ON AN EQUATION INVOLVING REPDIGITS AND PRODUCTS OF TWO K-FIBONACCI NUMBERS

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Received: 5/15/24, Revised: 1/6/25, Accepted: 5/27/25, Published: 6/27/25

Abstract

For an integer $k \geq 2$, let $(F_n^{(k)})_{n \geq -(k-2)}$ be the k-Fibonacci sequence. For this sequence, the first k terms are $0, \ldots, 0, 1$, and each term afterwards is the sum of the preceding k terms. In this paper, we will show that $F_n^{(k)} F_m^{(k)}$ can represent a repdigit, where n and m are two positive integers.

1. Introduction

A *repdigit* is a positive integer formed by the repetition of the same digit in its decimal expansion. In particular, it is a number of the form $\frac{a(10^\ell - 1)}{9}$, where $\ell \geq 1$ and $1 \leq a \leq 9$. Let $(F_n)_{n\geq 0}$ be the *sequence of Fibonacci*, defined by $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For an integer $k \geq 2$, we shall consider the following generalization for the Fibonacci sequence.

Let $(F_n^{(k)})_{n \geq -(k-2)}$ be the *k-generalized Fibonacci sequence* defined as

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad \text{for all } n \geq 2,$$

DOI: 10.5281/zenodo.15755971

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with the initial conditions

$$F^{(k)}_{-(k-2)} = F^{(k)}_{-(k-3)} = \cdots = F^{(k)}_0 = 0, \quad \text{and} \quad F^{(k)}_1 = 1.$$

This generalization represents a family of sequences where each new choice of k generates a distinct sequence. Table 1 displays the values of these numbers for the first few values of k and $n \geq 1$.

k	Name	First non-zero terms
2	Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ...
4	Tetranacci	1, 1,2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ...

Table 1: First terms of the sequence $(F^{(k)}_n)_{n\geq -(k-2)}$.

Several mathematicians have shown significant interest in the study of the generalized Fibonacci sequence with repdigits, leading to exploration of numerous related problems. Among the noticeable problems that have been addressed, 55 and 44 are the largest repdigits in the Fibonacci and Tribonacci sequences. This was proved by Luca [11] and Marques [12]. Furthermore, for $k > 3$, Marques conjectured that there are no repdigits with at least two digits belonging to $(F^{(k)}_n)_{n\geq -(k-2)}$. In [2], Bravo and Luca affirmed this conjecture. In [16], the second author determined all the k-Fibonacci and all the k-Lucas numbers expressible as the products of two repdigits.

Furthermore, in [9], Erduvan and Keskin investigated repdigits as products of two Fibonacci and two Lucas numbers. Additionally, in [6], Coufal and Trojovský investigated repdigits as products of terms of k-Bonacci sequences. In this work, we search for repdigits which are the product of two k-Fibonacci numbers. Our main result is given by the following theorem.

Theorem 1. *The only solution of the Diophantine equation*

$$F^{(k)}_n F^{(k)}_m = \frac{a(10^\ell - 1)}{9} \tag{1.1}$$

in positive integers $n, m, \ell, k,$ and a with $3 \leq m \leq n$, $k \geq 3$, $\ell \geq 2$, and $1 \leq a \leq 9$, is

$$(a, k, l, m, n) = (8, 3, 2, 3, 8).$$

We set the condition $m \geq 3$, as when $m \in \{1, 2\}$ then Equation (1.1) transforms into $F^{(k)}_n = \dfrac{a(10^\ell - 1)}{9}$ and this problem was already treated in [2, 11, 12].

The study of the above Diophantine equation and the proof of its obtained result given by Theorem 1 are mainly based on linear forms in logarithms of algebraic numbers and a modified version of the Baker-Davenport reduction method. Our approach begins with the introduction of essential results and crucial definitions for the subsequent sections of this study.

2. The Tools

2.1. Linear Forms in Logarithms

For any non-zero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X-\eta^{(j)}\right)$, we denote by

$$h(\eta)=\frac{1}{d}\left(\log|a|+\sum_{j=1}^{d}\log\max\left(1,|\eta^{(j)}|\right)\right)$$

the *usual absolute logarithmic height of* η. In particular, if $\eta=p/q$ is a rational number with $\gcd(p,q)=1$ and $q>0$, then $h(\eta)=\log\max\{|p|,q\}$. The following properties of the logarithmic height function, $h(\cdot)$, which will be used in the next sections without a special reference, are also known:

$$\begin{aligned} h(\eta\pm\gamma) &\leq h(\eta)+h(\gamma)+\log 2, && (2.1)\\ h(\eta\gamma^{\pm 1}) &\leq h(\eta)+h(\gamma), && (2.2)\\ h(\eta^{s}) &= |s|\,h(\eta) \quad (s\in\mathbb{Z}), && (2.3)\end{aligned}$$

where η and γ are algebraic numbers. We start by recalling Theorem 9.4 of [5], which is a modified version of a result of Matveev [13].

Theorem 2. *Let $\eta_1,\ldots,\eta_s$ be real algebraic numbers and let $b_1,\ldots,b_s$ be nonzero integers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1,\ldots,\eta_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j=\max\{d_{\mathbb{K}}h(\eta),|\log\eta|,0.16\},\quad \textit{for}\quad j=1,\ldots,s.$$

Assume that

$$B\geq\max\{|b_1|,\ldots,|b_s|\}.$$

If $\eta_1^{b_1}\cdots\eta_s^{b_s}-1\neq 0$, then

$$|\eta_1^{b_1}\cdots\eta_s^{b_s}-1|\geq\exp(-1.4\cdot 30^{s+3}\cdot s^{4.5}\cdot d_{\mathbb{K}}^{2}(1+\log d_{\mathbb{K}})(1+\log B)A_1\cdots A_s).$$

2.2. The Reduction Algorithm

Our secondary tool is based on a version of the reduction method of Baker and Davenport [1]. In this paper, we will use the version given by Bravo, Gómez, and Luca [3], which is an immediate variation of the result due to Dujella and Pethő from [8].

