ELEMENTARY PATTERNS FROM THE ERDŐS-STRAUS CONJECTURE

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Abstract

This paper makes the following conjecture: for every prime p there exists a positive integer x with $\lceil \frac{p}{4} \rceil \leq x \leq \lceil \frac{p}{2} \rceil$ and a positive divisor $d \mid x^2$ so that either $d \equiv -px \bmod (4x-p)$ or $d \leq x$ and $d \equiv -x \bmod (4x-p)$. Furthermore, this paper proves that the solutions to these modular equations are in one-to-one correspondence with the solutions of the diophantine equation used in the Erdős-Straus conjecture.

1. Introductory Material

The Erdős-Straus conjecture suggests that for any integer $n \geq 2$ there exist positive integers x, y, and z so that the following diophantine equation holds:

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}. \tag{1}$$

Introduced by Paul Erdős and Ernst Straus in the late 1940s [7], the problem was quickly picked up by other notable mathematicians such as Richard Obláth [13], Luigi Rosati [15], Koichi Yamamoto [26], and Louis Mordell [12]. Richard Guy included this problem in his book *Unsolved Problems in Number Theory* along with many other results on Egyptian fractions [8]. Notable papers use analytic number theory, abstractions, or computational methods to analyze this problem [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], but this paper introduces an insight that will govern how this problem will be resolved. In earlier work we described how it suffices to show the conjecture holds for any prime p [3]. In this previous work we insisted that $x \leq y \leq z$, and here we continue this convention. It was shown that $p \nmid x$, $p \mid z$ and p sometimes divides y. It was also shown that p^2 does not divide x, y, or z. Using the common nomenclature, a

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solution is of *Type I* if $p \nmid y$ and is of *Type II* if $p \mid y$. This paper discusses new results and serves to motivate further research. Some proofs for Type I solutions are found in the final section. The remaining proofs are left to the reader.

2. New Results

The results in this paper are very subtle but quite illuminating. Ultimately, for each prime p, both necessary and sufficient conditions are built to describe the solutions of Equation (1) solely through its smallest solution value, x. The following proposition and corollary derive a sufficient condition for finding Type I solutions to Equation (1).

Proposition 1. *Suppose for a prime p there exists a positive integer x with $\left\lceil \frac{p}{4} \right\rceil \leq x \leq \left\lceil \frac{p}{2} \right\rceil$ and a positive divisor $d \mid x^2$ so that $d \equiv -px \bmod (4x-p)$. Then letting*

$$y = \frac{px+d}{4x-p}$$
$$z = \frac{p\left(x + p\left(\frac{x^2}{d}\right)\right)}{4x-p},$$

we have that $x, y,$ and z are positive integers with $x \leq y \leq z$ and $p \nmid y$.

Corollary 1. *Suppose for a prime p there exists a positive integer x with $\left\lceil \frac{p}{4} \right\rceil \leq x \leq \left\lceil \frac{p}{2} \right\rceil$ and a positive divisor $d \mid x^2$ so that $d \equiv -px \bmod (4x-p)$. This is a Type I solution to Equation (1).*

The following proposition and corollary provide a sufficient condition for finding Type II solutions to Equation (1).

Proposition 2. *Suppose for a prime p there exists a positive integer x with $\left\lceil \frac{p}{4} \right\rceil \leq x \leq \left\lceil \frac{p}{2} \right\rceil$ and a positive divisor $d \mid x^2$ so that $d \leq x$ and $d \equiv -x \bmod (4x-p)$. Then letting*

$$y = \frac{p(x+d)}{4x-p}$$
$$z = \frac{p\left(x + \frac{x^2}{d}\right)}{4x-p},$$

we have that $x, y,$ and z are positive integers with $x \leq y \leq z$ and $p \mid y$.

