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MEAN SQUARE ESTIMATES FOR GCD-SUM FUNCTIONS

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Abstract

Let $\gcd(m,n)$ denote the greatest common divisor of two positive integers m and n, and let $A(n) := \frac{1}{n}\sum_{m=1}^{n}\gcd(m,n)$. For any real number $x > 3$ and any fixed positive integers k, we investigate mean square estimates for the error term $E_k(x)$ of the summatory function $\sum_{n\le x} A^k(n)$ under the Riemann Hypothesis.

1. Introduction and Main Results

Let $\gcd(n,m)$ denote the greatest common divisor of the positive integers n and m. In 1933, Pillai [2] introduced the gcd-sum function

$$P(n) := \sum_{k=1}^{n}\gcd(k,n) = n\sum_{d|n}\frac{\phi(d)}{d}$$

for any integer $n \ge 1$, where $\phi(n)$ denotes the Euler totient function defined by $\sum_{dl=n} d\mu(l)$ with μ being the Möbius function. Define $A(n) := \frac{P(n)}{n}$ for any integer $n \ge 1$. In 2010, the asymptotic formula of the summatory function $\sum_{n\le x} A^2(n)$ is derived by Tóth [4], who showed that the formula

$$\sum_{n\le x} A^2(n) = xP_4(\log x) + O\left(x^{1/2+\varepsilon}\right)$$

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holds for any real number $x > 3$, where $P_4(u)$ is a polynomial of degree 3 in u. Moreover, he listed some open problems concerning the gcd-sum function, one of which is to derive the asymptotic formula for $\sum_{n\leq x} P^k(n)$, where $k \geq 2$ is a fixed integer. The asymptotic formula of $\sum_{n\leq x} A^k(n)$ was considered by Zhang and Zhai [5], who used classical methods to obtain

$$\sum_{n\leq x} A^k(n) = xQ_{2^k-1}(\log x) + E_k(x),$$

where $E_k(x)$ is estimated by $O\left(x^{\beta_k+\varepsilon}\right)$, and $Q_{2^k-1}(u)$ is a polynomial of degree 2^k-1 in u. For example, the list of β_k is

$$\beta_2 = \frac{1}{2}, \quad \beta_3 = \frac{5}{8}, \quad \beta_4 = \frac{7}{9}, \quad \beta_5 = \frac{31}{36}, \quad \beta_6 = \frac{207}{224}, \quad \beta_k = 1 - \frac{1}{2^{\frac{2k}{3}}50} \quad \text{for } k \geq 7.$$

For any real number $T > 3$ and $k = 3, 4, 5$, they also proved that the mean value formula

$$J_k(T) := \int_1^T E_k(u)du \ll T^{1+\delta_k+\varepsilon} \tag{1}$$

holds, where $\delta_3 = \frac{1}{2}, \delta_4 = 0.6030739$, and $\delta_5 = 0.773114$.

Assume that the Riemann Hypothesis (RH) is true. For any fixed integer $k \geq 2$ and any real number $T > 3$, we consider the mean square estimate of $E_k(x)$,

$$I_k(T) := \int_1^T E_k^2(u)du,$$

under the RH. Then we investigate the integral of $E_k(x)$ defined by $J_k(T)$, which gives us an improvement on the estimate (1). Under the RH, we use some properties of the Riemann zeta-function, Parseval's identity, and the method of Mellin transforms to obtain the following theorem.

Theorem 1. *Assume that the RH is true. Suppose that a fixed number η_k depending on the integer k satisfies the inequality*

$$\frac{1}{2^k}\left(4^k - 3^k - \binom{2^k}{2}\right) + \frac{1}{2^{k+1}} + 1 \leq \eta_k \leq \frac{1}{2^{k+1}}\log\log T - 1.$$

For any sufficiently large real number $T \geq \exp\left(\exp\left(2^{k+1}(\eta_k+1)\right)\right)$, we have

$$I_k(T) \ll T^{2-\frac{1}{2^k}} \exp\left(2\eta_k \frac{\log T}{\log\log T}\right).$$

Remark 1. If $I_k(T)$ may be evaluated as

$$I_k(T) = c_k T^{2-\frac{1}{2^k}} (\log T)^{l_k} + O_k\left(T^{2-\frac{1}{2^k}-\delta}\right) \qquad (\delta > 0)$$

as $T \to \infty$, with a constant $c_k > 0$ and a number $l_k > 0$ depending on k, then we can deduce the Omega-result for $E_k(x)$, that is,

$$E_k(x) = \Omega\left(x^{\frac{1}{2}-\frac{1}{2^{k+1}}} (\log x)^{l_k/2}\right) \qquad (x \to \infty).$$

We use Theorem 1 and the Cauchy–Schwarz inequality to get the following corollary.

