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COMBINATORIAL INSIGHTS INTO LEONARDO p-NUMBERS AND LUCAS-LEONARDO p-NUMBERS

Amine Belkhir
Faculty of Mathematics, Recits Lab. USTHB, Bab Ezzouar, Algiers, Algeria
ambelkhir@gmail.com

Elif Tan
Department of Mathematics, Ankara University, Ankara, Türkiye
etan@ankara.edu.tr

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Abstract

In this paper, we present a combinatorial interpretation of Leonardo p-numbers in terms of colored linear tilings and provide combinatorial proofs for several identities involving them. We further explore the incomplete and hyper Leonardo p-numbers, presenting their combinatorial interpretations. Additionally, we present a combinatorial interpretation of the Lucas-Leonardo p-numbers, closely related to Leonardo p-numbers, in terms of colored circular tilings and investigate their combinatorial properties. Finally, we introduce hyper Lucas-Leonardo p-numbers and establish connections with their incomplete counterparts.

1. Introduction

The Fibonacci sequence is among the most well-known sequences in mathematics. The nth Fibonacci number, denoted as F_n, is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers and their extensions exhibit numerous fascinating properties and find diverse applications across science and art. For more details, see [9].

Many authors have explored non-homogeneous extensions of the Fibonacci recurrence relation. In particular, the Leonardo sequence $\{\mathcal{L}_n\}$, which was used by Dijkstra [8] as an integral part of his sorting algorithm, is defined by the non-homogeneous recurrence relation

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + 1,\ n \geq 2,$$

with initial values $\mathcal{L}_0 = \mathcal{L}_1 = 1$. For the history of Leonardo sequences, see [A001595] in [10]. The properties of the Leonardo numbers have been explored

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by Catarino and Borges [6], Alp and Kocer [1], and Shannon [12]. For a fixed positive integer k, Kuhapatanakul and Chobsorn [11] introduced the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}$ through the non-homogeneous recurrence relation

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k,\ n \geq 2$$

with initial values $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$. It is clear to see that when $k = 1$, it reduces to the classical Leonardo sequence $\{\mathcal{L}_n\}$. Additionally, Shattuck [13] provided combinatorial proofs of several identities satisfied by the generalized Leonardo numbers. He also explored some combinatorial aspects of incomplete generalized Leonardo numbers discussed in [7]. Furthermore, it is noteworthy that Bicknell [4] examined a similar type of sequence with arbitrary initial values.

Motivated by the aforementioned studies, Tan and Leung [16] defined a generalization of Leonardo numbers, namely the Leonardo p-numbers. For any given integer $p > 0$, the Leonardo p-numbers are defined by the non-homogeneous relation

$$\mathcal{L}_{p,n} = \mathcal{L}_{p,n-1} + \mathcal{L}_{p,n-p-1} + p,\ n > p, \tag{1}$$

with initial values $\mathcal{L}_{p,0} = \mathcal{L}_{p,1} = \cdots = \mathcal{L}_{p,p} = 1$. It is clear to see that when $p = 1$, the Leonardo p-sequence reduces to the classical Leonardo sequence. For the purposes of our paper, we require the following equations (2)-(4) related to the Leonardo p-numbers, which can be found in [16]. The non-homogeneous relation of the Leonardo p-sequence can be converted to the homogeneous relation

$$\mathcal{L}_{p,n} = \mathcal{L}_{p,n-1} + \mathcal{L}_{p,n-p} - \mathcal{L}_{p,n-2p-1},\ n > 2p. \tag{2}$$

A relation between Leonardo p-numbers and Fibonacci p-numbers is given by

$$\mathcal{L}_{p,n} = (p+1)\, F_{p,n+1} - p, \tag{3}$$

where $\{F_{p,n}\}$ is the Fibonacci p-sequence defined by the recurrence relation $F_{p,n} = F_{p,n-1} + F_{p,n-p-1}$, $n > p$, with initial values $F_{p,0} = 0$, $F_{p,i} = 1$ for $i = 1, 2, \ldots, p$. Note that when $p = 1$, the Fibonacci p-sequence reduces to the classical Fibonacci sequence. For details on Fibonacci p-sequences and their generalizations, we refer to [15]. The incomplete Leonardo p-numbers $\mathcal{L}_{p,n}(k)$ are defined as:

$$\mathcal{L}_{p,n}(k) = (p+1) \sum_{i=0}^{k} \binom{n-pi}{i} - p,\ 0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor. \tag{4}$$

It is clear to see that $\mathcal{L}_{p,n}\left(\left\lfloor \frac{n}{p+1} \right\rfloor\right) = \mathcal{L}_{p,n}$.

On the other hand, Zhong et al.[17] recently studied a companion sequence of the Leonardo p-sequence, called the Lucas-Leonardo p-sequence, and defined it using the non-homogeneous recurrence relation

$$\mathcal{R}_{p,n} = \mathcal{R}_{p,n-1} + \mathcal{R}_{p,n-p-1} + p,\ n > p \tag{5}$$

with initial values $\mathcal{R}_{p,0} = p^2 + p + 1, \mathcal{R}_{p,1} = \cdots = \mathcal{R}_{p,p} = 1$. For $p = 1$, it reduces to the Lucas-Leonardo sequence $\{\mathcal{R}_n\}$; see [A022319] in [10]. In a similar manner to the Leonardo p-numbers, we require the following equations (6)-(10) for the Lucas-Leonardo p-numbers, which can be found in [17].