Corollary 2. *Suppose for a prime p there exists a positive integer x with $\left\lceil \frac{p}{4} \right\rceil \leq x \leq \left\lceil \frac{p}{2} \right\rceil$ and a positive divisor $d \mid x^2$ so that $d \leq x$ and $d \equiv -x \bmod (4x-p)$. This is a Type II solution to Equation (1).*

The following two propositions provide the necessary conditions for finding Type I and Type II solutions to Equation (1).

Proposition 3. *Suppose for a prime p there exist positive integers* $x \le y \le z$ *that satisfy Equation (1) and* $p \nmid y$*. Then it is necessarily true that* $\lceil \frac{p}{4} \rceil \le x \le \lceil \frac{p}{2} \rceil$ *and a positive divisor* $d \mid x^2$ *exists so that*

$$\begin{aligned} &(i) \quad d \equiv -px \bmod (4x - p) \\ &(ii) \quad y = \frac{px + d}{4x - p} \\ &(iii) \quad z = \frac{p\left(x + p\left(\frac{x^2}{d}\right)\right)}{4x - p}. \end{aligned}$$

Proposition 4. *Suppose for a prime p there exist positive integers* $x \le y \le z$ *that satisfy Equation (1) and* $p \mid y$*. n Then it is necessarily true that* $\lceil \frac{p}{4} \rceil \le x \le \lceil \frac{p}{2} \rceil$ *and a positive divisor* $d \mid x^2$ *exists so that*

$$\begin{aligned} &(i) \quad d \le x \\ &(ii) \quad d \equiv -x \bmod (4x - p) \\ &(iii) \quad y = \frac{p(x + d)}{4x - p} \\ &(iv) \quad z = \frac{p\left(x + \frac{x^2}{d}\right)}{4x - p}. \end{aligned}$$

Indeed, Proposition 4 develops both the necessary and sufficient conditions for solving the Erdős-Straus conjecture. The solutions are in one-to-one correspondence with the modular identities in Propositions 1 and 2. This is a key result because it reduces the conjecture to one dimension. That is to say, to prove the Erdős-Straus conjecture we need to show that for every prime p there exists at least one pair, x and d, meeting the appropriate conditions as functions of p. This is summarized in the following conjecture.

Conjecture 1. For every prime p there exists a positive integer x with $\lceil \frac{p}{4} \rceil \le x \le \lceil \frac{p}{2} \rceil$ and a positive divisor $d \mid x^2$ so that either $d \equiv -px \bmod (4x - p)$ or $d \le x$ and $d \equiv -x \bmod (4x - p)$.

3. Computational Motivation Toward a Solution

The strength of this approach is that we have yet to employ different methodologies for different modular classes of prime numbers. At this point, one can use Conjecture 1 to derive Mordell's identities for all primes p except possibly for primes p such that

$p \bmod 840 \in \{1, 121, 169, 289, 361, 529\}$. It has been suggested that this problem can be solved through quadratic residues, and it may be no coincidence that our results suggest $d \mid x^2$. This approach might make it appear that x depends on a divisor of $\lceil \frac{p}{4} \rceil$, but such is not the case for all primes. For clarity, only primes less than 100 are graphed here. Figure 1 may serve as motivation to find a similar pattern.

