NEW REFINEMENTS OF MANDL'S INEQUALITY

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Abstract

In this paper, we study a particular sequence of prime numbers, $\mathcal{M}(n) := np_n/2 - \sum_{k\leq n} p_k$ $(n \in \mathbb{N})$, by using inequalities about p_n, and we obtain a new upper and a new lower bound on $\mathcal{M}(n)$.

1. Introduction

Let p_n denote the nth prime number. Mandl [1] speculated that for every integer $n \geq 9$, the expression $\mathcal{M}(n) := np_n/2 - \sum_{k\leq n} p_k$ is positive. In 1975, Rosser and Schoenfeld [12] discussed Mandl's inequality but did not provide a proof. In 1998, Dusart provided a comprehensive proof of Mandl's inequality, as documented in [7]. In 2013, Hassani [9] further refined Mandl's inequality, establishing that $n^2/12 < \mathcal{M}(n) < 9n^2/4$ for $n \geq 10$ and $n \geq 2$, respectively. Let

$$B_\eta(n) := \frac{n^2}{4}\left(1 + \frac{1}{\log n} - \frac{\log\log n - \eta}{\log^2 n}\right).$$

In the same vein, Axler [1] introduced a fresh estimate of $\mathcal{M}(n)$ in the year 2013, specifying $B_\alpha(n) < \mathcal{M}(n) < B_\beta(n)$ for $n \geq 348{,}247$ and $n \geq 26{,}219$, respectively, where $\alpha = 2.092$, $\beta = 5.228$. Then, in 2019, an enhancement to Axler's bound of $\mathcal{M}(n)$ emerged [4], refining the estimation to $B_\alpha(n) < \mathcal{M}(n) < B_\beta(n)$ for $n \geq 6{,}309{,}751$ and $n \geq 256{,}376$, respectively, where $\alpha = 2.9$, $\beta = 4.42$. Building upon Axler's result, we obtain a new lower and a new upper bound for $\mathcal{M}(n)$.

Theorem 1. *We have $B_\alpha(n) < \mathcal{M}(n) < B_\beta(n)$ for $n \geq 234{,}057{,}667{,}276{,}344{,}608$ and $n \geq 64{,}497{,}259{,}289$, respectively, where $\alpha = 3.151332$, $\beta = 4.168668$.*

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2. Several Inequalities

In this section, we collect certain prime number inequalities.

Lemma 1 ([10])**.** *For all positive integers n,*

$$p_n > n \log n. \tag{1}$$

Lemma 2 ([11])**.** *For $17 \le x$,*

$$\frac{x}{\log x} < \pi(x).$$

Then, for $n \ge 7$,

$$p_n < n \log p_n. \tag{2}$$

Lemma 3 ([7])**.** *For all $x \ge 60,184$,*

$$\pi(x) < \frac{x}{\log x - 1.1}.$$

Then, for $n \ge 6,076 = \pi(60184)$,

$$p_n > n(\log p_n - 1.1). \tag{3}$$

For all $x \ge 5,393$,

$$\pi(x) > \frac{x}{\log x - 1}.$$

Then, for $n \ge 711 = \pi(5393)$,

$$p_n < n(\log p_n - 1). \tag{4}$$

Lemma 4 ([2])**.** *For all $x \ge 468,049$,*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x}}.$$

Then, for $n \ge 39,071 = \pi(468049)$,

$$p_n < n(\log p_n - 1 - \tfrac{1}{\log p_n}). \tag{5}$$

Lemma 5 ([6])**.** *We have*

$$\pi(x) < \frac{x}{\log x - a_0 - \frac{a_1}{\log x} - \frac{a_2}{\log^2 x}}$$

for every $x \ge x_0$, where a_0, a_1, a_2, and x_0 are given as in Table 1.

a_0	1.0343	1	1
a_1	0	1.109	1
a_2	0	0	3.48
x_0	$106,640,139,304,611$	$81,250,795,096,339$	$145,413,088,724,077$

Table 1: Explicit values for a_0, a_1, a_2, and x_0.

We have

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.024334}{\log^2 x} - \frac{a_3}{\log^3 x} - \frac{a_4}{\log^4 x} - \frac{a_5}{\log^5 x}}$$

for every $x \geq x_0$, where a_3, a_4, a_5, and x_0 are given as in Table 2.

a_3	14.893	12.975666	12.975666
a_4	0	79.962	71.048668
a_5	0	0	533.594
x_0	$142,464,507,937,911$	22	32

Table 2: Explicit values for a_3, a_4, a_5, and x_0.

