ON A CONJECTURE OF RAMÍREZ ALFONSÍN AND SKAŁBA, III

Yuchen Ding[1]
School of Mathematical Science, Yangzhou University, Yangzhou, People's Republic of China
ycding@yzu.edu.cn

Takao Komatsu[2]
Institute of Mathematics, Henan Academy of Sciences, Zhengzhou, People's Republic of China
and
Department of Mathematics, Institute of Science Tokyo, Japan
komatsu@hnas.ac.cn

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Abstract

Let $1 < c < d$ be two relatively prime integers. For a non-negative integer ℓ, let $g_\ell(c,d)$ be the largest integer n such that $n = cx + dy$ has at most ℓ non-negative solutions (x, y). In this paper we prove that

$$\pi_{\ell,c,d} \sim \frac{\pi\big(g_\ell(c,d)\big)}{2\ell+2} \quad (\text{as } c \to \infty)\,,$$

where $\pi_{\ell,c,d}$ is the number of primes n having more than ℓ distinct non-negative solutions to $n = cx + dy$ with $n \le g_\ell(c,d)$, and $\pi(x)$ denotes the number of all primes less than or equal to x for any real number x. The case where $\ell = 0$ has been proved by Ding, Zhai, and Zhao recently, which was conjectured formerly by Ramírez Alfonsín and Skałba.

1. Introduction

Let $c_1, \dots, c_k$ $(k \ge 2)$ be a set of distinct integers with $c_i > 1$ $(i = 1, \dots, k)$. For a given non-negative integer ℓ, let $S_\ell(c_1, \dots, c_k)$ (written as S_ℓ for shorthand) be the set of all the elements n whose number of solutions to $c_1x_1 + \cdots + c_kx_k = n$ with $x_i \ge 0$ is more than ℓ. The set S_ℓ is called the *ℓ-numerical semigroup* if and

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[1]Corresponding author (initial)
[2]Corresponding author (final)

only if $\gcd(c_1,\dots,c_k)=1$ Then for the set of non-negative integers $\mathbb{N}_0$, the set $\mathbb{N}_0\backslash S_\ell$ is of all the elements of n whose number of solutions to $c_1x_1+\dots+c_kx_k=n$ ($x_1,\dots,x_k\in\mathbb{N}_0$) is less than or equal to ℓ. In [4, 5, 6], properties of the ℓ-numerical semigroups and explicit forms of crucial numbers are discussed[3].

The set $\mathbb{N}_0\backslash S_\ell$ is finite if and only if $\gcd(c_1,\dots,c_k)=1$. Then, there exists the largest element $g_\ell(c_1,\dots,c_k)$, which is called the *ℓ-Frobenius number*. When $\ell=0$, then $g(c_1,\dots,c_k)=g_0(c_1,\dots,c_k)$ is the original Frobenius number, which is the central topic on the classically well-known linear Diophantine problem of Frobenius; see, e.g., the excellent monograph [7] of Ramírez Alfonsín. In general, to find an explicit closed form of $g_\ell(c_1,\dots,c_k)$ is very difficult for $k\ge 3$, but when $k=2$, the ℓ-Frobenius number can be given explicitly: for any non-negative integer ℓ,

$$g_\ell(c,d)=(\ell+1)cd-c-d \tag{1}$$

(see [4, 5, 6] for more general formulas and related concepts).

For integers c,d with $1<c<d$ and $\gcd(c,d)=1$, let $\pi_{\ell,c,d}$ be the number of primes with $n\in S_\ell(c,d)$ and $n\le g_\ell(c,d)$. Also, let $\pi(x)$ denote the number of all primes less than or equal to x for any real number x.