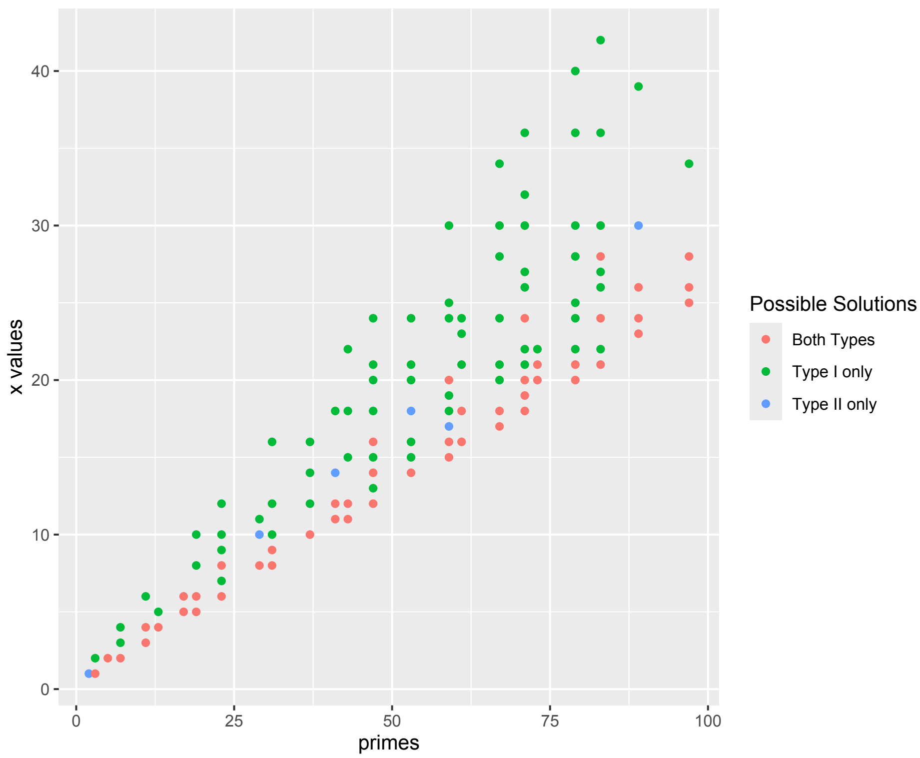

Figure 1: Primes less than 100 versus the possible solution values x in the Erdős-Straus conjecture.

The following two tables are included for increased clarity. Table 1 considers Type I solutions for $p < 100$. We define k to be the integer such that $x = \lceil \frac{p}{4} \rceil + k$. Notice that $0 \leq k \leq \lceil \frac{p}{4} \rceil$. The table provides for each prime p every possible value k that appears for Type I solutions. Notice that for primes $p \neq 2$ such that $p \bmod 24 \not\equiv 1$ we are guaranteed to have a Type I solution when $k = 0$. Also notice for primes p such that $p \mod 4 \equiv 3$ we are guaranteed to have Type I solutions when k is a divisor of $\lceil \frac{p}{4} \rceil$.

Table 2 considers Type II solutions for primes $p < 100$ with the same definition for k. For a given prime p, notice that there are some Type II solutions that have x values that have no Type I solutions. For example, we see that $p = 41$ has $k = 3$,

p	k										
2											
3	0	1									
5	0										
7	0	1	2								
11	0	1	3								
13	0	1									
17	0	1									
19	0	1	3	5							
23	0	1	2	3	4	6					
29	0	3									
31	0	1	2	4	8						
37	0	2	4	6							
41	0	1	7								
43	0	1	4	7	11						
47	0	1	2	3	4	5	9	12			
53	0	1	2	6	7	10					
59	0	1	3	4	5	9	10	15			
61	0	2	5	7	8						
67	0	1	3	4	7	11	13	17			
71	0	1	2	3	4	6	8	9	12	14	18
73	1	2	3								
79	0	1	2	4	5	8	10	16	20		
83	0	1	3	5	6	7	9	15	21		
89	0	1	3	16							
97	0	1	3	9							

Table 1: All possible k values for Type I solutions, $p < 100$.

which corresponds to $x = 14$. There are no Type I solutions when $p = 41$ and $x = 14$.

There are other patterns to consider. Tables for up to five-digit primes have previously been considered, but this paper outlines a motivation for finding a pattern and proving the conjecture.

4. Proofs

We now provide proofs of Propositions 1 and 3.