Lemma 1. *Let M be a positive integer and let A, B, μ, γ be given real numbers with $A > 0$ and $B > 1$. Assume that p/q is a convergent of the continued fraction of γ such that $q > 6M$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers $u, v,$ and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The above-mentioned lemma cannot be applied when μ is a linear combination of 1 and γ, since then $\varepsilon < 0$. Indeed, when μ is a linear combination of 1 and γ, then $\mu = a + b\gamma$ for some integers a and b. Substituting this into the inequality from the previous equation, it is simplified into

$$0 < |(u + a)\gamma - (v - b)| < AB^{-w}.$$

Consequently, we find that $\varepsilon = -M \cdot ||\gamma q|| < 0$ for any chosen q and M, making it impossible to apply Lemma 1. Hence, we turn to the following useful property of continued fractions.

Lemma 2. *Let p_i/q_i be the convergents of the continued fraction $[a_0, a_1, \ldots]$ of the irrational number γ. Let M be a positive integer and put $a_L := \max\{a_i | 0 \leq i \leq N + 1\}$ where $N \in \mathbb{N}$ is such that $q_N \leq M < q_{N+1}$. If $x, y \in \mathbb{Z}$ with $x > 0$, then*

$$|x\gamma - y| > \frac{1}{(a_L + 2)x}, \quad \text{for all } x < M.$$

2.3. Properties of k-Generalized Fibonacci Sequence

This subsection is focused on reviewing important facts and properties of the k-Fibonacci sequence, which will be used later. The defining characteristic polynomial for this sequence is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

$\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle (see, for example [14], [15], and [19]). This root is real, positive, and satisfies $\alpha(k) > 1$. The other roots are strictly inside the unit circle. In [19], Wolfram showed that

$$2(1 - 2^{-k}) < \alpha(k) < 2, \quad \text{for all } k \geq 2. \tag{2.4}$$

We simplify this notation by omitting the dependence on k of α. For $s \geq 2$, let

$$f_s(x) := \frac{x-1}{2+(s+1)(x-2)}.$$

In [3], Bravo, Gomez, and Luca proved that the following inequalities

$$1/2 < f_k(\alpha) < 3/4 \quad \text{and} \quad \left| f_k(\alpha^{(i)}) \right| < 1,\ 2 \leq i \leq k,$$

hold, where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. In the same paper, they proved that the number $f_k(\alpha)$ is not an algebraic integer. In addition, in [10], Gomez and Luca proved that the logarithmic height of $f_k(\alpha)$ satisfies

$$h(f_k(\alpha)) < 2 \log k, \quad \text{for all } k \geq 3. \tag{2.5}$$

With the above established notation, Dresden and Du demonstrated in [7] that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}) {\alpha^{(i)}}^{n-1} \tag{2.6}$$

and

$$|e_k(n)| < \frac{1}{2}, \quad \text{where } e_k(n) = F_n^{(k)} - f_k(\alpha)\alpha^{n-1}, \tag{2.7}$$

for all $n \geq 2-k$ and $k \geq 2$. Furthermore, for $n \geq 1$ and $k \geq 2$, it was proved in [2] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}. \tag{2.8}$$

Additionally, note that the initial $k+1$ non-zero terms in $(F_n^{(k)})_{n \geq -(k-2)}$ are powers of 2, namely

$$F_1^{(k)} = 1, \quad F_2^{(k)} = 1, \quad F_3^{(k)} = 2, \quad F_4^{(k)} = 4, \ldots, \quad F_{k+1}^{(k)} = 2^{k-1}.$$

We finish this section with the following important lemmas that will be used in the proofs.

Lemma 3 ([18], Lemma 2.2)**.** *Let $a, x \in \mathbb{R}$ and $0 < a < 1$. If $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} \cdot |x|.$$

Lemma 4 ([4], Lemma 3)**.** *If $n < 2^{k/2}$, then the following estimate holds:*

$$F_n^{(k)} = 2^{n-2}(1+\zeta_1), \quad \textit{where } |\zeta_1| < \frac{2}{2^{k/2}}.$$

3. Repdigits as Product of Two k-Fibonacci Numbers

This section is devoted to showing Theorem 1.

3.1. Preliminary Considerations

We start our examination of Equation (1.1) by considering the range $2 \leq m \leq n \leq k+1$. Within this context, we have $F_n^{(k)} = 2^{n-2}$ and $F_m^{(k)} = 2^{m-2}$. Hence, Equation (1.1) transforms into

$$2^{n+m-4} = \frac{a(10^\ell - 1)}{9}. \tag{3.1}$$

For any rational number x, let $\nu_2(x)$ denote the 2-adic valuation of x. Since $\nu_2(a(10^\ell - 1)/9) \leq 3$, then by comparing the 2-adic valuation on both sides of (3.1), one gets $2 \leq m \leq n \leq 7$. In this specified range, Equation (3.1) does not possess solutions. Thus, from now on, we proceed under the condition $n \geq k+2 \geq 5$.