Ramírez Alfonsín and Skałba [8] proved that for any $\varepsilon>0$, there is a constant $k_\varepsilon>0$ such that

$$\pi_{0,c,d}\ge k_\varepsilon\frac{g_0(c,d)}{\bigl(\log g_0(c,d)\bigr)^{2+\epsilon}}\,,$$

and conjectured the following:

$$\pi_{0,c,d}\sim\frac{\pi\bigl(g_0(c,d)\bigr)}{2}\quad(\text{as } c\to\infty)\,. \tag{2}$$

Ding [1] made some progress on Conjecture (2). More precisely, for all but at most

$$O\bigl(N(\log N)^{1/2}(\log\log N)^{1/2+\varepsilon}\bigr)$$

pairs c and d, one has

$$\pi_{0,c,d}=\frac{\pi\bigl(g_0(c,d)\bigr)}{2}+O\left(\frac{\pi\bigl(g_0(c,d)\bigr)}{\bigl(\log\log(cd)\bigr)^{\varepsilon}}\right)\,.$$

Since

$$\frac{\pi\bigl(g_0(c,d)\bigr)}{2}+O\left(\frac{\pi\bigl(g_0(c,d)\bigr)}{\bigl(\log\log(cd)\bigr)^{\varepsilon}}\right)\sim\frac{\pi\bigl(g_0(c,d)\bigr)}{2}\quad(\text{as } c\to\infty)\,,$$

[3] In [4, 5, 6] and other references, the terminology of p-numerical semigroup is frequently used. However, in this paper, we mainly deal with prime numbers, so we do not use p or q, but instead use ℓ.

and the total number of the pairs (c,d) such that $1<c<d$, $\gcd(c,d)=1$ and $cd \ll N$ is evaluated as $\gg N\log N$, Ding's result provided an "almost all" version of (2).

Though it seemed to be out of reach [8], recently Ding, Zhai, and Zhao [2] completely proved (2). In a very recent article [3], Huang and Zhu extended their result to the distributions of prime powers within the interval $\bigl[0, g_0(c,d)\bigr]$. More precisely, they established the asymptotic formula of

$$\#\Bigl\{p^k \le g_0(c,d) : p^k = cx+by,\ x,y\in\mathbb{Z}_{\ge 0},\ p\in\mathcal{P}\Bigr\},$$

where $k\ge 1$ is a given integer and $\mathcal{P}$ is the set of primes.

The main purpose of this paper is to show a more general result of (2) as follows.

Theorem 1. *Let ℓ be a non-negative integer. For integers c and d with $1<c<d$ and $\gcd(c,d)=1$, we have*

$$\pi_{\ell,c,d} \sim \frac{\pi\bigl(g_\ell(c,d)\bigr)}{2\ell+2} \quad (\textit{as } c\to\infty)\,.$$

Remark 1. When $\ell=0$, this is reduced to the main result in [2, Theorem 1.1]. The result itself is still never obvious because the speed of convergence is very slow.

2. Preliminaries

From now on, c and d will always denote two positive integers satisfying $1<c<d$ and $\gcd(c,d)=1$. The following result is straightforward.

Lemma 1. *Suppose that $cx+dy=cx'+dy'$ with $x,y,x',y'\in\mathbb{N}_0$. Then*

$$c|(y-y') \quad \textit{and} \quad d|(x-x').$$

Lemma 2. *Suppose that the number of solutions to*

$$n=cx+dy \quad (x,y\in\mathbb{N}_0)$$

is exactly ℓ. Then we have

$$n=(\ell-1)cd+cx_0+dy_0$$

for some $0\le x_0\le d-1$ and $0\le y_0\le c-1$.

Proof. Suppose that ℓ solutions are given as

$$n=cx_1+dy_1=cx_2+dy_2=\cdots=cx_\ell+dy_\ell\,.$$

Without loss of generality, assume that

$$x_1 < x_2 < \cdots < x_\ell.$$

So, $y_1 > y_2 > \cdots > y_\ell$. Now, by using Lemma 1 we obtain

$$x_2 = x_1 + d, \quad x_3 = x_1 + 2d, \quad \ldots, \quad x_\ell = x_1 + (\ell - 1)d$$

of $0 \le x_1 \le d - 1$ and $0 \le y_\ell \le c - 1$. Thus,

$$n = c\big(x_1 + (\ell - 1)d\big) + dy_\ell = (\ell - 1)cd + cx_1 + dy_\ell.$$

This completes the proof of Lemma 2. $\square$

Recall that $\mathcal{P}$ denotes the set of primes.