Corollary 1. *Let the notation be as above. We have*

$$J_k(T) \ll T^{\frac{3}{2}-\frac{1}{2^{k+1}}} \exp\left(\eta_k \frac{\log T}{\log\log T}\right).$$

Remark 2. Set $\widetilde{\alpha}_k := \frac{1}{2} - \frac{1}{2^{k+1}}$. Note that

$$\widetilde{\alpha}_2 = \frac{3}{8} = 0.375 < \widetilde{\alpha}_3 = \frac{7}{16} = 0.4375 < \widetilde{\alpha}_4 = \frac{15}{32} = 0.46875 < \widetilde{\alpha}_5 = \frac{31}{64} = 0.484375,$$

and $\widetilde{\alpha}_3 < \delta_3,\ \widetilde{\alpha}_4 < \delta_4,\ \widetilde{\alpha}_5 < \delta_5$.

We adopt the following notation. Let $s = \sigma + it$ be a complex variable, and let $\zeta(s)$ denote the Riemann zeta-function. Let ε denote an arbitrary small positive number which may be different at each occurrence.

2. Auxiliary Results

In order to prove our theorem, we need the following lemmas.

Lemma 1 ([1, Theorem 1.2 , Equations (1.23)–(1.25)] or [3, Theorem 2.1, Equations (2.1.9),(2.1.10)])**.** *The Riemann zeta-function $\zeta(s)$ can be analytically continued to a meromorphic function in the whole complex plane $\mathbb{C}$, its only singularity being a simple pole at $s = 1$ with residue 1. It satisfies a functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where $\chi(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\Gamma(1-s)$. Also, in any bounded vertical strip, using the Stirling formula, we deduce $\chi(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{i(t+\frac{\pi}{4})} \left(1 + O\left(\frac{1}{t}\right)\right)$ for $t \geq t_0 > 0$.*

The following lemma states the inversion formula.

Lemma 2 ([1, Equation (A.8)]). *Let $A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, be a Dirichlet series with a finite abscissa of absolute convergence. Then*

$$\sum_{n\leq x}{}' a_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{A(s)x^s}{s} ds,$$

where $c > 0$ is such a number that $A(s)$ is absolutely convergent for Re $s = c$ *and $\sum'$ indicates that if x is an integer, then the last term into the sum becomes $\frac{a_x}{2}$ instead of a_x.*

Suppose that $f(x)x^{\sigma-1}$ belongs to the space $L(0,\infty)$ and that $f(x)$ has bounded variation on every finite interval. Then

$$F(s) = \int_0^{\infty} x^{s-1} f(x) dx, \tag{2}$$

where $s = \sigma + it$ (with σ and t real), is called the Mellin transform of f.

The following lemma states the Parseval identity.

Lemma 3 ([1, Equation (A.5)]). *Assume that f and F are connected via (2),then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\sigma+it)|^2 dt = \int_0^{\infty} f^2(x) x^{2\sigma-1} dx.$$

The following lemma is a famous result concerning the boundary of the Riemann zeta-function under the RH.

Lemma 4 ([1, Equation (1.134)] or [3, Equations (14.2.5),(14.2.6)]). *Assume that the Riemann Hypothesis is true. For $|t| \geq t_0 > 0$, uniformly in σ, we have*

$$\zeta(\sigma+it) \ll (|t|+2)^{\varepsilon} \qquad \text{and} \qquad \frac{1}{\zeta(\sigma+it)} \ll (|t|+2)^{\varepsilon}$$

for any $\sigma > \frac{1}{2}$.

Lastly, the key lemma to proving our theorem is as follows.

Lemma 5. *Let s be a complex variable with* Re $(s) > 1$. *Then we have*

$$\sum_{n=1}^{\infty} \frac{A^k(n)}{n^s} = \zeta^{2^k}(s) G_k(s), \tag{3}$$

where $G_k(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is a Dirichlet series which is absolutely convergent for Re $s > 1/2$. *Moreover, we have*

$$\sum_{n=1}^{\infty} \frac{A^k(n)}{n^s} = \frac{\zeta^{2^k}(s)\widetilde{G}_k(s)}{\zeta^{4^k-3^k-\binom{2^k}{2}}(2s)}, \tag{4}$$

where $\widetilde{G}_k(s) = \sum_{n=1}^{\infty} \frac{\widetilde{g}(n)}{n^s}$ is a Dirichlet series which is absolutely convergent for Re $s > 1/3$.