Now, we will determine the correlation between the sizes of ℓ and n. Using inequalities (2.8) and $10^{\ell-1} < a(10^\ell - 1)/9$, we obtain

$$10^{\ell-1} < \frac{a(10^\ell - 1)}{9} = F_n^{(k)} F_m^{(k)} < \alpha^{n+m-2} < \alpha^{2n-2}.$$

Consequently, we get

$$\ell < (2n-2)\left(\frac{\log\alpha}{\log 10}\right) + 1 = n\left(\frac{2\log\alpha}{\log 10}\right) - \left(\frac{2\log\alpha}{\log 10}\right) + 1.$$

Moreover, utilizing (2.4), we get

$$\ell < n. \tag{3.2}$$

3.2. An Inequality for n Versus k

Now, we illustrate the following lemma which provides an inequality relating n to k.

Lemma 5. *If (a, k, ℓ, m, n) is a solution in integers of Equation (1.1) with $k \geq 3$ and $n \geq k+2$, then the inequality*

$$n < 1.64 \times 10^{29} k^8 \log^5 k$$

holds.

Proof. Employing estimate (2.6), Equation (1.1) can be expressed as follows:

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))(f_k(\alpha)\alpha^{m-1} + e_k(m)) = a\left(\frac{10^\ell - 1}{9}\right),$$

i.e.,

$$f_k^2(\alpha)\alpha^{n+m-2} - \frac{a10^\ell}{9} = -e_k(m)f_k(\alpha)\alpha^{n-1} - e_k(n)f_k(\alpha)\alpha^{m-1} - e_k(n)e_k(m) - \frac{a}{9}.$$

By taking the absolute value, dividing both sides by $f_k^2(\alpha)\alpha^{n+m-2}$, and using the fact that $f_k(\alpha) > 1/2$, along with (2.7), we arrive at

$$\left|\frac{a}{9f_k^2(\alpha)} \cdot \alpha^{-(n+m-2)} \cdot 10^\ell - 1\right| \le \frac{1}{\alpha^{m-1}} + \frac{1}{\alpha^{n-1}} + \frac{5}{\alpha^{n+m-2}} < \frac{7}{\alpha^{m-1}}.$$

Define

$$\Gamma_1 := \frac{a}{9f_k^2(\alpha)} \cdot \alpha^{-(n+m-2)} \cdot 10^\ell - 1. \tag{3.3}$$

Consequently, we obtain

$$|\Gamma_1| < \frac{7}{\alpha^{m-1}}. \tag{3.4}$$

We have $\Gamma_1 \neq 0$, because if we suppose that $\Gamma_1 = 0$, we would get

$$\frac{a10^\ell}{9} = f_k^2(\alpha)\alpha^{(n+m-2)}.$$

After applying an automorphism from the Galois group of the decomposition field $\Psi(x)$ over $\mathbb{Q}$ to the above relation and then taking absolute values, we conclude that for any $i \ge 2$, we have

$$\frac{100}{9} \le \frac{a10^\ell}{9} = |f_k(\alpha_i)|^2 \cdot |\alpha_i|^{n+m-2} < 1,$$

which is a contradiction. With the goal of applying Theorem 2 to Γ_1 given by (3.3), the parameters can be chosen as:

$$(\eta_1, b_1) = ((a/(9f_k^2(\alpha)), 1), \quad (\eta_2, b_2) = (\alpha, -(n+m-2)), \quad (\eta_3, b_3) = (10, \ell).$$

Since the algebraic numbers η_1, η_2, η_3 are members of $\mathbb{K} := \mathbb{Q}(\alpha)$, it follows that $d_{\mathbb{K}} = k$. Next, we estimate the usual absolute logarithmic heights of η_1 followed by that of η_2 and η_3. Using estimate (2.5) and the properties (2.2) and (2.3), we see that for all $k \ge 3$,

$$\begin{array}{rcl} h(\eta_1) & \le & h(a/9) + 2h(f_k(\alpha)) \\ & < & \log 9 + 4\log k \\ & < & 6.1\log k. \end{array}$$

Moreover, we have $h(\eta_2) = (\log\alpha)/k < (\log 2)/k$ and $h(\eta_3) = \log 10$. Then, we can choose

$$A_1 = 6.1k\log k = \max\{kh(\eta_1), |\log\eta_1|, 0.16\}$$

$$A_2 = \log 2 = \max\{kh(\eta_2), |\log\eta_2|, 0.16\}$$

and

$$A_3 = k \log 10 = \max\{kh(\eta_3), |\log \eta_3|, 0.16\}.$$

Finally, as $m \le n$ and using inequality (3.2), we can take $B = 2n$. Therefore, Theorem 2 gives

$$\begin{aligned} |\Gamma_1| &> \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2 (1+\log k)(1+\log 2n)(6.1k \log k)(\log 2)(k \log 10)\right) \\ &> \exp\left(-2.8 \cdot 10^{12} k^4 \log^2 k (1+\log 2n)\right), \end{aligned}$$

where we have used the fact that $1 + \log k < 2 \log k$, which holds for $k \ge 3$. Comparing this lower bound with the upper bound of $|\Gamma_1|$ as given in (3.4), we obtain

$$(m-1)\log \alpha < 2.9 \cdot 10^{12} k^4 \log^2 k (1 + \log 2n). \tag{3.5}$$

We return to Equation (1.1) and we use again (2.6) to reformulate it as

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))F_m^{(k)} = \frac{a(10^\ell - 1)}{9},$$

i.e.,

$$f_k(\alpha)\alpha^{n-1} - \frac{a10^\ell}{9F_m^{(k)}} = -\frac{a}{9F_m^{(k)}} - e_k(n). \tag{3.6}$$

Following the previous steps, by taking the absolute value and dividing through by $f_k(\alpha)\alpha^{n-1}$, we get

$$\left| \frac{a}{9F_m^{(k)} f_k(\alpha)} \cdot \alpha^{-(n-1)} \cdot 10^\ell - 1 \right| \le \frac{3}{2f_k(\alpha)\alpha^{n-1}} < \frac{3\alpha}{\alpha^n} < \frac{6}{\alpha^n}.$$