Proof of Proposition 1. Let p be a prime, x be a positive integer with $\lceil \frac{p}{4} \rceil \le x \le \lceil \frac{p}{2} \rceil$, and d be a positive divisor of x^2 so that $d \equiv -px \bmod (4x - p)$. It should be clear that $p \neq 2$; otherwise, both x and d equal 1 by definition, which contradicts

p	k			
2	0			
3	0			
5	0			
7	0			
11	0	1		
13	0			
17	0	1		
19	0	1		
23	0	2		
29	0	2		
31	0	1		
37	0			
41	0	1	3	
43	0	1		
47	0	2	4	
53	0	4		
59	0	1	2	5
61	0	2		
67	0	1		
71	0	1	2	6
73	1	2		
79	0	1		
83	0	3	7	
89	0	1	3	7
97	0	1	3	

Table 2: All possible k values for Type II solutions, $p < 100$.

the modular equation. If $d \equiv -px \bmod (4x-p)$, then $(px+d) \equiv 0 \bmod (4x-p)$. This implies that $(4x-p) \mid (px+d)$. By definition, p, x, and d are all positive, so $px+d$ is positive. Because $\lceil \frac{p}{4} \rceil \leq x \leq \lceil \frac{p}{2} \rceil$, it should be clear that $4x-p$ is also positive. Letting

$$y = \frac{px+d}{4x-p}$$

shows that y is a positive integer. We next prove that $p \nmid y$. Assume, for the sake of contradiction, that $p \mid y$. This implies that $p \mid (px+d)$, which further implies that $p \mid d$. If $p \mid d$, then $p \mid x^2$; however, from the definition of x, we see that $x < p$. Thus p cannot divide x or x^2. This creates a contradiction, implying that $p \nmid y$. Finally, note that $\left(\frac{p^2x^2}{d}\right) d = p^2x^2 = (-px)^2$ in $\mathbb{Z}$ and $d \equiv -px \bmod (4x-p)$, so we get the following modular equation:

$$\left(\frac{p^2x^2}{d}\right)\cdot(-px)\equiv(-px)\cdot(-px) \bmod (4x-p). \tag{2}$$

Because $p\neq 2$, we see that $-px$ and $4x-p$ are coprime. To show this, let m be a positive integer so that $m\mid(-px)$ and $m\mid(4x-p)$. We necessarily see that m must divide $(-4)(-px)+(-p)(4x-p)=p^2$. Thus $m=1,p$, or p^2. For the sake of contradiction, assume that $m=p$. This implies that $p\mid(4x-p)$, which further implies that $p\mid x$, but this is impossible as $x\leq\left\lceil\frac{p}{2}\right\rceil$. Next, for the sake of contradiction, assume that $m=p^2$. This implies that $p^2\mid(-px)$, which further implies that $p\mid x$. Again, this is impossible. We conclude that $m=1$, so we have that $-px$ and $4x-p$ are indeed coprime. This shows that $(-px) \bmod (4x-p)$ is a unit, with an inverse element in the group $(\mathbb{Z}/(4x-p)\mathbb{Z})^\times$. Applying this inverse element on the right to both sides of Equation (2), we have that $\left(\frac{p^2x^2}{d}\right)\equiv -px \bmod (4x-p)$. We then see that $\left(px+\left(\frac{p^2x^2}{d}\right)\right)\equiv 0 \bmod (4x-p)$. This implies $(4x-p)\mid\left(px+\left(\frac{p^2x^2}{d}\right)\right)$. Note that p,x,d, and $4x-p$ are positive integers. Letting

$$z=\frac{p\left(x+p\left(\frac{x^2}{d}\right)\right)}{4x-p}$$

shows us that z is a positive integer. To finish, we prove $x\leq y\leq z$. First consider when $x<\left\lceil\frac{p}{2}\right\rceil$. Because x is an integer, it should be clear that $x\leq\frac{p}{2}+\frac{d}{4x}$. This implies that $4x^2<2px+d$, $x(4x-p)<px+d$ and

$$x<\frac{px+d}{4x-p}=y.$$

Next, consider when $x=\left\lceil\frac{p}{2}\right\rceil$. Because $p\neq 2$, we have that $x=\frac{p+1}{2}$. We see that

$$\begin{aligned}
y&=\frac{p\left(\frac{p+1}{2}\right)+d}{4\left(\frac{p+1}{2}\right)-p}\\
&=\frac{p(p+1)+2d}{2(p+2)}\\
&=\frac{p+1}{2}+\frac{d-(p+1)}{p+2}.
\end{aligned}$$