A relation between Lucas-Leonardo p-numbers and Lucas p-numbers is given by

$$\mathcal{R}_{p,n} = (p+1)\, L_{p,n} - p, \tag{6}$$

where $\{L_{p,n}\}$ is the Lucas p-sequence defined by the recurrence relation $L_{p,n} = L_{p,n-1} + L_{p,n-p-1}$, $n > p$, with initial values $L_{p,0} = p+1$, $L_{p,i} = 1$ for $i = 1, 2, \ldots, p$. It is clear to see that when $p = 1$, the Lucas p-sequence reduces to the classical Lucas sequence $\{L_n\}$. For details on Lucas p-numbers and their generalizations, see [15]. A relationship between Leonardo p-numbers and Lucas-Leonardo p-numbers is given by

$$\mathcal{R}_{p,n} = (p+1)\, \mathcal{L}_{p,n} - p\mathcal{L}_{p,n-1}. \tag{7}$$

The incomplete Lucas-Leonardo p-numbers [17] are defined as

$$\mathcal{R}_{p,n}\,(k) = (p+1) \sum_{i=0}^{k} \frac{n}{n-pi} \binom{n-pi}{i} - p, \tag{8}$$

where $0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor$. For $0 \leq k \leq \left\lfloor \frac{n-p-1}{p+1} \right\rfloor$, the following relation is satisfied:

$$\mathcal{R}_{p,n}(k+1) = \mathcal{R}_{p,n-1}(k+1) + \mathcal{R}_{p,n-p-1}(k) + p. \tag{9}$$

A relationship between incomplete Leonardo p-numbers and incomplete Lucas-Leonardo p-numbers is

$$\mathcal{R}_{p,n}(k) = (p+1)\, \mathcal{L}_{p,n}(k) - p\mathcal{L}_{p,n-1}(k). \tag{10}$$

This paper offers a combinatorial perspective on Leonardo p-numbers, illustrating them through colored linear tilings. We provide combinatorial demonstrations of several identities associated with Leonardo p-numbers and incomplete Leonardo p-numbers. In particular, we provide combinatorial proofs of the identities (2)-(4) and more identities given in [16, Proposition 2-5]. Moreover, we establish a connection between incomplete Leonardo p-numbers and hyper Leonardo p-numbers. Similarly, we give a combinatorial interpretation of Lucas-Leonardo p-numbers and provide combinatorial proofs of the identities (6)-(10). We also introduce the concept of hyper Lucas-Leonardo p-numbers and give a relation between incomplete Lucas-Leonardo p-numbers and hyper Lucas-Leonardo p-numbers. Our findings provide combinatorial interpretations and present several novel identities associated with hyper Lucas-Leonardo p-numbers.

2. Combinatorial Interpretation of Leonardo p-numbers

Recall that a (linear) *n-board* is a board of length n with cells labeled $1, 2, \ldots, n$ from left to right. A (linear) *n-tiling* is a tiling of a (linear) n-board. Suppose that the board is covered by squares and p-minos, where a square covers a single cell and a p-mino covers $p+1$ cells. Here, a square or a p-mino is indistinguishable from other pieces of the same kind and is denoted by $\mathbf{s}$ or $\mathbf{p}$, respectively.

Let $f_{p,n}$ denote the set of linear n-tilings consisting of squares and p-minos. Considering whether the last piece within an n-tiling is $\mathbf{s}$ or $\mathbf{p}$ implies

$$|f_{p,n}| = F_{p,n+1}$$

for $n \geq 0$. Note that a member of $f_{p,n}$ containing exactly i p-minos must contain $n - (p+1)\,i$ squares, and hence there are $\binom{n-pi}{i}$ such members of $f_{p,n}$ for $0 \leq i \leq \left\lfloor \frac{n}{p+1} \right\rfloor$. See [14] for more on the combinatorial interpretation of Fibonacci p-numbers. Also, note that for $p = 1$, it reduces to the combinatorial interpretation of classical Fibonacci numbers in terms of squares and dominos. For details, we refer to the excellent book of Benjamin and Quinn [5].

To provide combinatorial proofs of identities involving Leonardo p-numbers, we extend the arguments given in [13]. We define a new tile, called *p-tile*:

Definition 1. A *p-tile* is a rectangular tile defined as follows:

- It comes in one of p colors, which must occur as the initial piece in a tiling if it is included in the arrangement at all.
- It has a length l with $l \geq p+1$, and is denoted by $\mathcal{P}_l$.

Let $\mathcal{K}_{p,n}$ denote the set of linear n-tilings using squares, p-minos, and p-tiles. For simplicity, we denote these members of $\mathcal{K}_{p,n}$ using sequences in $\mathbf{s}, \mathbf{p}$, and $\mathcal{P}_l$, respectively. For example, for $p = 3$ and $n = 4$, we have $\mathcal{K}_{3,4} = \left\{\mathbf{s}^4, \mathbf{p}, \mathcal{P}_4\right\}$. See Figure 1.

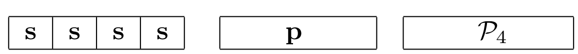

Figure 1: Tilings of length 4 for p = 3:

Since the $\mathcal{P}_4$ piece comes in one of 3 colors, we have

$$|\mathcal{K}_{3,4}| = 3 + 2 = 5 = \mathcal{L}_{3,4}.$$

For $p = 3, n = 5$, we have $\mathcal{K}_{3,5} = \left\{\mathbf{s}^5, \mathbf{sp}, \mathbf{ps}, \mathcal{P}_4\mathbf{s}, \mathcal{P}_5\right\}$. See Figure 2.

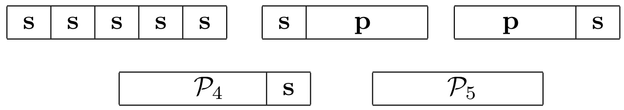

Figure 2: Tilings of length 5 for p = 3

Since $\mathcal{P}_4$ and $\mathcal{P}_5$ pieces come in one of 3 colors, we have

$$|\mathcal{K}_{3,5}| = 3.2 + 3 = 9 = \mathcal{L}_{3,5}.$$

Proposition 1. *If $n \geq 0$, then $|\mathcal{K}_{p,n}| = \mathcal{L}_{p,n}$.*

Proof. Considering whether the final piece of $\lambda \in \mathcal{K}_{p,n}$, where $n \geq p+1$, is **s** or **p** or if it equals $\mathcal{P}_n$ (in which case, it consists of a single p-tile of length n), we get

$$|\mathcal{K}_{p,n}| = |\mathcal{K}_{p,n-1}| + |\mathcal{K}_{p,n-p-1}| + p.$$

Since p-tiles and p-minos have length greater than p, we have $|\mathcal{K}_{p,n}| = 1 = \mathcal{L}_{p,n}$ for $n = 0, 1, \ldots, p$. $\square$