We have

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{a_2}{\log^2 x} - \frac{a_3}{\log^3 x} - \frac{a_4}{\log^4 x} - \frac{a_5}{\log^5 x} - \frac{a_6}{\log^6 x}}$$

for every $x \geq x_0$, where a_2, a_3, a_4, a_5, a_6, and x_0 are given as in Table 3.

a_2	2.975666	2.975666	2.975666
a_3	0	13.024334	13.024334
a_4	0	0	70.951332
a_5	0	0	0
a_6	0	0	0
x_0	$54,941,209$	$7,713,187,213$	$153,887,581,621$
a_2	2.975666	2.975666	
a_3	13.024334	13.024334	
a_4	70.951332	70.951332	
a_5	460.634397856444	460.634397856444	
a_6	0	3444.031844143556	
x_0	$1,035,745,443,241$	$1,751,189,194,177$	

Table 3: Explicit values for a_2, a_3, a_4, a_5, a_6, and x_0.

Corollary 1. *For $n \geq 2,621,506,158,680$,*

$$p_n > n(\log p_n - 1 - \tfrac{1.109}{\log p_n}). \tag{6}$$

For $n \geq 4,605,081,012,945$,

$$p_n > n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{3.48}{\log^2 p_n}). \tag{7}$$

For $n \geq 4,514,634,967,572$,

$$p_n > n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{3.024334}{\log^2 p_n} - \tfrac{14.893}{\log^3 p_n}). \tag{8}$$

For $n \geq 8$,

$$p_n > n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{3.024334}{\log^2 p_n} - \tfrac{12.975666}{\log^3 p_n} - \tfrac{79.962}{\log^4 p_n}). \tag{9}$$

Proof. According to the second column of Table 1 in Lemma 5, it can be inferred that

$$n = \pi(p_n) < \frac{p_n}{\log p_n - 1 - \frac{1.109}{\log p_n} - \frac{0}{\log^2 p_n}},$$

for every $p_n \geq 81,250,795,096,339$. This is equivalent to

$$p_n > n(\log p_n - 1 - \tfrac{1.109}{\log p_n}),$$

for every $n = \pi(p_n) \geq \pi(81,250,795,096,339) = 2,621,506,158,680$, which proves Inequality (6). Similar methodologies apply to the other inequalities. □

Corollary 2. *For $n \geq 3,278,841$,*

$$p_n < n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n}). \tag{10}$$

For $n \geq 355,193,892$,

$$p_n < n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n}). \tag{11}$$

For $n \geq 6,226,419,706$,

$$p_n < n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n} - \tfrac{70.951332}{\log^4 p_n}). \tag{12}$$

For $n \geq 38,900,733,192$,

$$p_n < n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n} - \tfrac{70.951332}{\log^4 p_n} - \tfrac{460.634397856444}{\log^5 p_n}). \tag{13}$$

For $n \geq 64,497,259,289$,

$$p_n < n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n} - \tfrac{70.951332}{\log^4 p_n} - \tfrac{460.634397856444}{\log^5 p_n} - \tfrac{3444.031844143556}{\log^6 p_n}). \tag{14}$$

Proof. Just as demonstrated in the proof of Corollary 1, this corollary can be established by referring to Table 3 provided in Lemma 5. □

Lemma 6 ([3])**.** *For all integers* $n \geq 234,057,667,276,344,608$*,*

$$\log n \geq 0.914 \log p_n. \tag{15}$$

Lemma 7 ([4])**.** *For all positive integers* n*,*

$$\log p_n \leq \log n + \log\log n + \tfrac{\log\log n - 1}{\log n} + \tfrac{\log\log n - 2}{\log^2 n}. \tag{16}$$

Lemma 8 ([8])**.** *For all positive integers* n*,*

$$\log p_n \geq \log n + \log\log n + \log(1 + \tfrac{\log\log n - 1}{\log n}). \tag{17}$$

Lemma 9 ([5])**.** *For all* $n \geq 440,200,309$*,*

$$\begin{aligned} np_n - \sum_{k \leq n} p_k \geq{}& \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + \frac{44.4p_n^2}{8\log^4 p_n} \\ &+ \frac{92.1p_n^2}{4\log^5 p_n} + \frac{937.5p_n^2}{8\log^6 p_n} + \frac{5674.5p_n^2}{8\log^7 p_n} + \frac{79789.5p_n^2}{16\log^8 p_n}. \end{aligned} \tag{18}$$

For every positive integer n*,*

$$\begin{aligned} np_n - \sum_{k \leq n} p_k \leq{}& \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + \frac{45.6p_n^2}{8\log^4 p_n} \\ &+ \frac{93.9p_n^2}{4\log^5 p_n} + \frac{952.5p_n^2}{8\log^6 p_n} + \frac{5755.5p_n^2}{8\log^7 p_n} + \frac{116371p_n^2}{16\log^8 p_n}. \end{aligned} \tag{19}$$

3. Auxiliary Propositions

Before proving Theorem 1, we first need to derive the following propositions.