Define

$$\Gamma_2 := \frac{a}{9F_m^{(k)} f_k(\alpha)} \cdot \alpha^{-(n-1)} \cdot 10^\ell - 1.$$

Hence, we see that

$$|\Gamma_2| < \frac{6}{\alpha^n}. \tag{3.7}$$

As above, we use the same argument to show that $\Gamma_2 \neq 0$. Now, we will apply Theorem 2 to Γ_2 by fixing the following parameters:

$$(\eta_1, b_1) = (a/(9F_m^{(k)} f_k(\alpha)), 1), \quad (\eta_2, b_2) = (\alpha, -(n-1)), \quad (\eta_3, b_3) = (10, \ell).$$

Once again, we consider $\mathbb{K} = \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. As before, we can take

$$A_2 = \log 2 \quad \text{and} \quad A_3 = k \log 10.$$

Now, we still need to determine A_1. Using the estimates (2.5) and (3.5), along with properties (2.1)-(2.3), for all $k \ge 3$, we deduce that

$$\begin{aligned} h(\eta_1) &\le h\left(\frac{a}{9}\right) + h(F_m^{(k)}) + h(f_k(\alpha)) \\ &< \log 9 + (m-1)\log \alpha + 2 \log k \\ &< 3 \times 10^{12} k^4 \log^2 k (1 + \log 2n). \end{aligned}$$

Conversely, given that

$$\eta_1 = \frac{a}{9F_m^{(k)} f_k(\alpha)} < 2 \quad \text{and} \quad \eta_1^{-1} = \frac{9F_m^{(k)} f_k(\alpha)}{a} < \frac{27\alpha^{m-1}}{4},$$

then, by (3.5), we see that

$$|\log \eta_1| < (m-1)\log\alpha + \log 6.75 < 3 \times 10^{12} k^4 \log^2 k (1 + \log 2n).$$

This indicates that

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 3 \times 10^{12} k^5 \log^2 k (1 + \log 2n) = A_1.$$

Taking for $B = 2n$, we can apply Theorem 2 to Γ_2 and then compare the resulting inequality with (3.7) to obtain

$$n \log \alpha < 6.1 \times 10^{24} k^8 \log^3 k \log^2 n,$$

where we have used the inequalities $1 + \log k < 2\log k$ and $(1 + \log 2n) < 2.1 \log n$, which are valid for $k \geq 3$ and $n \geq 5$. For smaller values of n (i.e., $n = 1, 2, 3, 4$), the inequality $(1 + \log 2n) < 2.1 \log n$ does not hold as the left-hand side exceeds the right-hand side in these cases. Therefore, we find

$$\frac{n}{\log^2 n} < 1.1 \times 10^{25} k^8 \log^3 k.$$

It is evident that the inequality

$$\frac{x}{\log^2 x} < A \text{ implies } x < 4A\log^2 A, \text{ whenever } A \geq 100, \tag{3.8}$$

which is derived from [17, Lemma 7] for $m = 2$. Hence, substituting $A := 1.1 \times 10^{25} k^8 \log^3 k$ in Inequality (3.8) and applying the inequality $57.7 + 8\log k + 3\log\log k < 61 \log k$, valid for all $k \geq 3$, we obtain

$$\begin{aligned} n &< 4(1.1 \times 10^{25} k^8 \log^3 k)(\log(1.1 \times 10^{25} k^8 \log^3 k))^2 \\ &< (4.4 \times 10^{25} k^8 \log^3 k)(57.7 + 8\log k + 3\log\log k)^2 \\ &< 1.64 \times 10^{29} k^8 \log^5 k. \end{aligned}$$

Based on this, the proof of Lemma 5 is complete. □

3.3. The Case $3 \leq k \leq 430$

Within this subsection, our focus is to examine the small values of k, specifically in the range $[3, 430]$. Define

$$\Lambda_1 := \log(\Gamma_1 + 1) = \ell \log 10 - (n + m - 2)\log\alpha + \log(a/(9f_k^2(\alpha))).$$

Assume that $m \geq 10$. With the help of estimate (3.4) and the use of the fact that $\alpha > 1.75$, we get $|\Gamma_1| < 0.05$. Putting $d = 0.05$ in Lemma 3, we obtain

$$|\Lambda_1| < \frac{-\log 0.95}{0.05} \cdot |\Gamma_1| < 14.4 \cdot \alpha^{-m}.$$

Thus, we get

$$\left| \ell \cdot \frac{\log 10}{\log \alpha} - (n+m-2) + \frac{\log(a/(9f_k^2(\alpha)))}{\log \alpha} \right| < 25.8 \cdot \alpha^{-m}. \tag{3.9}$$

We apply Lemma 1 to Λ_1, for each $a \in \{1, \cdots, 9\}$ and $k \in [3, 430]$, by taking as parameters

$$\gamma = \frac{\log 10}{\log \alpha}, \quad \mu_{(k,a)} = \frac{\log(a/(9f_k^2(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) = (25.8, \alpha).$$

For each $k \in [3, 430]$ and $a \in \{1, \ldots, 9\}$, we find a reliable approximation of γ. Additionally, we obtain a convergent p_i/q_i of the continued fraction of γ, satisfying the conditions $q_i > 6M_k$ and $\varepsilon = \varepsilon_{(k,a)} = ||\mu_{(k,a)} q_i|| - M_k ||\gamma q_i|| > 0$. Here, $M_k = \lfloor 1.64 \times 10^{29} k^8 \log^5 k \rfloor$, representing an upper bound for ℓ as derived from Lemma 5. Using Mathematica, we see that q_{185} fulfills the conditions specified in Lemma 1. After completing this step, Lemma 1 is applied to Inequality (3.9). By employing a computer program with Mathematica, it was determined for $k = 430$ and $a = 9$ that $\varepsilon > 1.02 \times 10^{-36}$ and the highest value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ across all $k \in [3, 430]$ and $a \in \{1, \ldots, 9\}$ is 425.623. This value serves as an upper bound of m as dictated by Lemma 1.

