k-1) = \frac{\pi^{k-1}}{2^{2k-1}} \left[8(k-1)I_k - k\right],$$

which yields the following surprising identity for $k = 2, 4, 6, 8$:

$$\beta(k-1) = 2^{(2-3k)/2}\, \pi^{k-1} \left[8(k-1)I_k - k\right]/k. \tag{17}$$

Furthermore, from Equations (12) and (17) the following exciting identity is also obtained for $k = 2, 4, 6, 8$:

$$I_k = \frac{k}{8(k-1)} \left\{ \frac{2^{(k-2)/2}\, |E_{k-2}|}{(k-2)!} + 1 \right\}. \tag{18}$$

Note that both the expressions for I_k presented in Equations (2) and (18) yield that $I_2 = 1/2$ and $I_4 = 1/3$. However, the formula for I_k given in Equation (18) is a generalization to the cases $k = 6$ and $k = 8$ of the identity for I_k in Equation (2), derived in [2].

3.3. Relation Between B_k and E_k for $k = 2, 4, 6, 8$

From Equations (15) and (18) it readily follows that for $k = 2, 4, 6, 8$,

$$2^k(2^k - 1)(2^{k/2} + 1)|B_k| = k^2 2^{(k-2)/2}|E_{k-2}| + k(k-2)!,$$

which also seems to be a new identity in the literature.

3.4. I_k Expressed as a Function of Both B_k and E_k for $k = 2, 4, 6, 8$

Rewrite Equation (13) in terms of x, i.e.,

$$\sum_{n=1}^{\infty} \frac{\sin^k(nx)}{n^k} = \pi x^{k-1} I_k - \frac{x^k}{2}, \tag{19}$$

for $k = 2, 4, 6, 8$. Since k is even, a well known trigonometric identity is given by

$$\sin^k(nx) = \frac{1}{2^k}\binom{k}{k/2} + \frac{2}{2^k}\sum_{i=0}^{(k-2)/2}(-1)^{(k-2i)/2}\binom{k}{i}\cos((k-2i)nx), \tag{20}$$

which can be deduced using De Moivre's identity, Euler's formula and the binomial theorem.

Substituting the expression for $\sin^k(nx)$ in Equation (20) into the left-hand side of Equation (19), and then integrating both sides of the resulting identity (term by term) with respect to x over the intervals $(0, t)$ (which is permissible by virtue of the uniform convergence of the sum and continuity of $\sin x$ and $\cos x$), we obtain

$$\begin{aligned} \frac{1}{2^k}\binom{k}{k/2} t\zeta(k) \quad &+ \quad \frac{2}{2^k}\sum_{i=0}^{(k-2)/2}\frac{(-1)^{(k-2i)/2}}{(k-2i)}\binom{k}{i}\sum_{n=1}^{\infty}\frac{\sin((k-2i)nt)}{n^{k+1}} \\ &= \quad \frac{\pi t^k I_k}{k} - \frac{t^{k+1}}{2(k+1)}. \end{aligned} \tag{21}$$

Choosing $t = \pi/4$ in Equation (21), we then have that

$$\begin{aligned} \frac{1}{2^k}\binom{k}{k/2}(\pi/4)\zeta(k) \quad &+ \quad \frac{1}{2^k}\sum_{j=1}^{k/2}\frac{(-1)^j}{j}\binom{k}{(k-2j)/2}\sum_{n=1}^{\infty}\frac{\sin(jn\pi/2)}{n^{k+1}} \\ &= \quad \frac{\pi^{k+1}}{4^k}\left\{\frac{I_k}{k} - \frac{1}{8(k+1)}\right\}. \end{aligned} \tag{22}$$

Applying Equation (10), the first term on the left-hand side of Equation (22) equals

$$\frac{\binom{k}{k/2}\pi^{k+1}|B_k|}{8(k!)}, \tag{23}$$

and it readily follows, since $k \leq 8$, that the second term on the left-hand side of Equation (22) becomes

$$\beta(k+1)\left\{\binom{1}{3}\binom{k}{(k-6)/2}-\binom{k}{(k-2)/2}\right\}/2^k, \tag{24}$$

with $\binom{k}{j}=0$ for $j<0$ and $\binom{k}{0}=1$.