Proposition 1. *For all* $n \in \mathbb{N}$*,*

$$937.5p_n^2 \log p_n + 5674.5p_n^2 > 118.676119576n^2 \log^3 p_n + 709.7856np_n \log^2 p_n.$$

Proof. First, for the case of $n \geq 6,076$,

$$\begin{aligned} 937.5p_n^2 \log p_n &= 937.5p_n \log p_n \cdot p_n \\ &> 937.5p_n \log p_n \cdot n(\log p_n - 1.1) \quad \text{(by Inequality (3))} \\ &= 937.5np_n \log^2 p_n - 1031.25np_n \log p_n \\ &= 227.7144np_n \log^2 p_n + 709.7856np_n \log^2 p_n - 1031.25np_n \log p_n. \end{aligned} \tag{20}$$

We have

$$109.038280424n^2\log^3 p_n - 250.48584n^2\log^2 p_n > 0,$$

since $\log p_n > 250.48584/109.038280424$ for $n \geq 6,076$. Consequently, we obtain

$$\begin{aligned}227.7144np_n\log^2 p_n &= 227.7144n\log^2 p_n \cdot p_n \\ &> 227.7144n\log^2 p_n \cdot n(\log p_n - 1.1) \quad \text{(by Inequality (3))} \\ &= 227.7144n^2\log^3 p_n - 250.48584n^2\log^2 p_n \\ &= 118.676119576n^2\log^3 p_n + 109.038280424n^2\log^3 p_n \\ &\quad - 250.48584n^2\log^2 p_n \\ &> 118.676119576n^2\log^3 p_n. \end{aligned} \tag{21}$$

Using Inequalities (20) and (21), we derive

$$937.5p_n^2\log p_n > 118.676119576n^2\log^3 p_n + 709.7856np_n\log^2 p_n - 1031.25np_n\log p_n. \tag{22}$$

Since $\log p_n > 1.1/0.8$ for all $n \geq 6,076$, we have

$$p_n > n(\log p_n - 1.1) > n(\log p_n - 0.8\log p_n) = 0.2n\log p_n$$

by Inequality (3), and then we get

$$5674.5p_n^2 > 5674.5p_n \cdot 0.2n\log p_n = 1134.9np_n\log p_n > 1031.25np_n\log p_n. \tag{23}$$

Using Inequalities (22) and (23), we derive

$$937.5p_n^2\log p_n + 5674.5p_n^2 > 118.676119576n^2\log^3 p_n + 709.7856np_n\log^2 p_n. \tag{24}$$

Thus the claim follows for every integer $n \geq 6,076$. We check the remaining cases with a computer. □

Proposition 2. *For all integers $n \geq 4,605,081,012,945$,*

$$\frac{np_n}{2} - \sum_{k\leq n} p_k > \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2\log\log n}{4\log^2 n} + \frac{\omega(n)n^2}{\log^2 n},$$

where

$$\omega(n) := \frac{3.151332\log^2 n}{4\log^2 p_n} + \frac{\log^2 n}{4\log p_n} + \frac{16.7\log^2 n}{4\log^3 p_n} - \frac{\log n}{4} + \frac{\log\log n}{4}.$$

We postpone the proof of this proposition to the Appendix.

Proposition 3. *For all integers $n \geq 2$, $n \in \mathbb{N}$, we have*

$$\frac{4.151332(\log\log n - 2)}{\log n} + \frac{3.151332(\log\log n - 1)}{\log n} + \frac{3.151332(\log\log n - 2)}{\log^2 n} \leq 1. \tag{25}$$

Proof. For all integers $n \geq 2$, $n \in \mathbb{N}$, let $f(x) := \log x - 4 \cdot 4.151332(\log\log x - 2)$. It is easy to see that $f^{'}(x) \geq 0$ for $x \geq e^{16.605328}$ and $f(e^{16.605328}) \geq 3.159$, so $f(x) \geq 0$ for $x > 1$. Then we get

$$\frac{4.151332(\log\log n - 2)}{\log n} + \frac{3.151332(\log\log n - 2)}{\log^2 n} \leq 2 \cdot \frac{4.151332(\log\log n - 2)}{\log n} \leq \frac{1}{2}. \tag{26}$$

Let $g(x) := \log x - 2 \cdot 3.151332(\log\log x - 1)$. It can be readily observed that $g^{'}(x) \geq 0$ for $x \geq e^{6.302664}$and $g(e^{6.302664}) \geq 1.002$. Hence, we conclude that $g(x) \geq 0$ for $x > 1$. It follows that

$$\frac{3.151332(\log\log n - 1)}{\log n} \leq \frac{1}{2}. \tag{27}$$

Combining Inequalities (26) and (27), we get the required inequality for all integers $n \geq 2$, $n \in \mathbb{N}$. □