Note that for the remaining values of k and a, it is observed that the corresponding results yield significantly smaller values compared to the chosen upper bound for m of 425.623.

Let us consider $3 \leq m < 426$ and

$$\Lambda_2 := \log(\Gamma_2 + 1) = \ell \log 10 - (n-1)\log \alpha + \log(a/(9F_m^{(k)} f_k(\alpha))).$$

Assuming $n \geq 10$, with the given estimate (3.7) and considering $\alpha > 1.75$, it follows that $|\Gamma_2| < 0.03$. Substituting $d = 0.03$ in Lemma 3, we see that

$$|\Lambda_2| < \frac{-\log 0.97}{0.03} \cdot |\Gamma_2| < 6.1 \cdot \alpha^{-n}.$$

Hence, we get

$$\left| \ell \cdot \frac{\log 10}{\log \alpha} - (n-1) - \frac{\log(a/(9F_m^{(k)} f_k(\alpha)))}{\log \alpha} \right| < 11 \cdot \alpha^{-n}. \tag{3.10}$$

In view to apply Lemma 1 to Λ_2, for all $a \in \{1, \cdots, 9\}$ and $3 \leq m \leq 425$, we consider

$$\gamma = \frac{\log 10}{\log \alpha}, \quad \mu_{(k,m,a)} = \frac{\log(a/(9F_m^{(k)} f_k(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) = (11, \alpha).$$

Again, considering each pair $(k, m) \in [3, 430] \times [3, 425]$ and $a \in \{1, \ldots, 9\}$, we find a reliable approximation of γ and a convergent p_i/q_i of the continued fraction of γ ensuring $q_i > 6M_k$ and $\varepsilon = \varepsilon_{(k,m,a)} = ||\mu_{(k,m,a)} q_i|| - M_k ||\gamma q_i|| > 0$, where $M_k = \lfloor 1.64 \times 10^{29} k^8 \log^5 k \rfloor$, serving as an upper bound of ℓ obtained from Lemma 5. Again, we use Mathematica to verify that q_{179} satisfies the conditions of Lemma 1. Next, we apply Lemma 1 on Inequality (3.10). Our Mathematica computation revealed for $k = 409$, $a = 9$, and $m = 3$ that $\varepsilon > 2.22 \times 10^{-34}$ and the highest value attained by $\dfrac{\log(Aq/\varepsilon)}{\log B}$ across all $(k, m) \in [3, 430] \times [3, 425]$ and $a \in \{1, \ldots, 9\}$ is 425.634, serving as an upper bound for n by Lemma 1. The other upper bounds for n obtained with the remaining values of k, m, and a fall substantially below the established upper bound for n of 425.634.

Therefore, we conclude that the possible solutions (a, k, l, m, n) of Equation (1.1), where $k \in [3, 430]$ and $a \in \{1, \cdots, 9\}$, satisfy $m \leq n \leq 425$. Hence, utilizing Inequality (3.2), we derive $\ell \leq 424$.

Finally, we use Mathematica to manage a comparative analysis between $F_n^{(k)} F_m^{(k)}$ and $\dfrac{a(10^\ell - 1)}{9}$ over the intervals $k + 2 \leq n \leq 425$, $m \leq n$, and $2 \leq \ell \leq 424$, where $\ell < n$, confirming that the solutions to Equation (1.1) are exclusively those enumerated in Theorem 1.

3.4. The Case $k > 430$

In this subsection, we undertake an examination of the large values of k, precisely when $k > 430$. A simple verification for $k > 430$ affirms that

$$m \leq n < 1.64 \times 10^{29} k^8 \log^5 k < 2^{k/2}.$$

Thus, following Lemma 4, we derive

$$F_n^{(k)} = 2^{n-2}(1 + \zeta_1), \quad \text{where } |\zeta_1| < \frac{2}{2^{k/2}}. \tag{3.11}$$

and

$$F_m^{(k)} = 2^{m-2}(1 + \zeta_2), \quad \text{where } |\zeta_2| < \frac{2}{2^{k/2}}. \tag{3.12}$$

By substituting (3.11) and (3.12) into Equation (1.1), we get

$$2^{n+m-4} - \frac{a10^\ell}{9} = 2^{n+m-4} \left(-\zeta_1 - \zeta_2 - \zeta_1\zeta_2\right) - \frac{a}{9}.$$

Hence, we obtain

$$\left|2^{n+m-4}-\frac{a10^{\ell}}{9}\right| \leq 2^{n+m-4}\left(|\zeta_1|+|\zeta_2|+|\zeta_1\zeta_2|\right)+\frac{a}{9} \leq \frac{2^{n+m-2}}{2^{k/2}}+\frac{2^{n+m-2}}{2^{k}}+1.$$

Consequently, dividing through by 2^{n+m-4} and using the fact that $n \geq m$ give us

$$\begin{aligned}\left|1-\frac{a}{9}\cdot 10^{\ell}\cdot 2^{-(n+m+4)}\right| &< \frac{4}{2^{k/2}}+\frac{4}{2^{k}}+\frac{1}{2^{2m-4}}\\ &< \frac{8.5}{2^{\min\{k/2,m-2\}}}.\end{aligned}$$