Proof. The Dirichlet series (3) is due to Lemma 4 in [5]. Expanding $G(s)$ of Lemma 4 in [5] we have

$$G_k(s) = \frac{\widetilde{G}_k(s)}{\zeta^{4^k-3^k-\binom{2^k}{2}}(2s)},$$

where

$$\begin{aligned}
\widetilde{G}_k(s) = \prod_p \left(1-\frac{1}{p^{2s}}\right)^{3^k-4^k+\binom{2^k}{2}} &\Bigg(1+\frac{3^k-4^k+\binom{2^k}{2}}{p^{2s}} - \frac{k2^{k-1}}{p^{s+1}} + \ldots + (-1)^k\frac{1}{p^{s+k}} \\
&+ \frac{4^k-6^k+\binom{2^k}{2}2^k-\binom{2^k}{3}}{p^{3s}} + \frac{k2^{k-1}-2k3^{k-1}}{p^{2s+1}} + \ldots \\
&+ \frac{2k\cdot 2^k3^{k-1}-3k\cdot 4^{k-1}-k\cdot 2^{k-1}\binom{2^k}{2}}{p^{3s+1}} + \ldots\Bigg) \\
= \prod_p \Bigg(1 + \frac{4^k-6^k+\binom{2^k}{2}2^k-\binom{2^k}{3}}{p^{3s}} &- \frac{k2^{k-1}}{p^{s+1}} + \ldots + (-1)^k\frac{1}{p^{s+k}} \\
&+ \frac{k2^{k-1}-2k3^{k-1}}{p^{2s+1}} + \ldots + \frac{2k\cdot 2^k3^{k-1}-3k\cdot 4^{k-1}-k\cdot 2^{k-1}\binom{2^k}{2}}{p^{3s+1}} + \ldots\Bigg).
\end{aligned}$$

Here the product $\widetilde{G}_k(s)$ converges absolutely in the half-plane Re $s > 1/3$. □

3. Proof of Theorem 1

Proof of Theorem 1. Assume that the Riemann Hypothesis is true. Without loss of generality we can assume that $x \in \mathbb{Z} + \frac{1}{2}$. We shall consider the mean square estimate of $E_k(x)$ for any fixed integer $k \geq 2$ under the RH. We use Lemmas 2 and 5 to deduce

$$\sum_{n\leq x} A^k(n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta^{2^k}(s)G_k(s)}{s}x^s ds. \tag{5}$$

We define $\alpha_{0,k}$ as the infimum of $\alpha_k > 0$ for which

$$\int_{-\infty}^{\infty} \frac{|\zeta(\alpha_k+it)|^{2^{k+1}}|G_k(\alpha_k+it)|^2}{|\alpha_k+it|^2}dt \ll 1.$$

We use Equation (5) and the Cauchy residue theorem to obtain

$$\sum_{n\leq x} A^k(n) = M_k(x) + \frac{1}{2\pi i}\int_{\alpha_k-i\infty}^{\alpha_k+i\infty} \frac{\zeta^{2^k}(s)G_k(s)}{s}\left(\frac{1}{x}\right)^{-s} ds,$$

where $M_k(x)$ is the main term, which is $\operatorname*{Res}_{s=1} \zeta^{2^k}(s)G_k(s)\frac{x^s}{s}$. Then we have

$$E_k(x) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\alpha_k-iT}^{\alpha_k+iT}\frac{\zeta^{2^k}(s)G_k(s)}{s}\left(\frac{1}{x}\right)^{-s}ds \tag{6}$$

for $\alpha_{0,k} < \alpha_k < 1$ and close to 1. Since $\zeta^{2^k}(s)G_k(s)s^{-1} \to 0$ uniformly in the strip $\alpha_{0,k} < \alpha'_k < \alpha_k < 1$ as $t \to \pm\infty$, it is seen on integrating over the rectangle $\alpha'_k \pm iT$, $\alpha_k \pm iT$ ($\alpha_{0,k} < \alpha'_k < \alpha_k < 1$), that Equation (6) holds for any $\alpha_k > \alpha_{0,k}$. Using Lemma 3 we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}&\frac{|\zeta(\alpha_k+it)|^{2^{k+1}}|G_k(\alpha_k+it)|^2}{|\alpha_k+it|^2}dt\\ &= \int_0^\infty E_k^2\left(\frac{1}{x}\right)x^{2\alpha_k-1}dx = \int_0^\infty E_k^2(x)x^{-2\alpha_k-1}dx.\end{aligned}$$