Proposition 4. *For all integers* $n \geq 64,497,259,289$*,*

$$\frac{np_n}{2} - \sum_{k \leq n} p_k < \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2 \log\log n}{4\log^2 n} + \frac{\Omega(n) n^2}{\log^2 n},$$

where

$$\Omega(n) := \frac{\log^2 n}{4\log p_n} + \frac{3.848668\log^2 n}{4\log^2 p_n} - \frac{\log n}{4} + \frac{\log\log n}{4} + \frac{\tau(\log p_n)\log^2 n}{8\log^6 p_n},$$

$$\tau(x) := 34.6x^3 + 205.592008x^2 + 1420.426661948448x + 28890.293784512888.$$

We postpone the proof of this proposition to the Appendix.

4. Proof of Theorem 1

Now we use Proposition 2 and Proposition 4 to give a proof of Theorem 1.

Proof. Firstly, we prove the left inequality in Theorem 1. By Proposition 2 it suffices to show that $\omega(n) \geq 3.151332/4$. It is easy to see that $x^2 - 7.302664x + 16.7 \cdot 0.914^2 \geq 0$ for $x \geq 0$. Then we get

$$\begin{aligned}((\log\log n)^2 - 7.302664\log\log n + 16.7 \cdot 0.914^2)\log^2 p_n \\ + 3.151332\log p_n(\log\log n)^2 - 3.151332\log p_n(\log\log n - 1) \geq 0.\end{aligned} \tag{28}$$

Combining (28) with Inequality (15), we obtain

$$16.7\log^2 n + ((\log\log n)^2 - 7.302664\log\log n)\log^2 p_n$$

$$+ 3.151332 \log p_n (\log\log n)^2 - 3.151332 \log p_n (\log\log n - 1) \geq 0. \quad (29)$$

Using Inequality (1), we obtain

$$-\log p_n + \log\log n \leq -\log n. \quad (30)$$

Then we have

$$\begin{aligned} &-3.151332 \log^2 p_n \log\log n + 3.151332 \log p_n (\log\log n)^2 \\ &\qquad \leq -3.151332 \log p_n \log n \log\log n. \end{aligned}$$

Using Inequality (29), we obtain

$$\begin{aligned} 0 &\leq 16.7 \log^2 n + (\log\log n - 4.151332) \log\log n \log^2 p_n - 3.151332 \log\log n \log^2 p_n \\ &\quad + 3.151332 \log p_n (\log\log n)^2 - 3.151332 \log p_n (\log\log n - 1) \\ &\leq 16.7 \log^2 n + (\log\log n - 4.151332) \log\log n \log^2 p_n - 3.151332 \log\log n \log^2 p_n \\ &\quad + 3.151332 \log^2 p_n \log\log n - 3.151332 \log p_n \log n \log\log n \\ &\quad - 3.151332 \log p_n (\log\log n - 1). \end{aligned}$$

In other words,

$$\begin{aligned} &16.7 \log^2 n + (\log\log n - 4.151332) \log\log n \log^2 p_n \\ &\qquad - 3.151332 \log p_n \log n \log\log n - 3.151332 \log p_n (\log\log n - 1) \geq 0. \end{aligned} \quad (31)$$

Multiplying both sides of Inequality (25) by $\log^2 p_n$ and combining with Inequality (31), we get

$$\begin{aligned} &\log^2 p_n + 16.7 \log^2 n + (\log\log n - 4.151332) \log\log n \log^2 p_n \\ &\quad \geq \frac{(\log^2 p_n + 3.151332 \log^2 p_n)(\log\log n - 2)}{\log n} \\ &\qquad + 3.151332 \log p_n (\log n \log\log n + \log\log n - 1) \\ &\qquad + \frac{3.151332 \log^2 p_n}{\log n} \left(\log\log n - 1 + \frac{\log\log n - 2}{\log n} \right). \end{aligned}$$

It is easy to see that $\log^2 p_n > \log p_n$ and $\log\log n - 2 > 0$. Then

$$\begin{aligned} &\log^2 p_n + 16.7 \log^2 n + (\log\log n - 4.151332) \log\log n \log^2 p_n \\ &\quad \geq \frac{(\log^2 p_n + 3.151332 \log p_n)(\log\log n - 2)}{\log n} \\ &\qquad + 3.151332 \log p_n (\log n \log\log n + \log\log n - 1) \\ &\qquad + \frac{3.151332 \log^2 p_n}{\log n} \left(\log\log n - 1 + \frac{\log\log n - 2}{\log n} \right). \end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\log^2 p_n + 16.7\log^2 n + 3.151332\log p_n \log^2 n + (\log\log n - 4.151332)\log\log n \log^2 p_n \\
&\quad \geq 3.151332\log p_n \log n \left(\log n + \log\log n + \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2}{\log^2 n}\right) \\
&\qquad + \frac{\log^2 p_n(\log\log n - 2)}{\log n} + \frac{3.151332\log^2 p_n}{\log n}\left(\log\log n - 1 + \frac{\log\log n - 2}{\log n}\right).
\end{aligned}$$