Lemma 3. *Let $\pi_{\ell,c,d}$ be defined as in the introduction. Then we have*

$$\pi_{\ell,c,d} := \sum_{\substack{\ell cd < n \le g_\ell(c,d) \\ n = \ell cd + cx_0 + dy_0 \in \mathcal{P} \\ 0 \le x_0 \le d;\, 0 \le y_0 \le c}} 1\,.$$

Proof. If n has more than $\ell + 1$ solutions with the form $cx + dy$ $(x, y \in \mathbb{N}_0)$, then by Lemma 2, we have

$$n = (\ell + k)cd + cx_0 + dy_0 \qquad (0 \le x_0 \le d - 1;\, 0 \le y_0 \le c - 1)\,,$$

for some $k \in \mathbb{Z}^+$. For such an n, we will have

$$n \ge (\ell + 1)cd > g_\ell(c, d).$$

Thus, if $n \le g_\ell(c, d)$ with more than ℓ solutions, then n has exactly $\ell + 1$ solutions (expressions). Applying again Lemma 2, we obtain that

$$n = \ell cd + cx_0 + dy_0 \qquad (0 \le x_0 \le d - 1;\, 0 \le y_0 \le c - 1)\,,$$

provided that $n \le g_\ell(c, d)$.

Furthermore, if $n = \ell cd + cx_0 + dy_0 \le g_\ell(c, d)$ with

$$0 \le x_0 \le d \quad \text{and} \quad 0 \le y_0 \le c,$$

then we clearly have $x_0 \le d - 1$ and $y_0 \le c - 1$.

Now, the lemma follows from the trivial fact that $\ell cd \notin \mathcal{P}$. $\square$

3. A Weighted Version

As usual, the von Mangoldt function $\Lambda(n)$ is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \ (\alpha > 0); \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1. *Let ℓ be a non-negative integer. For integers c and d with $1 < c < d$ and $\gcd(c,d) = 1$, we have*

$$\psi_{\ell,c,d} \sim \frac{g_0(c,d)}{2} \quad (\textit{as } c \to \infty)\,,$$

where

$$\psi_{\ell,c,d} = \sum_{\substack{\ell cd < n \le g_\ell(c,d) \\ n = \ell cd + cx_0 + dy_0 \\ 0 \le x_0 \le d;\, 0 \le y_0 \le c}} \Lambda(n)\,.$$

Proof. Our proof follows from the argument of [2, Theorem 1.1] with some adjustments. For convenience, we will let $g = g_0(c,d)$. Throughout the proof, the integer c is supposed to be sufficiently large.

By the definitions of $\psi_{\ell,c,d}$, we have

$$\psi_{\ell,c,d} = \sum_{\substack{\ell cd < n \le g_\ell(c,d) \\ n = \ell cd + cx_0 + dy_0 \\ 0 \le x_0 \le d;\, 0 \le y_0 \le c}} \Lambda(n) = \sum_{\substack{n \le g \\ n = cx_0 + dy_0 \\ 0 \le x_0 \le d;\, 0 \le y_0 \le c}} \Lambda(\ell cd + n) + O_\ell(\log g)\,,$$

where the big-O_ℓ with subscript ℓ means that the implied constant depends at most on ℓ. For any real α, let

$$f(\alpha) = \sum_{0 \le n \le g} \Lambda(\ell cd + n) e(\alpha n)\,,$$
$$h(\alpha) = \sum_{\substack{0 \le x \le d \\ 0 \le y \le c}} e\big(\alpha(cx + dy)\big)\,,$$

where $e(t)$ denotes $e^{2\pi i t}$ for any number t, as usual. Then by the orthogonal relation, we have

$$\psi_{\ell,c,d} = \int_0^1 f(\alpha) h(-\alpha) d\alpha\,. \tag{3}$$

We are in a position to introduce the Hardy–Littlewood method to evaluate the above integral.