We see that if the integral

$$\int_{-\infty}^{\infty}\frac{|\zeta(\alpha_k+it)|^{2^{k+1}}|G_k(\alpha_k+it)|^2}{|\alpha_k+it|^2}dt \tag{7}$$

is bounded for some α_k ($\alpha_{0,k} < \alpha_k < 3/2$), then it is sufficient to show

$$\int_T^{2T} E_k^2(x)dx \ll T^{2\alpha_k+1}$$

for any real number $T \geq 3$, because we know that

$$\int_1^T E_k^2(u)du = \sum_{k\leq\frac{\log T}{\log 2}}\int_{2^{k-1}}^{2^k}E_k^2(u)du \ll T^{2\alpha_k+1}.$$

Next, it remains to evaluate the bound for the integral (7). For any real number $T \geq 3$ and a fixed integer $k \geq 2$, we set

$$L_k(T) := \int_{T/2}^{T}\frac{|\zeta(\alpha_k+it)|^{2^{k+1}}|G_k(\alpha_k+it)|^2}{|\alpha_k+it|^2}dt. \tag{8}$$

For any real number $T \geq 3$, we use Equations (4) and (8) to get

$$L_k(T) := \int_{T/2}^{T}\frac{|\zeta(\alpha_k+it)|^{2^{k+1}}}{|\alpha_k+it|}\cdot\frac{|\widetilde{G}_k(\alpha_k+it)|^2}{|\zeta(2\alpha_k+2it)|^{2(4^k-3^k-\binom{2^k}{2})}|\alpha_k+it|}dt.$$

Let $\varepsilon > 0$ be any sufficiently small number that satisfies Lemma 4. We set

$$\alpha_k := \frac{1}{2} - \frac{1}{2^{k+1}} + \eta_k\varepsilon$$

with a constant $\eta_k > 0$ depending on k, and $\varepsilon := \frac{1}{\log\log T}$. From Lemma 5, we see that the series $\widetilde{G}_k(\sigma+it)$ converges absolutely for $\sigma > \alpha_k > \frac{1}{3}$. For any real number $T \geq 3$, we use Lemmas 1, 4, and 5 and insert α_k into Equation (8) to deduce

$$\begin{aligned} L_k(T) &= \int_{T/2}^{T} \frac{|\chi(\alpha_k+it)|^{2^{k+1}}|\zeta(1-\alpha_k-it)|^{2^{k+1}}}{|\alpha_k+it|} \cdot \frac{|\widetilde{G}_k(\alpha_k+it)|^2}{|\zeta(2\alpha_k+2it)|^{2(4^k-3^k-\binom{2^k}{2})}|\alpha_k+it|} dt \\ &\ll \int_{T/2}^{T} \frac{|\zeta(1-\alpha_k-it)|^{2^{k+1}}}{|\zeta(2\alpha_k+2it)|^{2(4^k-3^k-\binom{2^k}{2})} t^{1+2^{k+1}\eta_k\varepsilon}} dt \\ &\ll \int_{T/2}^{T} \frac{1}{t^{1+\{2^{k+1}(\eta_k-1)-2(4^k-3^k-\binom{2^k}{2})\}\varepsilon}} dt. \end{aligned}$$

Since

$$1-\alpha_k \geq \frac{1}{2}+\varepsilon \quad \text{and} \quad 2^{k+1}(\eta_k-1) - 2\left(4^k-3^k-\binom{2^k}{2}\right) \geq 1,$$

we suppose that a fixed number $\eta_k > 0$ satisfies the inequality

$$\frac{1}{2^k}\left(4^k-3^k-\binom{2^k}{2}\right) + \frac{1}{2^{k+1}} + 1 \leq \eta_k \leq \frac{1}{2^{k+1}}\log\log T - 1.$$

Then we have

$$L_k(T) \ll 1$$

for any sufficiently large positive real number $T \geq \exp(\exp(2^{k+1}(\eta_k+1)))$.

Combining the above results, we have

$$\int_1^T E_k^2(u)du \ll T^{2\alpha_k+1},$$

which yields

$$\int_1^T E_k^2(u)du \ll T^{2-\frac{1}{2^k}} \exp\left(2\eta_k \frac{\log T}{\log\log T}\right). \tag{9}$$

Hence, the theorem is derived. □

4. Proof of Corollary 1

Proof of Corollary 1. We use Equation (9) and the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} \int_{T/2}^{T} E_k(u)du &\ll \left(\int_{T/2}^{T} E_k^2(u)du\right)^{1/2} \left(\int_{T/2}^{T} du\right)^{1/2} \\ &\ll T^{\frac{3}{2}-\frac{1}{2^{k+1}}} \exp\left(\eta_k \frac{\log T}{\log\log T}\right). \end{aligned}$$

□

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