From Inequality (16), it follows that

$$\begin{aligned}
&\log^2 p_n + 16.7\log^2 n + 3.151332\log p_n \log^2 n + (\log\log n - 4.151332)\log\log n \log^2 p_n \\
&\quad \geq 3.151332\log^2 p_n \log n + \frac{\log^2 p_n(\log\log n - 2)}{\log n} \\
&\qquad + \frac{3.151332\log^2 p_n}{\log n}\left(\log\log n - 1 + \frac{\log\log n - 2}{\log n}\right).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\log^2 p_n + 16.7\log^2 n + 3.151332\log p_n \log^2 n + (\log\log n - 1)\log\log n \log^2 p_n \\
&\quad \geq 3.151332\log^2 p_n \left(\log n + \log\log n + \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2}{\log^2 n}\right) \\
&\qquad + \frac{\log^2 p_n(\log\log n - 2)}{\log n}.
\end{aligned}$$

Reapplying Inequality (16) gives

$$\begin{aligned}
&\log^2 p_n + 16.7\log^2 n + 3.151332\log p_n \log^2 n + (\log\log n - 1)\log\log n \log^2 p_n \\
&\quad \geq 3.151332\log^3 p_n + \frac{\log^2 p_n(\log\log n - 2)}{\log n}.
\end{aligned}$$

Using Inequality (30), we obtain

$$\begin{aligned}
&16.7\log^2 n + 3.151332\log p_n \log^2 n + (\log p_n - \log n)\log\log n \log^2 p_n \\
&\quad \geq \log^2 p_n(\log\log n - 1) + 3.151332\log^3 p_n + \frac{\log^2 p_n(\log\log n - 2)}{\log n}.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\log^2 p_n \log^2 n + 16.7\log^2 n + 3.151332\log p_n \log^2 n \\
&\quad \geq \log^2 p_n \log n \left(\log n + \log\log n + \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2}{\log^2 n}\right) \\
&\qquad - \log^3 p_n \log\log n + 3.151332\log^3 p_n.
\end{aligned}$$

Reapplying Inequality (16) gives

$$\begin{aligned}&\log^2 p_n \log^2 n + 16.7\log^2 n + 3.151332\log p_n \log^2 n\\&\qquad \geq \log^3 p_n \log n - \log^3 p_n \log\log n + 3.151332\log^3 p_n.\end{aligned}$$

Finally, we divide both sides of this last inequality by $4\log^3 p_n$. This yields

$$\frac{\log^2 n}{4\log p_n} + \frac{16.7\log^2 n}{4\log^3 p_n} + \frac{3.151332\log^2 n}{4\log^2 p_n} \geq \frac{\log n}{4} - \frac{\log\log n}{4} + \frac{3.151332}{4}.$$

According to the expression for $\omega(n)$ in Proposition 2, we get $\omega(n) \geq 3.151332/4$. Hence, the claim follows from Proposition 2 for every $n \geq 234,057,667,276,344,608$.

Next, we prove the right inequality in Theorem 1. Let $a_1 = 0.08$ and for all positive $x \geq e^{24.889}$, $f(x) := 4a_1(x+\log x)+(x+4a_1-\log x)\log(1+\frac{\log x-1}{x})-\log^2 x$. Then $f(\log n) \geq 0$ for $\log n = \log(64,497,259,289) > 24.889$, so that

$$(\log n + \log\log n + \log(1 + \tfrac{\log\log n-1}{\log n}))(4a_1 + \log n - \log\log n) \geq \log^2 n. \tag{32}$$

Combining Inequalities (32) and (17), we obtain

$$8a_1\log^8 p_n \geq 2\log^7 p_n \log^2 n - 2\log^8 p_n \log n + 2\log^8 p_n \log\log n. \tag{33}$$

Let $a_2 = 0.962167$ and define $g(x) := 16a_2x^3\log x + 8a_2x^2\log^2 x - \tau(x)$. For $x \geq \log(1,751,189,194,177) > 28.20$, we have $g(x) \geq 0$. Consequently,

$$\begin{aligned}&16a_2\log^5 p_n \log^2 n \log\log p_n + 8a_2\log^4 p_n \log^2 n(\log\log p_n)^2\\&\qquad - \tau(\log p_n)\log^2 p_n \log^2 n + (8a_2 - 7.697336)\log^6 p_n \log^2 n\\&\quad = \log^2 p_n \log^2 n \cdot g(\log p_n) \geq 0.\end{aligned} \tag{34}$$