Proposition 2. *For $n \geq 2p+1$, we have*

$$\mathcal{L}_{p,n} = \mathcal{L}_{p,n-1} + \mathcal{L}_{p,n-p} - \mathcal{L}_{p,n-2p-1},$$

with $\mathcal{L}_{p,0} = \mathcal{L}_{p,1} = \cdots = \mathcal{L}_{p,p} = 1$ and

$$\begin{aligned}
\mathcal{L}_{p,p+1} &= \mathcal{L}_{p,p} + \mathcal{L}_{p,0} + p = 2 + p,\\
\mathcal{L}_{p,p+2} &= \mathcal{L}_{p,p+1} + \mathcal{L}_{p,1} + p = 3 + 2p,\\
\mathcal{L}_{p,p+3} &= \mathcal{L}_{p,p+2} + \mathcal{L}_{p,2} + p = 4 + 3p,\\
&\vdots\\
\mathcal{L}_{p,2p} &= \mathcal{L}_{p,2p-1} + \mathcal{L}_{p,p-1} + p = p^2 + p + 1.
\end{aligned}$$

Proof. The initial conditions follow from the definitions.

Suppose $n \geq 2p+1$ and note that there are $\mathcal{L}_{p,n-1}$ members of $\mathcal{K}_{p,n}$ that ends in **s**.

Let S denote the subset of $\mathcal{K}_{p,n-p}$ consisting of those tilings that do not end in **p**. By subtraction, we have $|S| = \mathcal{L}_{p,n-p} - \mathcal{L}_{p,n-2p-1}$.

If $\lambda \in S$ ends in **s**, then let λ' be obtained from λ by replacing the final **s** with a **p**. Otherwise, $n \geq 2p+1$ implies $\lambda = \mathcal{P}_{n-p}$ is also possible, where $\mathcal{P}_{n-p}$ comes one of p colors. In this case, we let $\lambda' = \mathcal{P}_n$, keeping the color the same. Then the mapping $\lambda \to \lambda'$ is a bijection from S to the subset of $\mathcal{K}_{p,n}$ whose members do not end in $\mathbf{s}$, and hence they number $\mathcal{L}_{p,n-p} - \mathcal{L}_{p,n-2p-1}$, which completes the proof. $\square$

For example, consider

$$\mathcal{K}_{3,7} = \left\{\mathbf{s}^7, \mathbf{ps}^3, \mathbf{sps}^2, \mathbf{s}^2\mathbf{ps}, \mathbf{s}^3\mathbf{p}, \mathcal{P}_4\mathbf{s}^3, \mathcal{P}_5\mathbf{s}^2, \mathcal{P}_6\mathbf{s}, \mathcal{P}_7\right\}.$$

There are $\mathcal{L}_{3,6} = 13$ members of $\mathcal{K}_{3,7}$ that end in $\mathbf{s}$. Now consider the subset S of $\mathcal{K}_{3,4} = \left\{\mathbf{s}^4, \mathbf{p}, \mathcal{P}_4\right\}$ that do not end in $\mathbf{p}$. So $S = \left\{\mathbf{s}^4, \mathcal{P}_4\right\}$. Since there is a bijection from S to the subset of $\mathcal{K}_{3,7}$ whose members do not end in $\mathbf{s}$, it is clear to see that $\mathbf{s}^4 \in S$ maps to $\mathbf{s}^3\mathbf{p} \in \mathcal{K}_{3,7}$ and $\mathcal{P}_4 \in S$ maps to $\mathcal{P}_7 \in \mathcal{K}_{3,7}$. Thus

$$|\mathcal{K}_{3,7}| = |\mathcal{K}_{3,6}| + |S| = 13 + 4 = 17.$$

Proposition 3. *For $n \geq 0$, we have*

$$\mathcal{L}_{p,n} = (p+1)\, F_{p,n+1} - p.$$

Proof. Let $\mathcal{K}^*_{p,n} := \mathcal{K}_{p,n} - \{s^n\}$ denote the subset of tilings that do not contain only squares. There are $\mathcal{L}_{p,n} - 1$ tilings in $\mathcal{K}^*_{p,n}$. Now, $\mathcal{K}^*_{p,n}$ can be partitioned into two disjoints subsets, $\mathcal{K}^*_{p,n} = \mathcal{K}^{*1}_{p,n} \cup \mathcal{K}^{*2}_{p,n}$, where the set $\mathcal{K}^{*1}_{p,n}$ contains tilings with squares and at least one p-mino, and the set $\mathcal{K}^{*2}_{p,n}$ contains tilings with a p-tile, squares, and p-minos. Then, $|\mathcal{K}^{*1}_{p,n}| = F_{p,n+1} - 1$ and

$$|\mathcal{K}^{*2}_{p,n}| = p \sum_{l=p+1}^{n} F_{p,n+1-l} = p(F_{p,n+1} - 1).$$

Hence, $|\mathcal{K}^*_{p,n}| = (p+1)(F_{p,n+1} - 1)$ which gives the desired result. $\square$

Proposition 4. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} \mathcal{L}_{p,i} = \mathcal{L}_{p,n+p+1} - (n+1)p - 1.$$

Proof. Consider the $(n+p+1)$-tilings that do not contain any p-minos. Such tilings are either in the form of s^{n+p+1} or $\mathcal{P}_l s^{n+p+1-l}$ for $p+1 \leq l \leq n+p+1$, resulting in $p(n+1)+1$ such tilings. Since the number of $(n+p+1)$-tilings is $\mathcal{L}_{p,n+p+1}$, excluding those that do not contain any p-minos yields the right-hand side of the identity.