Define

$$\Gamma_3 := \frac{a}{9}\cdot 2^{-(n+m-4)}\cdot 10^{\ell}-1.$$

Thus, we deduce that

$$|\Gamma_3| < \frac{8.5}{2^{\min\{k/2,m-2\}}}. \tag{3.13}$$

We have $\Gamma_3 \neq 0$, because assuming $\Gamma_3 = 0$ leads to $a\cdot 10^{\ell} = 9\cdot 2^{n+m-4}$. Consequently, this would imply that 5 divides $9\cdot 2^{n+m-4}$, which is an impossibility, for $3 \leq m \leq n$. We are now in a position to apply Theorem 2 to Γ_3, taking into account the following parameters:

$$(\eta_1,b_1)=(a/9,1),\quad (\eta_2,b_2)=(2,-(n+m-4)),\quad (\eta_3,b_3)=(10,\ell).$$

Then, the usual absolute logarithmic heights of these numbers are given by

$$h(\eta_1)=\log 9,\quad h(\eta_2)=\log 2,\quad \text{and}\quad h(\eta_3)=\log 10.$$

Observing that η_1,η_2,η_3 belong to $\mathbb{K}=\mathbb{Q}$, we take $d_{\mathbb{K}}=1$. Consequently, we opt for:

$$A_1=\log 9,\ A_2=\log 2,\ A_3=\log 10.$$

Finally, we choose $B=2n$ and we apply Theorem 2 to Γ_3, which gives us

$$\begin{aligned}|\Gamma_3| &> \exp\left(-1.4\cdot 30^6\cdot 3^{4.5}\cdot(1+\log 2n)(\log 9)(\log 2)(\log 10)\right)\\ &> \exp\left(-1.1\cdot 10^{12}\log n\right),\end{aligned}$$

where we have used the fact that $1+\log 2n < 2.1\log n$, for all $n \geq 5$. By comparing the resulting inequality with (3.13), we obtain

$$\min\{k/2,m-2\} < 1.6\cdot 10^{12}\log n.$$

As specified by Lemma 5 and considering that $67.3+8\log k+5\log\log k < 20\log k$, valid for all $k > 430$, we get

$$\begin{aligned}\min\{k/2,m-2\} &< 1.6\cdot 10^{12}\log(1.64\cdot 10^{29}k^8\log^5 k)\\ &< 1.6\cdot 10^{12}(67.3+8\log k+5\log\log k)\\ &< 3.2\cdot 10^{13}\log k.\end{aligned}$$

`Case 1`: $min\{k/2, m-2\} = k/2$. In this case, we get $k < 6.4 \cdot 10^{13} \log k$. Solving this inequality and applying Lemma 5, we see that

$$k < 2.3 \cdot 10^{15} \quad \text{and} \quad n < 7.12 \cdot 10^{159}. \tag{3.14}$$

`Case 2`: $min\{k/2, m-2\} = m-2$. In this case, we obtain

$$m < 3.21 \cdot 10^{13} \log k. \tag{3.15}$$

Returning now to (3.6), we proceed to rephrase it as follows:

$$\frac{a10^\ell}{9F_m^{(k)}} - 2^{n-2} = 2^{n-2}\zeta_1 + \frac{a}{9F_m^{(k)}}.$$

Thus, we obtain

$$\left|\frac{a10^\ell}{9F_m^{(k)}} - 2^{n-2}\right| \leq \frac{2^{n-1}}{2^{k/2}} + 1.$$

Consequently, dividing through by 2^{n-2} and using the fact that $n \geq k+2$ lead to

$$|\Gamma_4| \leq \frac{2}{2^{k/2}} + \frac{1}{2^{n-2}} < \frac{2}{2^{k/2}} + \frac{1}{2^k} < \frac{3}{2^{k/2}}, \tag{3.16}$$

where

$$\Gamma_4 := \frac{a}{9F_m^{(k)}} \cdot 2^{-(n-2)} \cdot 10^\ell - 1.$$

We must ensure that $\Gamma_4 \neq 0$. Otherwise, we would derive the equation $\frac{a10^\ell}{9F_m^{(k)}} = 2^{n-2}$. If $a \in \{1, \ldots, 8\}$, then it is evident that the expression on the left cannot yield an integer value. For the case where $a = 9$, we have $\frac{10^\ell}{F_m^{(k)}} = 2^{n-2}$. In this case, as $m - 2 < k/2$, it implies that $m \leq k+1$, consequently leading to $F_m^{(k)} = 2^{m-2}$. Substituting this into the equation results in $\frac{10^\ell}{2^{m-2}} = 2^{n-2}$, which inevitably leads to a contradiction. Consequently, $\Gamma_4 \neq 0$. Now, we apply Theorem 2 to Γ_4 by setting

$$(\eta_1, b_1) = (a/(9F_m^{(k)}), 1), \quad (\eta_2, b_2) = (2, -(n-2)), \quad (\eta_3, b_3) = (10, \ell).$$

As previously calculated, we define $A_2 = \log 2$, $A_3 = \log 10$, and $B = n$. Subsequently, we proceed to estimate $h(\eta_1)$. Utilizing the fact that $F_m^{(k)} < \alpha^{m-1}$ and Inequality (3.15), we derive

$$h(\eta_1) \leq h(a/9) + h(F_m^{(k)}) < \log 9 + (m-1)\log\alpha < 2.23 \cdot 10^{13} \log k.$$