It is easy to see that $\frac{\log\log n}{\log n} > \frac{\log\log p_n}{\log p_n}$ for $\log p_n > \log n > e$. Combining this with Inequality (34), it follows that

$$8a_2\log^6 p_n(\log n+\log\log n)^2 - \tau(\log p_n)\log^2 p_n \log^2 n - 7.697336\log^6 p_n \log^2 n \geq 0. \tag{35}$$

We can see from Inequality (17) that $\log p_n \geq \log n + \log\log n$. We combine this with Inequality (35) to obtain

$$8a_2\log^8 p_n \geq \tau(\log p_n)\log^2 p_n \log^2 n + 7.697336\log^6 p_n \log^2 n.$$

Combining the last inequality with Inequality (33), we have $\Omega(n) \leq 4.168668/4$ for every integer $n \geq 64,497,259,289$. Hence, the claim follows from Proposition 4 for every integer $n \geq 64,497,259,289$. □

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Appendix

Below are the detailed proofs of Proposition 2 and Proposition 4.

Proof of Proposition 2. We consider the case where $n \geq 4,605,081,012,945$. To prove the statement, we first prove the following inequality:

$$\begin{aligned} &\frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + \frac{44.4p_n^2}{8\log^4 p_n} \\ &\quad + \frac{92.1p_n^2}{4\log^5 p_n} + \frac{937.5p_n^2}{8\log^6 p_n} + \frac{5674.5p_n^2}{8\log^7 p_n} + \frac{79789.5p_n^2}{16\log^8 p_n} \\ &\quad > \frac{np_n}{2} + \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2\log\log n}{4\log^2 n} + \frac{\omega(n)n^2}{\log^2 n}. \end{aligned} \tag{36}$$

According to the expression for $\omega(n)$ in Proposition 2, we obtain

$$2n^2\log^7 p_n \log^2 n + 6.302664n^2\log^6 p_n \log^2 n + 33.4n^2\log^5 p_n \log^2 n$$

$$= 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\omega(n)n^2 \log^8 p_n. \tag{37}$$

Multiplying both sides of Inequality (24) by $\log p_n \log^2 n$ and combining with Inequality (37), we obtain

$$\begin{aligned}
2n^2 \log^7 p_n \log^2 n &+ 6.302664n^2 \log^6 p_n \log^2 n + 33.4n^2 \log^5 p_n \log^2 n \\
&+ 937.5p_n^2 \log^2 p_n \log^2 n + 5674.5p_n^2 \log p_n \log^2 n \\
&> 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\omega(n)n^2 \log^8 p_n \\
&+ 118.676119576n^2 \log^4 p_n \log^2 n + 709.7856np_n \log^3 p_n \log^2 n.
\end{aligned}$$

Then

$$\begin{aligned}
2n^2 \log^7 p_n \log^2 n &+ 6.302664n^2 \log^6 p_n \log^2 n + 33.4n^2 \log^5 p_n \log^2 n \\
&+ 937.5p_n^2 \log^2 p_n \log^2 n + 5674.5p_n^2 \log p_n \log^2 n + 39894.75p_n^2 \log^2 n \\
&> 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\omega(n)n^2 \log^8 p_n \\
&+ 118.676119576n^2 \log^4 p_n \log^2 n + 709.7856np_n \log^3 p_n \log^2 n.
\end{aligned} \tag{38}$$

We have

$$\begin{aligned}
55.751332&np_n \log^4 p_n \log^2 n \\
&= 55.751332n \log^4 p_n \log^2 n \cdot p_n \\
&> 55.751332n \log^4 p_n \log^2 n \cdot n(\log p_n - 1.1) \quad \text{(by Inequality (3))} \\
&= 55.751332n^2 \log^5 p_n \log^2 n - 61.3264652n^2 \log^4 p_n \log^2 n.
\end{aligned} \tag{39}$$

Subtracting $33.4n^2 \log^5 p_n \log^2 n$ from both sides of Inequality (38) and adding Inequality (39), we derive

$$\begin{aligned}
2n^2 \log^7 p_n \log^2 n &+ 6.302664n^2 \log^6 p_n \log^2 n + 55.751332np_n \log^4 p_n \log^2 n \\
&+ 937.5p_n^2 \log^2 p_n \log^2 n + 5674.5p_n^2 \log p_n \log^2 n + 39894.75p_n^2 \log^2 n \\
&> 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\omega(n)n^2 \log^8 p_n \\
&+ 22.351332n^2 \log^5 p_n \log^2 n + 57.349654376n^2 \log^4 p_n \log^2 n \\
&+ 709.7856np_n \log^3 p_n \log^2 n.
\end{aligned} \tag{40}$$