Let λ be a tiling of length $n+p+1$ containing at least one p-mino. Conditioning on the position of the last p-mino, which covers cells $i+1, \ldots, i+p+1$ $(0 \leq i \leq n)$, such a tiling must have the form $\lambda = \lambda' \mathbf{p} s^{n-i}$ for $(1 \leq i \leq n)$ where $\lambda' \in \mathcal{K}_{p,i}$, and there are $\mathcal{L}_{p,i}$ such tilings. Summing over i gives the left-hand side of the identity. $\square$

Next, we provide a combinatorial proof of the identity in [16, Theorem 2] for $m = 0$.

Proposition 5. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} \mathcal{L}_{p,(p+1)i} = \mathcal{L}_{p,(p+1)n+1} - pn.$$

Proof. Consider the $(p+1)n+1$-tilings that do not contain any squares. Such tilings are in the form of $\mathcal{P}_{(p+1)i+1}\mathbf{p}^{n-i}$ for $1 \leq i \leq n$, resulting in pn such tilings. Since the number of $(p+1)n+1$-tilings is $\mathcal{L}_{p,(p+1)n+1}$, excluding those that do not contain any squares yields the right-hand side of the identity.

Let λ be a tiling of length $n+p+1$ containing at least one square. Conditioning on the position of the last square, which covers cell $(p+1)i$ where $(0 \leq i \leq n)$, such a tiling must have the form $\lambda = \lambda' s\mathbf{p}^{n-i}$ for $1 \leq i \leq n$, where $\lambda' \in \mathcal{K}_{p,(p+1)i}$, and there are $\mathcal{L}_{p,(p+1)i}$ such tilings. Summing over i gives the left-hand side of the identity. □

Now, we provide a combinatorial interpretation of the incomplete Leonardo p-numbers $\mathcal{L}_{p,n}(k)$.

Let $\mathcal{A}_{p,n}$ denote the set consisting of tilings λ of length n such that the first piece of λ is assigned one of $p+1$ colors provided λ is not all the square tiling, in which case the first piece of λ is not assigned a color. Then, from Proposition 3, we have

$$|\mathcal{A}_{p,n}| = (p+1)(F_{p,n+1} - 1) + 1 = (p+1) F_{p,n+1} - p = \mathcal{L}_{p,n}.$$

For $0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor$, let $\mathcal{A}_{p,n}(k)$ denote the subset of $\mathcal{A}_{p,n}$ containing at most k p-minos. Then

$$|\mathcal{A}_{p,n}(k)| = (p+1) \sum_{i=0}^{k} \binom{n-pi}{i} - p. \tag{11}$$

Comparing (11) with (4), we get $|\mathcal{A}_{p,n}(k)| = \mathcal{L}_{p,n}(k)$.

It is clear to see that $\left|\mathcal{A}_{p,n}\left(\left\lfloor \frac{n}{p+1} \right\rfloor\right)\right| = \mathcal{L}_{p,n}$.

Proposition 6. *For $0 \leq k \leq \left\lfloor \frac{n}{p+1} \right\rfloor$ and $n > p$, we have*

$$\mathcal{L}_{p,n}(k+1) = \mathcal{L}_{p,n-1}(k+1) + \mathcal{L}_{p,n-p-1}(k) + p.$$

Proof. Let $\lambda \in \mathcal{A}_{p,n}(k+1)$, we condition on the last tile. If the last tile is square, there are $\mathcal{L}_{p,n-1}(k+1)$ ways to tile the remaining $n-1$ cells with at most $k+1$ p-minos. If the last tile is p-mino, there are $\mathcal{L}_{p,n-p-1}(k)$ ways to tile the first $n-p-1$ cells with at most k p-minos. Note that the tiling of the form $s^{n-p-1}\mathbf{p}$ is counted without coloring. Hence, we must add p to account for these missed tilings. □

Proposition 7. *For* $0 \leq k \leq \frac{n-p-r}{p+1}$, *we have*

$$\sum_{i=0}^{r} \binom{r}{i} \mathcal{L}_{p,n+pi}(k+i) + (2^r - 1)p = \mathcal{L}_{p,n+(p+1)r}(k+r).$$

Proof. Let $\lambda \in \mathcal{A}_{p,n+(p+1)r}(k+r)$ be a tiling of length $n+(p+1)r$ containing at most $k+r$ p-minos. Suppose there are i p-minos among the last r tiles. Thus, λ is of the form $\lambda'\lambda''$ where $\lambda' \in \mathcal{A}_{p,n+p(r-i)}(k+r-i)$ and λ'' is a tiling with exactly i p-minos. There are $\binom{r}{i}$ ways to tiles λ'' and $\mathcal{L}_{p,n+pi}(k+i)$ possible ways to tiles λ'. Note that the tilings of the form $s^{n+p(r-i)}\lambda''$ are counted without coloring. Thus, we must multiply the number of such tilings by p, which gives us $p(2^r - 1)$. □

Proposition 8. *For* $n \geq (p+1)(k+1)$, *we have*

$$\sum_{i=0}^{r-1} \mathcal{L}_{p,n-p+i}(k) + rp = \mathcal{L}_{p,n+r}(k+1) - \mathcal{L}_{p,n}(k+1).$$

Proof. We will show that both sides of the identity count tilings of length $n+r$ with at most $k+1$ p-minos such that the last p-mino from the right-hand side occupies position $(i+1, \ldots, i+p+1)$ $(n-p \leq i \leq n+r-p-1)$. If there is a p-mino at position $(i+1, \ldots, i+p+1)$ then there are $\mathcal{L}_{p,i}(k)$ ways to tile the first i cells. Note that when the first i cells are covered only with squares, the tiling is counted without coloring. There are r such tilings. □

In [2], the authors define the hyper Leonardo p-numbers, $\mathcal{L}_{p,n}^{(k)}$, and provide the following explicit formula

$$\mathcal{L}_{p,n}^{(k)} = (p+1) \sum_{i=0}^{\lfloor \frac{n}{p+1} \rfloor} \binom{n+k-pi}{i+k} - p\binom{n+k}{k}. \tag{12}$$