Let $Q < c^{1/3}$ denote a parameter depending only on c and d, which will be determined later. Define the major arcs to be

$$\mathfrak{M}(Q)=\bigcup_{1\le q\le Q}\ \bigcup_{\substack{1\le a\le q\\ \gcd(a,q)=1}}\left\{\alpha:\left|\alpha-\frac{a}{q}\right|\le\frac{Q}{qg}\right\}.$$

By our assumption, we have $Q<g^{1/6}$, from which it follows trivially that the above subsets are pairwise disjoint (see, e.g., [2, Section 2]). In addition, we note that

$$\mathfrak{M}(Q)\subseteq\left[\frac{1}{Q}-\frac{Q}{qg},1+\frac{Q}{qg}\right]\subseteq\left[\frac{Q+1}{g},1+\frac{Q+1}{g}\right].$$

The minor arcs are then defined to be (see [2, Equation (2.1)])

$$\mathfrak{m}(Q)=\left[\frac{Q+1}{g},1+\frac{Q+1}{g}\right]\setminus\mathfrak{M}(Q).$$

From Equation (3), it is clear that

$$\begin{aligned}\psi_{\ell,c,d}&=\int_{\frac{Q+1}{g}}^{1+\frac{Q+1}{g}}f(\alpha)h(-\alpha)d\alpha\\&=\int_{\mathfrak{M}(Q)}f(\alpha)h(-\alpha)d\alpha+\int_{\mathfrak{m}(Q)}f(\alpha)h(-\alpha)d\alpha.\end{aligned}\tag{4}$$

3.1. Estimates of the Minor Arcs

Note that

$$\begin{aligned}|f(\alpha)|&=\left|\sum_{0\le n\le g}\Lambda(\ell cd+n)e(\alpha n)\right|\\&=\left|\sum_{\ell cd\le m\le\ell cd+g}\Lambda(m)e(\alpha m)e(-\alpha\ell cd)\right|\quad(m=\ell cd+n)\\&\le\left|\sum_{\ell cd\le m\le\ell cd+g}\Lambda(m)e(\alpha m)\right|.\end{aligned}$$

By a remarkable theorem of Vinogradov (see, e.g., [9, Theorem 3.1]) as well as the Dirichlet approximation theorem (see, e.g., [9, Lemma 2.1]), we can obtain that

$$\sup_{\alpha\in\mathfrak{m}(Q)}\left|\sum_{m}\Lambda(m)e(\alpha m)\right|\ll_\ell\frac{g(\log g)^4}{Q^{1/2}}+g^{4/5}(\log g)^4,$$

where the implied constant depends on ℓ (see [2, Lemma 3.1]), which means that

$$\sup_{\alpha\in\mathfrak{m}(Q)}|f(\alpha)|\ll_\ell\frac{g(\log g)^4}{Q^{1/2}}+g^{4/5}(\log g)^4.$$

By using the following estimate given in [2, Lemma 3.2]

$$\int_0^1 |h(-\alpha)|d\alpha \ll (\log g)^2\,,$$

we have

$$\begin{aligned}\int_{\mathfrak{m}(Q)} f(\alpha)h(-\alpha) &\le \sup_{\alpha\in\mathfrak{m}(Q)} |f(\alpha)| \int_{\mathfrak{m}(Q)} |h(-\alpha)|d\alpha \\ &\ll_\ell \left(\frac{g(\log g)^4}{Q^{1/2}} + g^{4/5}(\log g)^4\right)\int_0^1 |h(-\alpha)|d\alpha \\ &\ll_\ell \frac{g(\log g)^6}{Q^{1/2}} + g^{4/5}(\log g)^6\,. \end{aligned} \tag{5}$$