Note that replacing k by $k+2$ in Equation (12) we have that

$$\beta(k+1)=\left(\frac{\pi^{k+1}}{2^{k+2}}\right)\frac{|E_k|}{k!}. \tag{25}$$

Hence, from Equations (22), (23), (24), and (25) we obtain the following unexpected new identity for $k=2,4,6,8$:

$$I_k=\frac{4^k\binom{k}{k/2}|B_k|}{8(k-1)!}+\frac{|E_k|}{4(k-1)!}\left\{\binom{1}{3}\binom{k}{(k-6)/2}-\binom{k}{(k-2)/2}\right\}+\frac{k}{8(k+1)}.$$

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References

[1] D.M. Bradley and C.R. Gupta, On the distribution of the sum of n non-identically distributed uniform random variables, *Ann. Inst. Statist. Math.* **54** (3) (2002), 689-700.

[2] R. Butler, On the evaluation of $\int_0^\infty (\sin^m t)/t^m dt$ by the trapezoidal rule, *Amer. Math. Monthly* **67** (6) (1960), 566-569.

[3] P. Cerone, On a double inequality for the Dirichlet beta function, *Tamsui Oxf. J. Math. Sci.* **24** (3) (2008), 269-276.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricome, *Tables of Integral Transforms*, vol.1, McGraw-Hill, New York, 1954.

[5] L. Euler, De seriebus quibusdam considerationes, *Comm. Acad. Sci. Petrop.* **12** (1740), 53-96, *Opera Omnia* **14** (1925), 407-462.

[6] S.R. Finch, *Mathematical Constants*, Cambridge Univ. Press, Cambridge, 2003.

[7] L.S. Goddard, The accumulation of chance effects and the Gaussian frequency distribution, *Philos. Mag.* **36** (257) (1945), 428-433.

[8] A.H.R. Grimsey, On the accumulation of chance effects and the Gaussian frequency distribution, *Philos. Mag.* **36** (255) (1945), 294-295.

[9] K. Harumi, S. Katsura, and J.W. Wrench Jr., Values of $(2/\pi)\int_0^\infty ((\sin t)/t)^n dt$, *Math. Comp.* **14** (1960), 379.

[10] L.B.W. Jolley, *Summation of Series*, 3rd edition, Chapman and Hall, Ltd., London, 1925.

[11] R.G. Medhurst and J.H. Roberts, Evaluation of the integral $I_n(b) = (2/\pi)\int_0^\infty((\sin x)/x)^n\cos(bx)dx$, *Math. Comp.* **19** (1965), 113-117.

[12] S.K. Mitra, On the probability distribution of the sum of uniformly distributed random variables, *SIAM J. Appl. Math.* **20** (2) (1971), 195-198.

[13] A.M. Mood, F.A. Graybill, and D.C. Boes, *Introduction to the theory of statistics*, 3rd edition, McGraw-Hill, London, 1974.

[14] E.G. Olds, A note on the convolution of uniform distributions, *Ann. Math. Stat.* **23** (1952), 282-285.

[15] S.A. Roach, The frequency distribution of the sample mean where each member of the sample is drawn from a different rectangular distribution, *Biometrika* **50** (1963), 508-513.

[16] S.M. Sadooghi-Alvandi, A.R. Nematollahi, and R. Habibi, On the distribution of the sum of independent uniform random variables, *Statist. Papers* **50** (1) (2009), 171-175.

[17] L. Silberstein, The accumulation of chance effects and the Gaussian frequency distribution, *Philos. Mag.* **35** (1944), 395-404.

[18] J.W.H. Swanepoel, On a generalization of a theorem by Euler, *J. Number Theory* **149** (2015), 46-56.

[19] R. Thompson, Evaluation of $I_n(b) = 2\pi^{-1}\int_0^\infty((\sin x)/x)^n\cos(bx)dx$, and of similar integrals, *Math. Comp.* **20** (94) (1966), 330-332.

[20] E.C. Titchmarsh, *The Theory of Functions*, 2nd edition, Oxford Univ. Press, London, 1952.

[21] J. Wolstenholme, *Mathematical Problems*, 3rd edition, Macmillan Publishers, London, 1891.