Now, we give a combinatorial interpretation of hyper Leonardo p-numbers. Let $\mathcal{A}_{p,n}^{(k)}$ be the set of tilings λ of length $n+(p+1)k$ that contain at least k p-minos, where the first tile of λ is assigned one of $p+1$ colors, except when λ contains exactly k p-minos, in which case the first tile of λ is not assigned a color. Then, we have

$$\begin{aligned}
|\mathcal{A}_{p,n}^{(k)}| &= (p+1) \sum_{i=k+1}^{\lfloor \frac{n}{p+1} \rfloor + k} \binom{n+(p+1)k-pi}{i} + \binom{n+(p+1)k-pk}{k} \\
&= (p+1) \sum_{i=k}^{\lfloor \frac{n}{p+1} \rfloor + k} \binom{n+k-p(i-k)}{i} - p\binom{n+k}{k} \\
&= (p+1) \sum_{i=0}^{\lfloor \frac{n}{p+1} \rfloor} \binom{n+k-pi}{i+k} - p\binom{n+k}{k} = \mathcal{L}_{p,n}^{(k)}.
\end{aligned}$$

Proposition 9. *For $m, n \geq 1$ with $m \leq k$, we have*

$$\mathcal{L}_{p,n+m}^{(k)} = \sum_{i=0}^{m} \binom{m}{i} \mathcal{L}_{p,n+i}^{(k-i)}.$$

Proof. The left-hand side corresponds to the number of $(n+m+(p+1)k)$-tilings with at least k p-minos. There are $\binom{m}{i}$ ways to select the positions of i p-minos that appear among the first m tiles. The number of ways to tile the remaining $n+p(k-i)+k$-tilings with at least $k-i$ p-minos is $\mathcal{L}_{p,n+i}^{(k-i)}$. Summing over all values of i yields the right-hand side of the identity. □

Next, we provide tiling proofs of the following two recurrences relations from [2].

Proposition 10. *For $n, k \geq 1$, we have*

$$\mathcal{L}_{p,n}^{(k)} = \mathcal{L}_{p,n-1}^{(k)} + \mathcal{L}_{p,n}^{(k-1)}.$$

with $\mathcal{L}_{p,n}^{(0)} = \mathcal{L}_{p,n}$ and $\mathcal{L}_{p,0}^{(k)} = \mathcal{L}_{p,0}$.

Proof. Let $\lambda \in \mathcal{A}_{p,n}^{(k)}$ be a $n+(p+1)k$-tiling with at least k p-minos. For $k = 0$, we get $|\mathcal{A}_{p,n}^{(0)}| = |\mathcal{A}_{p,n}| = \mathcal{L}_{p,n}$, and for $n = 0$, we obtain $|\mathcal{A}_{p,0}^{(k)}| = 1 = \mathcal{L}_{p,0}$. Now, assuming $n, k \geq 1$, if the last tile of λ is a square, then there are $\mathcal{L}_{p,n-1}^{(k)}$ ways to tile the remaining $n-1+(p+1)k$-tiling with at least k p-minos. Otherwise, if the last tile is a p-mino, then there are $\mathcal{L}_{p,n}^{(k-1)}$ ways to tile the remaining $n+(p+1)(k-1)$-tilings with at least $k-1$ p-minos. Considering both cases, we obtain the desired result. □

Proposition 11. *For $n > p$ and $k \geq 1$, we have*

$$\mathcal{L}_{p,n+(p+1)k} = \mathcal{L}_{p,n+(p+1)k}(k-1) + \mathcal{L}_{p,n}^{(k)} + p\binom{n+k}{k}.$$

Proof. For $0 \leq i \leq \left\lfloor \frac{n}{p+1} \right\rfloor + k$, let $\mathcal{A}_{p,n+(p+1)k,i} \in \mathcal{A}_{p,n+(p+1)k}$ be the subset of $n+(p+1)k$-tilings using exactly i p-minos. It is clear that

$$\mathcal{A}_{p,n+(p+1)k} = \bigcup_{i} \mathcal{A}_{p,n+(p+1)k,i}.$$

Using combinatorial intepretation of incomplete Leonardo p-numbers we get

$$\sum_{i=0}^{k-1} |\mathcal{A}_{p,n+(p+1)k,i}| = \mathcal{L}_{p,n+(p+1)k}(k-1). \tag{13}$$

Note that the first piece in the tilings of $\mathcal{A}_{p,n+(p+1)k,k}$ is assigned $p+1$ colors. Then, from the combinatorial interpretation of hyper Leonardo numbers, we obtain

$$\sum_{i=k}^{\lfloor \frac{n}{p+1} \rfloor + k} |\mathcal{A}_{p,n+(p+1)k,i}| = \mathcal{L}_{p,n}^{(k)} + p\binom{n+k}{k}. \tag{14}$$

Combining (13) and (14), we obtain the desired result. □

3. Combinatorial Interpretation of Lucas-Leonardo p-numbers

In this section, we give a combinatorial interpretation of the Lucas-Leonardo p-numbers. We provide combinatorial proofs of the identities (6)-(10). We also introduce the hyper Lucas-Leonardo p-numbers, which is a new family of sequences in OEIS [10].

Recall that a *circular n-board* is obtained from an n-board by attaching the right side of the nth cell to the left side of the first cell. A bracelet length n (or *n-bracelet*) is a tiling of a circular n-board. Here, the cells and tiles are labeled in a clockwise direction, with the first tile designated as the one that covers cell 1.