3.2. Calculations of the Major Arc

We now calculate the integral on the major arcs:

$$\begin{aligned}\int_{\mathfrak{M}(Q)} f(\alpha)h(-\alpha) &= \sum_{1\le q\le Q}\ \sum_{\substack{1\le a\le q\\ \gcd(a,q)=1}} \int_{\frac{a}{q}-\frac{Q}{qg}}^{\frac{a}{q}+\frac{Q}{qg}} f(\alpha)h(-\alpha)d\alpha \\ &= \sum_{1\le q\le Q}\ \sum_{\substack{1\le a\le q\\ \gcd(a,q)=1}} \int_{-\frac{Q}{qg}}^{\frac{Q}{qg}} f\left(\theta+\frac{a}{q}\right) h\left(-\theta-\frac{a}{q}\right)d\theta \\ &= \int_{-\frac{Q}{g}}^{\frac{Q}{g}} f(\theta)h(-\theta)d\theta + \mathcal{R}\,, \end{aligned} \tag{6}$$

where

$$\mathcal{R} = \sum_{1<q\le Q}\ \sum_{\substack{1\le a\le q\\ \gcd(a,q)=1}} \int_{-\frac{Q}{qg}}^{\frac{Q}{qg}} f\left(\theta+\frac{a}{q}\right) h\left(-\theta-\frac{a}{q}\right)d\theta.$$

We shall see later that the set $\mathcal{R}$ still contributes to the 'error term'.

For any real θ, we have

$$\begin{aligned} f(\theta) &= \sum_{0\le n\le g} \Lambda(n+\ell cd)e(\theta n) \\ &= \sum_{\ell cd\le m\le g+\ell cd} \Lambda(m)e\big(\theta(m-\ell cd)\big) \quad (m=n+\ell cd) \\ &= e(-\theta\ell cd)\widetilde{f(\theta)}\,, \end{aligned}$$

where

$$\widetilde{f(\theta)} = \sum_{\ell cd\le m\le g+\ell cd} \Lambda(m)e(\theta m)\,.$$

Let $\rho(m) = \Lambda(m) - 1$. Then

$$\widetilde{f(\theta)} - \sum_{\ell cd \leq m \leq g+\ell cd} e(\theta m) = \sum_{\ell cd < m \leq g+\ell cd} \rho(m) e(\theta m) + O_\ell\left(\log g\right) .$$

By partial summation, we have

$$\begin{aligned}\sum_{\ell cd < m \leq g+\ell cd} \rho(m) e(\theta m) &= e\big((\ell cd + g)\theta\big) \sum_{m \leq \ell cd+g} \rho(m) \\ &\quad - e(\ell cd\theta) \sum_{m \leq \ell cd} \rho(m) - 2\pi i \theta \int_{\ell cd}^{\ell cd+g} \left(\sum_{m \leq t} \rho(m) \right) e(t\theta) dt . \qquad (7)\end{aligned}$$

By using the Prime Number Theorem, there exists some absolute constant $\kappa_1 > 0$ such that

$$\sum_{m \leq t} \rho(m) = \psi(t) - t \ll t e^{-\kappa_1 \sqrt{\log t}}.$$

Inserting this into Equation (7), it follows that

$$\begin{aligned}\sum_{\ell cd < m \leq g+\ell cd} \rho(m) e(\theta m) &\ll_\ell g e^{-\kappa_1 \sqrt{\log g}} + |\theta| \int_{\ell cd}^{\ell cd+g} t e^{-\kappa_1 \sqrt{\log t}} dt \\ &\ll_\ell g e^{-\kappa_1 \sqrt{\log g}} + |\theta| e^{-\kappa_1 \sqrt{\log g}} \int_{\ell cd}^{\ell cd+g} t \, dt \\ &\ll_\ell g(1 + |\theta| g) e^{-\kappa_1 \sqrt{\log g}} .\end{aligned}$$

Thus,

$$\widetilde{f(\theta)} = \sum_{\ell cd \leq m \leq g+\ell cd} e(\theta m) + O_\ell\big(g(1+|\theta|g) e^{-\kappa_1 \sqrt{\log g}}\big) .$$