Consequently, we set $A_1 = 2.23 \cdot 10^{13} \log k$. Thus, according to Theorem 2, we obtain

$$|\Gamma_4| > \exp(-8.67 \cdot 10^{24} \log k \log n), \tag{3.17}$$

where we have used the fact that $1 + \log n < 1.7 \log n$, for all $n \geq 5$. By considering both (3.16) and (3.17), it follows that

$$k < 2.51 \cdot 10^{25} \log k \log n.$$

According to Lemma 5 and using the fact that $67.4 + 8 \log k + 5 \log\log k < 20 \log k$, for all $k > 430$, we get

$$\begin{aligned} k &< 2.51 \cdot 10^{25} \log k \left(\log(1.64 \cdot 10^{29} k^8 \log^5 k)\right) \\ &< 2.51 \cdot 10^{25} \log k (67.3 + 8 \log k + 5 \log\log k) \\ &< 5.1 \cdot 10^{26} \log^2 k. \end{aligned}$$

Solving this inequality and applying Lemma 5, we obtain

$$k < 2.5 \cdot 10^{30} \quad \text{and} \quad n < 4.21 \cdot 10^{281}. \tag{3.18}$$

From (3.14) and (3.18), it is evident that (3.18) consistently remains valid. However, the resulting bounds are exceedingly large. Therefore, our subsequent step involves their reduction. Let us put

$$\Lambda_3 := \log(\Gamma_3 + 1) = \ell \log 10 - (n + m - 4) \log 2 + \log(a/9).$$

Assuming that $m \geq 10$, it then follows that $|\Gamma_3| < 0.04$. Setting $d = 0.04$ in Lemma 3, we obtain

$$|\Lambda_3| < \frac{-\log 0.96}{0.04} \cdot |\Gamma_3| < 8.7 \cdot 2^{-\min\{k/2, m-2\}}.$$

Consequently, we have

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n + m - 4) + \frac{\log(a/9)}{\log 2} \right| < 12.6 \cdot 2^{-\min\{k/2, m-2\}}. \tag{3.19}$$

We apply Lemma 1 to Λ_3, for $a \in \{1, \ldots, 8\}$, with the following parameters:

$$\gamma = \frac{\log 10}{\log 2}, \quad \mu = \frac{\log(a/9)}{\log 2}, \quad \text{and} \quad (A, B) = (12.6, 2).$$

We aim to reduce our excessively large bounds utilizing Lemma 1. Setting $M = 4.21 \cdot 10^{281}$ as an upper bound on ℓ by (3.2) and (3.18), we employ Lemma 1 on Inequality (3.19) to derive an upper bound on k. After conducting a computer search with Maple, we confirm that q_{571} satisfies the conditions of Lemma 1 for $a \in \{1, \ldots, 8\}$. As a result, applying Lemma 1 leads to the results shown in Table 2.

a	1	2	3	4	5	6	7	8
$\varepsilon \geq$	0.19	0.19	0.41	0.19	0.19	0.41	0.25	0.19
$\min\{k/2, m-2\} \leq$	947	947	945	947	947	945	946	947

Table 2: First reduction.

From the obtained results, it follows that $\min\{k/2, m-2\} < 948$, a condition that holds in all cases.

For the case $a = 9$, it follows that

$$\Lambda_3 = \log(\Gamma_3 + 1) = \ell \log 10 - (n + m - 4) \log 2. \tag{3.20}$$

Therefore, Inequality (3.19) becomes

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n + m - 4) \right| < 12.6 \cdot 2^{-\min\{k/2, m-2\}}. \tag{3.21}$$

Consider the continued fraction expression of $\frac{\log 10}{\log 2}$, represented as

$$[a_0, a_1, a_2, \ldots] = [3, 3, 9, 2, 2, 4, 6, 2, 1, 1, 3, 1, 18, 1, \ldots].$$

Let p_s/q_s denote its convergent. Recall that $\ell < 4.21 \cdot 10^{281}$. Using Maple, we determine that

$$q_{567} < 4.21 \cdot 10^{281} < q_{568}$$

and

$$a_L = \max\{a_i : i = 1, 2, \ldots, 568\} = a_{135} = 5393.$$

Thus, from the known properties of the continued fractions given by Lemma 2, we obtain that

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n + m - 4) \right| > \frac{1}{(a_L + 2)\ell}. \tag{3.22}$$

Putting the above inequality together with (3.21) and using the fact that $\ell < 4.21 \cdot 10^{281}$, we get

$$2^{\min\{k/2, m-2\}} < 12.6 \cdot 5395 \cdot 4.21 \cdot 10^{281}.$$

Hence, we obtain $\min\{k/2, m-2\} < 952$. So, in all cases, we have $\min\{k/2, m-2\} < 952$. Let us now continue the procedure of reduction with each case individually in order to achieve the reduced bound on k.

`Case 1`: $min\{k/2, m-2\} = k/2$. In this case, it results

$$k < 1904.$$

`Case 2`: $min\{k/2, m-2\} = m-2$. In this case, we obtain $m \leq 953$. Set $3 \leq m \leq 953$ and we put

$$\Lambda_4 := \log(\Gamma_4 + 1) = \ell \log 10 - (n-2)\log 2 + \log(a/(9F_m^{(k)})).$$

Since $k > 430$, then by (3.16), we have $|\Gamma_4| < 0.01$. Thus, applying Lemma 3 with $d = 0.01$, we obtain

$$|\Lambda_4| < -\frac{\log(0.99)}{0.01} \cdot |\Gamma_4| < 3.02 \cdot 2^{-k/2}.$$

So, we get

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n-2) + \frac{\log(a/(9F_m^{(k)}))}{\log 2} \right| < 4.4 \cdot 2^{-k/2}. \tag{3.23}$$

For all $a \in \{1, \dots, 8\}$ and $3 \leq m \leq 953$, we apply Lemma 1 to Λ_4 by considering

$$\gamma = \frac{\log 10}{\log 2}, \quad \mu = \frac{\log(a/(9F_m^{(k)}))}{\log 2}, \quad M = 4.21 \cdot 10^{281}, \quad \text{and} \quad (A, B) = (4.4, 2).$$

The inequality $m - 2 < k/2$ implies that $m \leq k+1$ for $k \geq 2$, and we can replace $F_m^{(k)}$ by 2^{m-2} in our calculations. Utilizing Maple once more, it follows that q_{571} meets the conditions of Lemma 1, for all $a \in \{1, \dots, 8\}$ and $3 \leq m \leq 953$. Hence, the application of Lemma 1 yields the results presented in Table 3, which are valid for $3 \leq m \leq 953$.

a	1	2	3	4	5	6	7	8
$\varepsilon \geq$	0.19	0.19	0.41	0.19	0.19	0.41	0.25	0.19
$k/2 \leq$	945	945	944	945	945	944	945	945

Table 3: First reduction for second case.