Let $l_{p,n}$ denote the set of n-bracelets that can be made using curved squares and p-minos. A bracelet is said to have phase r (or *r-phase*) where $1 \leq r \leq p+1$ if the rth cell of the first tile covers cell 1. We denote by $\mathbf{p}_r$ a curved p-mino whose rth cell covers cell 1. Then, since there is only one way to tile a circular n-board using only squares, we have $|l_{p,n}| = 1$ for $1 \leq n \leq p$. Since a circular $(p+1)$-board can be tiled with squares or with one p-mino arranged in $p+1$ different phases, we have $|l_{p,p+1}| = p+2$ for $n = p+1$. We define $|l_{p,0}| = p+1$ and interpret this as the number of phases. For $n \geq p+1$, considering whether the last tile, the one that precedes the first tile, of the n-bracelet is square or p-mino implies $|l_{p,n}| = |l_{p,n-1}| + |l_{p,n-p-1}|$. Thus, we have

$$|l_{p,n}| = L_{p,n}$$

for $n \geq 0$. That is, the Lucas p-numbers count the number of ways to tile a circular n-board using curved squares and p-minos. Note that for $p = 1$, it reduces to the combinatorial interpretation of classical Lucas numbers. For details, we refer to [5].

In the Figure 3, we illustrate all 4-bracelets that can be made using curved squares and 2-minos $\{\mathbf{p}_1 s, s\mathbf{p}, s^4, \mathbf{p}_2 s,\ \mathbf{p}_3 s\}$.

Now, we provide a combinatorial interpretation of Lucas-Leonardo p-numbers. Throughout this section, we consider all tiles as curved tiles. We define a (curved) p-tile as follows:

- A p-tile is a circular piece with length $l \geq p+1$ coming in one of p colors, which must occur as the first tile in an n-bracelet if it is included in the arrangement at all.

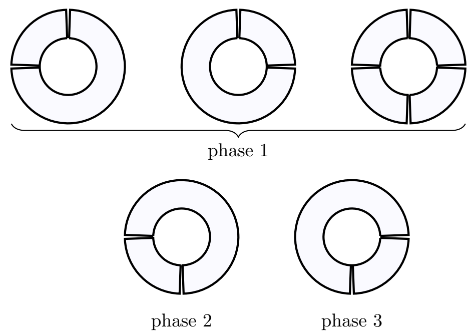

Figure 3: All 4-bracelets that can be made using squares and 2-minos

- If the p-tile has length $p+1$, then it can be in $p+1$ phases. We denote by $\mathcal{P}_{p+1}^{r}$ a p-tile whose rth cell covers cell 1 of the bracelet. Otherwise, the p-tile of length l with $l > p+1$, denoted by $\mathcal{P}_l$, must start from cell 1.

Let $\mathcal{B}_{p,n}$ denote the set of n-bracelets that can be made using curved squares, p-minos, and p-tiles. In Figure 4, we illustrate 13 possible bracelets for $n = 4$, $p = 2$. The p-tiles $\mathcal{P}_3^r$ $(1 \le r \le 3)$ and $\mathcal{P}_4$ come in one of 2 possible colors.

$$\mathcal{B}_{2,4} = \{\mathbf{p}_1 s, s\mathbf{p}, s^4, \mathbf{p}_2 s, \mathbf{p}_3 s, \mathcal{P}_3^1 s, \mathcal{P}_3^2 s, \mathcal{P}_3^3 s, \mathcal{P}_4\}$$

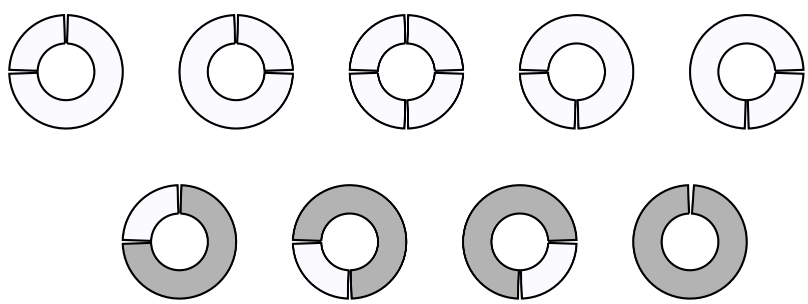

Figure 4: All 4-bracelets that can be made using squares and 2-minos

Proposition 12. *For $n \ge 0$, we have $|\mathcal{B}_{p,n}| = \mathcal{R}_{p,n}$.*

Proof. Considering whether the final piece of $\lambda \in \mathcal{B}_{p,n}$ where $n \ge p+1$ is $\mathbf{s}$ or $\mathbf{p}$ or if it equals $\mathcal{P}_n$ (in which case, it consists of a single p-tile of length n), we get

$$|\mathcal{B}_{p,n}| = |\mathcal{B}_{p,n-1}| + |\mathcal{B}_{p,n-p-1}| + p.$$

Since p-tiles and p-minos have length greater than p, we have $|\mathcal{B}_{p,n}| = 1 = \mathcal{R}_{p,n}$ for $n = 1, \ldots, p$, and $|\mathcal{B}_{p,0}| = p^2 + p + 1 = \mathcal{R}_{p,0}$, where $p+1$ corresponds to the number of phases of a p-mino, and p^2 is the number of colored phases of a p-tile excluding phase 1. $\square$

Proposition 13. *For $n \geq 0$, we have*

$$\mathcal{R}_{p,n} = (p+1)\, L_{p,n} - p.$$

Proof. Let $\mathcal{B}^*_{p,n} := \mathcal{B}_{p,n} - \{s^n\}$ denote the subset of n-bracelets without only squares. There are $\mathcal{R}_{p,n} - 1$ bracelets in $\mathcal{B}^*_{p,n}$. Now, $\mathcal{B}^*_{p,n}$ can be partitioned into two disjoint subsets, $\mathcal{B}^*_{p,n} = \mathcal{B}^{*1}_{p,n} \cup \mathcal{B}^{*2}_{p,n}$, where $\mathcal{B}^{*1}_{p,n}$ contains n-bracelets with squares and at least one p-mino, and where $\mathcal{B}^{*2}_{p,n}$ contains n-bracelets with p-tiles, squares, and p-minos. Then, we have $|\mathcal{B}^{*1}_{p,n}| = L_{p,n} - 1$. To count the number of bracelets in $B^{*2}_{p,n}$, we consider the following two cases.