We have

$$\begin{aligned}
184.2&p_n^2 \log^3 p_n \log^2 n \\
&= 184.2p_n \log^3 p_n \log^2 n \cdot p_n \\
&> 184.2p_n \log^3 p_n \log^2 n \cdot n(\log p_n - 1.1) \quad \text{(by Inequality (3))} \\
&= 184.2np_n \log^4 p_n \log^2 n - 202.62np_n \log^3 p_n \log^2 n.
\end{aligned} \tag{41}$$

Subtracting $55.751332np_n \log^4 p_n \log^2 n$ from both sides of Inequality (40) and adding Inequality (41), we derive

$$\begin{aligned}
&2n^2 \log^7 p_n \log^2 n + 6.302664n^2 \log^6 p_n \log^2 n + 184.2p_n^2 \log^3 p_n \log^2 n \\
&\quad + 937.5p_n^2 \log^2 p_n \log^2 n + 5674.5p_n^2 \log p_n \log^2 n + 39894.75p_n^2 \log^2 n \\
&> 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\omega(n)n^2 \log^8 p_n \\
&\quad + 22.351332n^2 \log^5 p_n \log^2 n + 57.349654376n^2 \log^4 p_n \log^2 n \\
&\quad + 128.448668np_n \log^4 p_n \log^2 n + 507.1656np_n \log^3 p_n \log^2 n. \qquad (42)
\end{aligned}$$

We have

$$\begin{aligned}
&12.302664np_n \log^5 p_n \log^2 n \\
&= 12.302664n \log^5 p_n \log^2 n \cdot p_n \\
&> 12.302664n \log^5 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1.109}{\log p_n}) \quad \text{(by Inequality (6))} \\
&= 12.302664n^2 \log^6 p_n \log^2 n - 12.302664n^2 \log^5 p_n \log^2 n \\
&\quad - 13.643654376n^2 \log^4 p_n \log^2 n. \qquad (43)
\end{aligned}$$

Subtracting $6.302664n^2 \log^6 p_n \log^2 n$ from both sides of Inequality (42) and adding Inequality (43), we derive

$$\begin{aligned}
&2n^2 \log^7 p_n \log^2 n + 12.302664np_n \log^5 p_n \log^2 n + 184.2p_n^2 \log^3 p_n \log^2 n \\
&\quad + 937.5p_n^2 \log^2 p_n \log^2 n + 5674.5p_n^2 \log p_n \log^2 n + 39894.75p_n^2 \log^2 n \\
&> 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\omega(n)n^2 \log^8 p_n \\
&\quad + 6n^2 \log^6 p_n \log^2 n + 10.048668n^2 \log^5 p_n \log^2 n \\
&\quad + 43.706n^2 \log^4 p_n \log^2 n + 128.448668np_n \log^4 p_n \log^2 n \\
&\quad + 507.1656np_n \log^3 p_n \log^2 n. \qquad (44)
\end{aligned}$$

We have

$$\begin{aligned}
&44.4p_n^2 \log^4 p_n \log^2 n \\
&= 44.4p_n \log^4 p_n \log^2 n \cdot p_n \\
&> 44.4p_n \log^4 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1.109}{\log p_n}) \quad \text{(by Inequality (6))} \\
&= 44.4np_n \log^5 p_n \log^2 n - 44.4np_n \log^4 p_n \log^2 n \\
&\quad - 49.2396np_n \log^3 p_n \log^2 n. \qquad (45)
\end{aligned}$$

Subtracting $12.302664np_n \log^5 p_n \log^2 n$ from both sides of Inequality (44) and adding Inequality (45), we derive

$$\begin{aligned}
&2n^2 \log^7 p_n \log^2 n + 44.4p_n^2 \log^4 p_n \log^2 n + 184.2p_n^2 \log^3 p_n \log^2 n \\
&\quad + 937.5p_n^2 \log^2 p_n \log^2 n + 5674.5p_n^2 \log p_n \log^2 n + 39894.75p_n^2 \log^2 n
\end{aligned}$$

$$
\begin{aligned}
&> 2n^2\log^8 p_n \log n - 2n^2\log^8 p_n \log\log n + 8\omega(n)n^2\log^8 p_n \\
&\quad + 6n^2\log^6 p_n \log^2 n + 10.048668n^2\log^5 p_n \log^2 n \\
&\quad + 43.706n^2\log^4 p_n \log^2 n + 32.097336np_n\log^5 p_n \log^2 n \\
&\quad + 84.048668np_n\log^4 p_n \log^2 n \\
&\quad + 457.926np_n\log^3 p_n \log^2 n. \qquad (46)
\end{aligned}
$$