The data from Table 3 confirm that $k < 1891$ in all instances. When $a = 9$, then Inequality (3.23) becomes

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n + m - 4) \right| < 4.4 \cdot 2^{-k/2}. \tag{3.24}$$

According to (3.22) and (3.24), we deduce that $k < 1901$. Therefore, in all cases we have $k < 1904$.

With this refined bound, we deduce $n < 1.5 \cdot 10^{40}$. Subsequently, we once again employ Lemma 1 with the same dataset but with a revised upper bound of $M = 1.5 \cdot 10^{40}$. Utilizing Maple, we confirm that q_{120} satisfies the conditions stipulated in Lemma 1 for all $a \in \{1, \dots, 8\}$. The results of this application are presented in Table 4.

a	1	2	3	4	5	6	7	8
$\varepsilon \geq$	0.11	0.11	0.45	0.11	0.11	0.45	0.18	0.11
$\min\{k/2, m-2\} \leq$	191	191	189	191	191	189	190	191

Table 4: Second reduction.

These obtained results affirm that $\min\{k/2, m-2\} < 192$ holds in all cases. For $a = 9$, we find that $\min\{k/2, m-2\} < 143$. Thus, we deduce that $\min\{k/2, m-2\} < 192$ is valid in all cases.

As previously, for the first case, we ascertain that $k < 384$, while for the second case, we once again see that q_{120} satisfies the conditions specified in Lemma 1 for all $a \in \{1, \ldots, 8\}$ and $3 \leq m \leq 193$. Then, we derive the subsequent results, valid for all $3 \leq m \leq 193$.

a	1	2	3	4	5	6	7	8
$\varepsilon \geq$	0.11	0.11	0.45	0.11	0.11	0.45	0.18	0.11
$k/2 \leq$	189	189	187	189	189	187	188	189

Table 5: Second reduction for second case.

The obtained results from Table 5 state that $k < 380$ for all cases. When $a = 9$, we find that $k < 283$. Consequently, in all cases, we establish that $k < 384$. However, this contradicts our assumption that $k > 430$. This achieves the proof of Theorem 1

Acknowledgement. We appreciate the anonymous referee for their valuable suggestions which improved the content of the paper.

References

[1] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. Math. Oxford Ser.* **20** (3) (1969), 129–137.

[2] J.J. Bravo and F. Luca, On a conjecture about repdigits in k-generalized Fibonacci sequences, *Publ. Math. Debrecen* **82** (2013), 623-639.

[3] J. J. Bravo, C. A. Gómez, and F. Luca, Powers of two as sums of two k-Fibonacci numbers, *Miskolc Math. Notes* **17** (2016), 85-100.

[4] J. J. Bravo, C. A. Gómez, and F. Luca, A Diophantine equation in k-Fibonacci numbers and repdigits, *Colloq. Math.* **152** (2018), 299–315.

[5] Y. Bugeaud, M. Maurice, and S. Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. of Math.* **163** (2006), 969–1018.

[6] P. Coufal and P. Trojovský, Repdigits as product of terms of k-Bonacci sequences, *Mathematics* **9** (6) (2021), 682.

[7] G. Dresden and Z. Du, A simplified Binet formula for k-generalized Fibonacci numbers, *J. Integer Seq.* **17** (2014), Article 14.4.7.

[8] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser.* **49** (2) (1998), 291–306.

[9] F. Erduvan and R. Keskin, Repdigits as products of two Fibonacci or Lucas numbers, *Proc. Indian Acad. Sci. Math. Sci.* **130** (2020), 1–14.

[10] C. A. Gómez and F. Luca, Power of two-classes in k-generalized Fibonacci sequences, *Rev. Colomb. Mat.* **48** (2) (2014), 219–234.

[11] F. Luca, Fibonacci and Lucas numbers with only one distinct digit, *Port. Math.* **57** (2000), 243–254.

[12] D. Marques, On k-generalized Fibonacci numbers with only one distinct digit, *Util. Math.* **98** (2015), 23–31.

[13] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II, *Izv. Math.* **64** (6) (2000), 1217—1269.

[14] E. P. Jr. Miles, Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* **67** (1960), 745—752.

[15] M. D. Miller, Mathematical notes: on generalized Fibonacci numbers, *Amer. Math. Monthly* **78** (1971), 1108–1109.

[16] S. E. Rihane, k-Fibonacci and k-Lucas numbers as product of two Repdigits, *Results Math.* **76** (2021), 1–20.

[17] S. G. Sánchez and F. Luca, Linear combinations of factorials and S-units in a binary recurrence sequences, *Ann. Math. Qué.* **38** (2014), 169-188.

[18] B. M. M. de Weger, *Algorithms for Diophantine Equations*, PhD. Thesis, Eindhoven University of Technology, Eindhoven, the Netherlands, (1989).

[19] D. A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* **36** (2) (1998), 129—145.