If the p-tile has length $p+1$, then there are $p+1$ different phases to arrange it, each in one of the p colors. The number of ways to tile the remaining $n-p-1$ cells is given by $F_{p,n-p}$. Then, $\left|\mathcal{B}^{*2}_{p,n}\right| = p(p+1)F_{p,n-p}$.

If the p-tile has length $l \geq p+2$, the number of ways to tile the remaining $n-p-2$ cells is

$$\sum_{j=0}^{n-p-2} F_{p,j+1} = F_{p,n+1} - F_{p,n-p} - 1.$$

Thus, we have

$$|\mathcal{B}^{*2}_{p,n}| = p(p+1)F_{p,n-p} + p(F_{p,n+1} - F_{p,n-p} - 1) = p(L_{p,n} - 1).$$

Hence, $|\mathcal{B}^*_{p,n}| = (p+1)L_{p,n} - p - 1$ which gives the desired result. $\square$

Proposition 14. *For $n \geq 0$, we have*

$$\mathcal{R}_{p,n} = (p+1)\, \mathcal{L}_{p,n} - p\mathcal{L}_{p,n-1}.$$

Proof. Let λ be an n-bracelet. There are two cases to consider.

If λ is breakable at cell n (i.e., the first tile starts at cell 1), then λ can be converted into a linear tiling, and there are $\mathcal{L}_{p,n}$ such tilings.

If λ is unbreakable at cell n, then the first tile is either a p-mino or a p-tile of length $p+1$ at phase r $(2 \leq r \leq p+1)$. If the first tile is a p-mino, then there are p ways to arrange it, and $F_{p,n-p}$ ways to tile the remaining cells. If the first tile is a p-tile of length $p+1$, then there are p ways to arrange it, each in one of the p colors, and $F_{p,n-p}$ ways to tile the remaining cells. So, we have

$$p(pF_{p,n-p} + F_{p,n-p}) = p(\mathcal{L}_{p,n} - \mathcal{L}_{p,n-1}).$$

Considering both cases, we obtain the desired results. $\square$

Proposition 15. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} \mathcal{R}_{p,i} = \mathcal{R}_{p,n+p+1} - p(n+1) - 1. \tag{15}$$

Proof. Consider the $(n+p+1)$-bracelets that do not contain any p-minos. Such tilings are either in the form of s^{n+p+1} or $\mathcal{P}^1_{p+1}s^n$, $\mathcal{P}_l s^{n+p+1-l}$ for $p+2 \leq l \leq n+p+1$, resulting in $p(n+1)+1$ such bracelets. Since the number of $(n+p+1)$-bracelets is $\mathcal{R}_{p,n+p+1}$, excluding those that do not contain any p-minos yields the right-hand side of the identity.

The remaining bracelets must have at least one p-mino and have one of the following forms:

1. $\mathcal{P}^r_{p+1}s^n$ for $2 \leq r \leq p+1$,

2. $\mathbf{p}^r s^n$ for $1 \leq r \leq p+1$,

3. $\lambda'\mathbf{p}s^{n-i}$ for $1 \leq i \leq n$, where $\lambda' \in \mathcal{B}_{p,i}$.

There are p^2 bracelets of the first form and $p+1$ bracelets of the second form, giving a total of $p^2+p+1 = \mathcal{R}_{p,0}$. For the third type of bracelets, we condition on the position of the last p-mino, which covers cells $i+1, \ldots, i+p+1$ for $1 \leq i \leq n$. These bracelets can be transformed into i-bracelets by removing the pieces $\mathbf{p}s^{n-i}$. Gluing the remaining pieces together results in $\mathcal{R}_{p,i}$ such bracelets (see Figure 5). Summing over i gives $\sum_{i=1}^{n} \mathcal{R}_{p,i}$. Combining these three cases, we obtain the left-hand side of the identity. $\square$

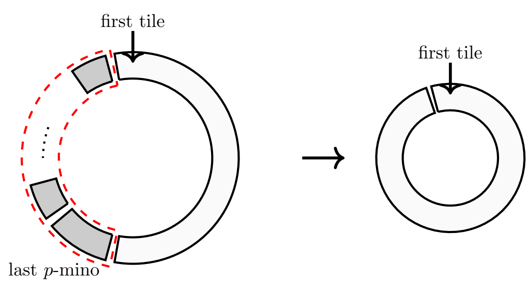

Figure 5: Transforming a bracelet in the form of $\lambda'\mathbf{p}s^{n-i}$ to λ'

Proposition 16. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} \mathcal{R}_{p,(p+1)i} = \mathcal{R}_{p,(p+1)n+1} - p(n-p-1). \tag{16}$$

Proof. The proof of the identity (16) is similar to that of Proposition 5, considering the position of the last square within $((p+1)n+1)$-bracelets. $\square$

Now, we give a combinatorial interpretation of incomplete Lucas-Leonardo p-numbers.

Let $\mathcal{C}_{p,n}$ denote the set consisting of n-bracelets λ. The first piece of λ is assigned one of $p+1$ colors, provided λ is not the entire square bracelet. If λ is the entire square bracelet, then the first piece of λ is not assigned a color. Then, from the proof of relation (6), we have

$$|\mathcal{C}_{p,n}| = (p+1)\left(L_{p,n}-1\right)+1 = (p+1)\,L_{p,n}-p = \mathcal{R}_{p,n}.$$

For $0 \le k \le \left\lfloor \frac{n}{p+1} \right\rfloor$, let $\mathcal{C}_{p,n}(k)$ denote the subset of $\mathcal{C}_{p,n}$ containing at most k p-minos. It is well known that $L_{p,n} = \sum_{i=0}^{\lfloor \frac{n}{p+1} \rfloor} \frac{n}{n-pi}\binom{n-pi}{i}$, then

$$|\mathcal{C}_{p,n}(k)| = (p+1)\sum_{i=0}^{k} \frac{n}{n-pi}\binom{n-pi}{i} - p. \tag{17}$$

Comparing (17) with (8), we get $|\mathcal{C}_{p,n}(k)| = \mathcal{R}_{p,n}(k)$.