Hence,

$$\begin{aligned}f(\theta) &= e(-\theta \ell cd) \sum_{\ell cd \leq m \leq g+\ell cd} e(\theta m) + O_\ell\big(g(1+|\theta|g) e^{-\kappa_1 \sqrt{\log g}}\big) \\ &= \sum_{0 < n \leq g} e(\theta n) + O_\ell\big(g(1+|\theta|g) e^{-\kappa_1 \sqrt{\log g}}\big) .\end{aligned}$$

By the estimates of [2, Lemma 4.4], we have

$$\int_{|\theta| \leq \frac{Q}{g}} f(\theta) h(-\theta) d\theta = \frac{g}{2} + O_\ell\left(\frac{g}{Q} (\log g)^2 + g Q^2 e^{-\kappa_1 \sqrt{\log g}} \right) . \qquad (8)$$

For $Q < c^{1/3}$, by [2, Lemma 4.5] we also have

$$\mathcal{R} = \sum_{2 \leq q \leq Q} \sum_{\substack{1 \leq a \leq q \\ \gcd(a,q)=1}} \int_{|\theta| \leq \frac{Q}{qg}} f\left(\frac{a}{q} + \theta \right) h\left(-\frac{a}{q} - \theta \right) d\theta \ll_\ell d Q^3 . \qquad (9)$$

Bringing together (6), (8) and (9), we conclude that for $Q < c^{1/3}$

$$\int_{\mathfrak{M}(Q)} f(\alpha)h(-\alpha) = \frac{g}{2} + O_\ell\left(\frac{g}{Q}(\log g)^2 + gQ^2 e^{-\kappa_1\sqrt{\log g}} + dQ^3\right). \tag{10}$$

3.3. The Asymptotic Formula

It can be concluded from (4), (5), and (10) that for $Q < c^{1/3}$,

$$\psi_{\ell,c,d} = \frac{g}{2} + O_\ell\left(\frac{g}{Q}(\log g)^2 + gQ^2 e^{-\kappa_1\sqrt{\log g}} + dQ^3 + \frac{g(\log g)^6}{Q^{1/2}} + g^{4/5}(\log g)^6\right).$$

We now choose $Q = (\log g)^{14}$. Then we can obtain

$$\psi_{\ell,c,d} = \frac{g}{2} + O_\ell\left(\frac{g}{\log g}\right), \tag{11}$$

provided that $c \geq (\log g)^{43}$.

For $c \leq (\log g)^{43}$, we have

$$\begin{aligned}
\psi_{\ell,c,d} &= \sum_{\substack{\ell cd < n \leq g_\ell(c,d) \\ n = \ell cd + cx_0 + dy_0 \\ 0 \leq x_0 \leq d;\, 0 \leq y_0 \leq c}} \Lambda(n) = \sum_{\substack{0 < m \leq g \\ m = cx + dy \\ x,y \in \mathbb{N}_0}} \Lambda(\ell cd + m) \\
&= \sum_{\substack{1 \leq y \leq c \\ \gcd(y,c)=1}} \sum_{\substack{m + \ell cd \equiv dy \pmod{c} \\ dy \leq m + \ell cd \leq g}} \Lambda(m + \ell cd) + O_\ell(\log g) \\
&= \sum_{\substack{1 \leq y \leq c \\ \gcd(y,c)=1}} \sum_{\substack{n \equiv dy \pmod{c} \\ dy + \ell cd \leq n \leq g + \ell cd}} \Lambda(n) + O_\ell(\log g) \\
&= \sum_{\substack{1 \leq y \leq c \\ \gcd(y,c)=1}} \big(\psi(g + \ell cd; c, dy) - \psi(dy + \ell cd; c, dy)\big) + O_\ell(\log g).
\end{aligned}$$