Because y is an integer, it follows that $(p+2)\mid(d-(p+1))$. We cannot have $0<d<(p+1)$, because this would imply

$$-1<\frac{d-(p+1)}{p+2}<0.$$

In fact $d\geq(p+1)$ and $x=\frac{p+1}{2}\leq y$. In either scenario we are guaranteed to have $x\leq y$. Because $d\mid x^2$, we see that $d\leq x^2\leq px$. This implies that $d^2\leq p^2x^2$.

Dividing by d shows that

$$d \leq \left(\frac{p^2x^2}{d}\right),$$

which will imply that

$$\frac{px+d}{4x-p} \leq \frac{px+\left(\frac{p^2x^2}{d}\right)}{4x-p}.$$

We conclude that $y \leq z$. $\square$

Proof of Proposition 3. Let p be prime and let $x \leq y \leq z$ be positive integers that satisfy Equation (1) with $p \nmid y$. First, it was shown in [3] that a necessary condition for Type I solutions is that $\lceil \frac{p}{4} \rceil \leq x \leq \lceil \frac{p}{2} \rceil$. Next, slightly changing the notation in [3], we let $m = \gcd(x, y, z), a = \gcd(x, y)/m, b = \gcd(x, z)/m$ and $c = \gcd(y, z)/m$. Using this new notation, it was shown in [3] that $x = abm$, $y = acm$, and $z = bcm$. For this Type I solution, let $d = (4x - p)y - px$. We need to show that $d \mid x^2$. In [3] it was shown for Type I solutions that

$$p = \frac{4abcm - a}{b + c}.$$

This makes

$$\begin{aligned} d &= (4x-p)y - px \\ &= 4xy - (x+y)p \\ &= 4a^2bcm^2 - (abm + acm)\left(\frac{4abcm - a}{b+c}\right) \\ &= 4a^2bcm^2 - 4a^2bcm^2 + a^2m \\ &= a^2m. \end{aligned}$$

Because $a^2m \mid a^2b^2m^2$, we see that $d \mid x^2$. From the definition of $d = (4x-p)y-px$, it should be clear that $d \equiv -px \bmod (4x-p)$. It can be concluded that

$$\begin{aligned} \frac{px+d}{4x-p} &= \frac{px + (4x-p)y - px}{4x-p} \\ &= \frac{(4x-p)y}{4x-p} \\ &= y. \end{aligned}$$

Similarly we find that

$$\begin{aligned}
\frac{p\left(x+p\left(\frac{x^2}{d}\right)\right)}{4x-p} &= \frac{px+\frac{p^2x^2}{(4x-p)y-px}}{4x-p}\\
&= \frac{(4x-p)xyp-p^2x^2+p^2x^2}{(4x-p)(4xy-(x+y)p)}\\
&= \frac{(4x-p)xyp}{(4x-p)(4xy-(x+y)p)}\\
&= \frac{xyp}{4xy-(x+y)p}\\
&= \frac{1}{\frac{4}{p}-\frac{1}{x}-\frac{1}{y}}\\
&= z.
\end{aligned}$$

□

References

[1] A. A. Abdulaziz, On the Egyptian method of decomposing $\frac{2}{n}$ into unit fractions, *Historia Math.* **35** (2008), 1-18.

[2] M. Bello-Hernández, M. Benito and E. Fernández, On Egyptian fractions, preprint, `arXiv: 1010.2035v2`.

[3] K. Bradford, A note on the Erdős-Straus conjecture, *Integers* **21** (2021), #A24, 1-3.