Proposition 17. *For $0 \le k \le \left\lfloor \frac{n}{p+1} \right\rfloor$ and $n > p$, we have*

$$\mathcal{R}_{p,n}(k+1) = \mathcal{R}_{p,n-1}(k+1) + \mathcal{R}_{p,n-p-1}(k) + p.$$

Proof. We condition on the last tile. If the last tile is square, there are $\mathcal{R}_{p,n-1}(k+1)$ ways to tile the remaining $n-1$ cells with at most $k+1$ p-minos. If the last tile is a p-mino, there are $\mathcal{R}_{p,n-p-1}(k)$ ways to tile the first $n-p-1$ cells with at most k p-minos. Note that the tiling of the form $s^{n-p-1}\mathbf{p}$ is counted without coloring. Hence, we must add p to account for these missed tilings. $\square$

Proposition 18. *For $0 \le n \le \frac{n-p-r}{p+1}$, we have*

$$\sum_{i=0}^{r}\binom{r}{i}\mathcal{R}_{p,n+pi}(k+i) + (2^r-1)p = \mathcal{R}_{p,n+(p+1)r}(k+r). \tag{18}$$

Proof. The proof is similar to the Proposition 7. $\square$

Now we establish a link between incomplete Lucas-Leonardo p-numbers and Leonardo p-numbers. To do this, we introduce the concept of hyper Lucas-Leonardo p-numbers.

The hyper Lucas-Leonardo p-number $\mathcal{R}_{p,n}^{(k)}$ counts the number of circular tilings of length $n+(p+1)k$ with at least k p-minos. Such a tiling λ, the first tile is assigned one of $p+1$ colors, except when λ has phase 1 bracelet with exactly k p-minos. In this case, the first tile of λ is not assigned a color. Thus, we give the following definition.

Definition 2. For $n \geq 1$ and $k \geq 1$, the hyper Lucas-leonardo p-numbers are defined by

$$\mathcal{R}_{p,n}^{(k)} = (p+1) \sum_{i=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \frac{n+(p+1)\,k}{n+k-pi} \binom{n+k-pi}{i+k} - p\binom{n+k}{k},$$

with $\mathcal{R}_{p,n}^{(0)} = \mathcal{R}_{p,n}$, $\mathcal{R}_{p,0}^{(k)} = \mathcal{R}_{p,0} = p^2 + p + 1$.

Some special cases for the hyper Lucas-Leonardo p-sequence can be given as follows. For $p = 1$, we get the hyper Lucas-Leonardo sequence $\left\{\mathcal{R}_n^{(k)}\right\}$. In particular, we have

$$\begin{aligned}
\left\{\mathcal{R}_n^{(1)}\right\} &= \{3, 4, 9, 16, 29, 50, 85, 122, \ldots\}, \\
\left\{\mathcal{R}_n^{(2)}\right\} &= \{3, 7, 16, 32, 61, 101, 186, 308, \ldots\}, \\
\left\{\mathcal{R}_n^{(3)}\right\} &= \{3, 10, 26, 58, 119, 220, 406, 714, \ldots\}.
\end{aligned}$$

For $p = 2$, we get the hyper Lucas-Leonardo 2-sequence. In particular, we have

$$\begin{aligned}
\left\{\mathcal{R}_{2,n}^{(1)}\right\} &= \{7, 8, 9, 19, 32, 48, 76, 119, 180, \ldots\}, \\
\left\{\mathcal{R}_{2,n}^{(2)}\right\} &= \{7, 15, 24, 43, 75, 123, 199, 318, 498, \ldots\}, \\
\left\{\mathcal{R}_{2,n}^{(3)}\right\} &= \{7, 22, 46, 89, 164, 287, 486, 804, 1302, \ldots\}.
\end{aligned}$$

We note that for $p > 1$, we observe that the hyper Lucas-Leonardo p-sequences are new additions in OEIS [10].

Proposition 19. *For $n > p \geq 1$ and $k \geq 1$, we have*

$$\mathcal{R}_{p,n}^{(k)} = \mathcal{R}_{p,n+(p+1)k} - \mathcal{R}_{p,n+(p+1)k}(k-1) - p\binom{n+k}{k}.$$

Proof. By using Definition 2 and relation (9), we get the desired result. $\square$

Finally, we give a relationship between hyper Leonardo p-numbers and hyper Lucas-Leonardo p-numbers.

Proposition 20. *For $n > p \geq 1$ and $k \geq 1$, we have*

$$\mathcal{R}_{p,n}^{(k)} = (p+1)\mathcal{L}_{p,n}^{(k)} - p\mathcal{L}_{p,n-1}^{(k)} + p^2\binom{n+k-1}{k}.$$

Proof. From Proposition 11 and using the relations (7) and (10), we have

$$\begin{aligned}(p+1)\,\mathcal{L}_{p,n}^{(k)} - p\mathcal{L}_{p,n-1}^{(k)} &= \big((p+1)\,\mathcal{L}_{p,n+(p+1)k} - p\mathcal{L}_{p,n+(p+1)k-1}\big)\\ &\quad - \big((p+1)\,\mathcal{L}_{p,n+(p+1)k}(k-1) - p\mathcal{L}_{p,n+(p+1)k-1}(k-1)\big)\\ &\quad - \left(p^2+p\right)\binom{n+k}{k} + p^2\binom{n+k-1}{k}\\ &= \mathcal{R}_{p,n+(p+1)k} - \mathcal{R}_{p,n+(p+1)k}(k-1) - p^2\binom{n+k-1}{k-1}\\ &\quad -p\binom{n+k}{k}.\end{aligned}$$

By using Proposition 19, we get the desired result. □

4. Concluding Remarks

This paper provides combinatorial interpretations of Leonardo p-numbers and Lucas-Leonardo p-numbers while also extending the incomplete and hyper analogs. Our results offer combinatorial interpretations and reveal several new identities related to hyper Lucas-Leonardo p-numbers. In future work, we plan to investigate additional identities and properties of hyper Lucas-Leonardo p-numbers.

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