Since $c \leq (\log g)^{43} \ll (\log d)^{43}$, by the Siegel–Walfisz theorem we have

$$\psi(g + \ell cd; c, dy) - \psi(dy + \ell cd; c, dy) = \frac{g - dy}{\varphi(c)} + O_\ell\left(g e^{-\kappa_2\sqrt{\log g}}\right),$$

where $\kappa_2 > 0$ is an absolute constant. Thus, for $c \leq (\log g)^{43}$ we have

$$\begin{aligned}
\psi_{\ell,c,d} &= \sum_{\substack{1 \leq y \leq c \\ \gcd(y,c)=1}} \frac{g - dy}{\varphi(c)} + O_\ell\left(g(\log g)^{43} e^{-\kappa_2\sqrt{\log g}}\right) \\
&= g - \frac{1}{2}cd + O_\ell\left(g(\log g)^{43} e^{-\kappa_2\sqrt{\log g}}\right) \\
&= g/2 + O_\ell\left(g/c + g(\log g)^{43} e^{-\kappa_2\sqrt{\log g}}\right).
\end{aligned} \tag{12}$$

Therefore, from (11) and (12) we know that

$$\psi_{\ell,c,d} \sim g/2 \quad (\text{as } c \to \infty).$$

This completes the proof of proposition 1. $\square$

4. Proof of Theorem 1

Proof of Theorem 1. From now on, the symbol p will always denote a prime. For $\ell cd < p \le g_\ell(c,d)$, let

$$\vartheta_{\ell,a,b}(t) = \sum_{\substack{\ell cd < p \le t \\ p = \ell cd + cx_0 + dy_0 \\ 0 \le x_0 \le d;\, 0 \le y_0 \le c}} \log p,$$

and $\vartheta_{\ell,a,b} = \vartheta_{\ell,a,b}\big(g_\ell(c,d)\big)$. From Lemma 3, we obtain that

$$\pi_{\ell,a,b} = \sum_{\substack{\ell cd < p \le g_\ell(c,d) \\ p = \ell cd + cx_0 + dy_0 \\ 0 \le x_0 \le d;\, 0 \le y_0 \le c}} 1 = \frac{\vartheta_{\ell,a,b}}{\log g_\ell(c,d)} + \int_{\ell cd}^{g_\ell(c,d)} \frac{\vartheta_{\ell,a,b}(t)}{t \log^2 t} dt \tag{13}$$

via partial summations. By the Chebyshev estimate, we have

$$\vartheta_{\ell,a,b}(t) \leqslant \sum_{p \leqslant t} \log p \ll t,$$

from which it follows that

$$\int_{\ell cd}^{g_\ell(c,d)} \frac{\vartheta_{\ell,a,b}(t)}{t \log^2 t} dt \ll \int_{\ell cd}^{g_\ell(c,d)} \frac{1}{\log^2 t} dt \ll_\ell \frac{g}{(\log g)^2}. \tag{14}$$

Again, using the Chebyshev estimate, we have

$$\vartheta_{\ell,a,b} = \psi_{\ell,a,b} + O_\ell\big(\sqrt{g}\big). \tag{15}$$

Thus, by Proposition 1 and Equations (13), (14), (15), we conclude that

$$\pi_{\ell,a,b} = \frac{\psi_{\ell,a,b}}{\log g_\ell(c,d)} + O_\ell\left(\frac{\sqrt{g}}{\log g} + \frac{g}{(\log g)^2}\right) \sim \frac{1}{2} \frac{g}{\log g_\ell(c,d)}, \tag{16}$$

as $c \to \infty$. Recall that

$$\pi(g_\ell(c,d)) \sim \frac{g_\ell(c,d)}{\log g_\ell(c,d)} \sim \frac{(\ell+1)g}{\log g_\ell(c,d)} \quad (\text{as } c \to \infty). \tag{17}$$

Now, Theorem 1 follows immediately from (16) and (17). $\square$

5. Final Comment

We further expect that for a fixed $c \geq 3$

$$\pi_{\ell,c,d} \sim \frac{c-2}{2c\ell + 2(c-1)} \pi\big(g_\ell(c,d)\big) \quad (\text{as } d \to \infty)\,.$$

Hence, when $c \to \infty$, our main result is reduced. Further results will follow later.

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