[4] K. Bradford and E. Ionascu, A geometric reduction of the Erdős-Straus conjecture, *Adv. Mod. and Optim.* **17** (1) (2015), 41-54.

[5] E. S. Croot III, *Egyptian Fractions*, Ph. D. thesis, University of Georgia, 2000.

[6] C. Elsholtz and T. Tao, Counting the number of solutions to the Erdős-Straus equation on unit fractions, *J. Aust. Math. Soc.* **94** (1) (2013), 50-105.

[7] P. Erdős, *Az* $1/x_1+\cdots+1/x_n=a/b$ *egyenlet egész számú megoldásairól*, *Mat. Lapok* **1** (1950).

[8] R. Guy, *Unsolved Problems in Number Theory*, Springer, New York, 2004.

[9] E. J. Ionascu and A. Wilson, On the Erdős-Straus conjecture, *Rev. Roumaine Math. Pures Appl.* **56** (1) (2011), 21-30.

[10] D. Li, On the equation 4 /n = 1 /x + 1 /y + 1 /z, *J. Number Theory* **13** (1981), 485-494.

[11] G. G. Martin, *The Distribution of Prime Primitive Roots and Dense Egyptian Fractions*, Ph.D. thesis, University of Michigan, 1997.

[12] L. G. Mordell, *Diophantine Equations*, Academic Press, London, 1969.

[13] M. R. Obláth, Sur l' équation diophantienne $4/n = 1/x_1 + 1/x_2 + 1/x_3$, *Mathesis* **59** (1950), 308-316.

[14] Y. Rav, On the representation of rational numbers as a sum of a fixed number of unit fractions, *J. Reine Angew. Math.* **222** (1966), 207-213.

[15] L. A. Rosati, Sull'equazione diofantea $4/n = 1/x_1 + 1/x_2 + 1/x_3$, *Bollettino dell'Unione Math. Ital.* **9** (1) (1954), 59-63.

[16] J. W. Sander, On $4/n = 1/x + 1/y + 1/z$ and Rosser's sieve, *Acta Arith.* **49** (1988), 281-289.

[17] J. W. Sander, On $4/n = 1/x + 1/y + 1/z$ and Iwaniec' Half Dimensional Sieve, *Acta Arith.* **59** (1991), 183-204.

[18] J. W. Sander, Egyptian fractions and the Erdős-Straus Conjecture, *Nieuw Arch. Wiskd. (4)* **15** (1997), 43-50.

[19] A. Schinzel, On sums of three unit fractions with polynomial denominators, *Funct. Approx. Comment. Math.* **28** (2000), 187-194.

[20] A. Swett, The Erdős Straus conjecture, *Current Research on ESC, rev.10/28/99. http://math.uindy.edu/swett.esc.htm.*

[21] D. G. Terzi, On a conjecture by Erdős-Straus, *Nordisk Tidskr. Info. (BIT)* **11** (1971), 212-216.

[22] R. C. Vaughan, On a problem of Erdos, Straus and Schinzel, *Mathematika* **17** (1970), 193-198.

[23] W. Webb, On $4/n = 1/x + 1/y + 1/z$, *Proc. Amer. Math. Soc.* **25** (1970), 578-584.

[24] W. Webb, On a theorem of Rav concerning Egyptian fractions, *Canad. Math. Bull.* **18** (1) (1975), 155-156.

[25] W. Webb, On the diophantine equation $k/n = a_1/x_1 + a_2/x_2 + a_3/x_3$, *Časopis pro pěstováni matematiy, roč* **10** (1976), 360-365.

[26] K. Yamamoto, On the diophantine equation $4/n = 1/x + 1/y + 1/z$, *Memoirs of the Faculty of Science, Kyushu University, Ser. A* **19** (1) (1965), 37-47.

[27] X. Q. Yang, A note on $4/n = 1/x + 1/y + 1/z$, *Proc. Amer. Math. Soc.* **85** (1982), 496-498.