We have

$$
\begin{aligned}
&4np_n\log^6 p_n \log^2 n \\
&\quad = 4n\log^6 p_n \log^2 n \cdot p_n \\
&\quad > 4n\log^6 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{3.48}{\log^2 p_n}) \quad \text{(by Inequality (7))} \\
&\quad = 4n^2\log^7 p_n \log^2 n - 4n^2\log^6 p_n \log^2 n \\
&\qquad - 4n^2\log^5 p_n \log^2 n - 13.92n^2\log^4 p_n \log^2 n. \qquad (47)
\end{aligned}
$$

Subtracting $2n^2\log^7 p_n \log^2 n$ from both sides of Inequality (46) and adding Inequality (47), we derive

$$
\begin{aligned}
&4np_n\log^6 p_n \log^2 n + 44.4p_n^2\log^4 p_n \log^2 n + 184.2p_n^2\log^3 p_n \log^2 n \\
&\qquad + 937.5p_n^2\log^2 p_n \log^2 n + 5674.5p_n^2\log p_n \log^2 n + 39894.75p_n^2\log^2 n \\
&\quad > 2n^2\log^8 p_n \log n - 2n^2\log^8 p_n \log\log n + 8\omega(n)n^2\log^8 p_n \\
&\qquad + 2n^2\log^7 p_n \log^2 n + 2n^2\log^6 p_n \log^2 n + 6.048668n^2\log^5 p_n \log^2 n \\
&\qquad + 29.786n^2\log^4 p_n \log^2 n + 32.097336np_n\log^5 p_n \log^2 n \\
&\qquad + 84.048668np_n\log^4 p_n \log^2 n + 457.926np_n\log^3 p_n \log^2 n. \qquad (48)
\end{aligned}
$$

We have

$$
\begin{aligned}
&14p_n^2\log^5 p_n \log^2 n \\
&\quad = 14p_n\log^5 p_n \log^2 n \cdot p_n \\
&\quad > 14p_n\log^5 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{3.48}{\log^2 p_n}) \quad \text{(by Inequality (7))} \\
&\quad = 14np_n\log^6 p_n \log^2 n - 14np_n\log^5 p_n \log^2 n \\
&\qquad - 14np_n\log^4 p_n \log^2 n - 48.72np_n\log^3 p_n \log^2 n. \qquad (49)
\end{aligned}
$$

Subtracting $4np_n\log^6 p_n \log^2 n$ from both sides of Inequality (48) and adding Inequality (49), we derive

$$
\begin{aligned}
&14p_n^2\log^5 p_n \log^2 n + 44.4p_n^2\log^4 p_n \log^2 n + 184.2p_n^2\log^3 p_n \log^2 n \\
&\qquad + 937.5p_n^2\log^2 p_n \log^2 n + 5674.5p_n^2\log p_n \log^2 n + 39894.75p_n^2\log^2 n \\
&\quad > 2n^2\log^8 p_n \log n - 2n^2\log^8 p_n \log\log n + 8\omega(n)n^2\log^8 p_n \\
&\qquad + 2n^2\log^7 p_n \log^2 n + 2n^2\log^6 p_n \log^2 n + 6.048668n^2\log^5 p_n \log^2 n
\end{aligned}
$$

$$+ 29.786n^2 \log^4 p_n \log^2 n + 10np_n \log^6 p_n \log^2 n$$
$$+ 18.097336np_n \log^5 p_n \log^2 n + 70.048668np_n \log^4 p_n \log^2 n$$
$$+ 409.206np_n \log^3 p_n \log^2 n. \quad (50)$$

We have

$$\begin{aligned}2np_n \log^7 p_n \log^2 n &= 2n \log^7 p_n \log^2 n \cdot p_n \\ &> 2n \log^7 p_n \log^2 n \cdot n(\log p_n - 1 \\ &\quad - \tfrac{1}{\log p_n} - \tfrac{3.024334}{\log^2 p_n} - \tfrac{14.893}{\log^3 p_n}) \quad \text{(by Inequality (8))} \\ &= 2n^2 \log^8 p_n \log^2 n - 2n^2 \log^7 p_n \log^2 n - 2n^2 \log^6 p_n \log^2 n \\ &\quad - 6.048668n^2 \log^5 p_n \log^2 n - 29.786n^2 \log^4 p_n \log^2 n. \quad (51)\end{aligned}$$

Then add Inequality (50) to Inequality (51) to get

$$\begin{aligned}&2np_n \log^7 p_n \log^2 n + 14p_n^2 \log^5 p_n \log^2 n + 44.4p_n^2 \log^4 p_n \log^2 n \\ &\quad + 184.2p_n^2 \log^3 p_n \log^2 n + 937.5p_n^2 \log^2 p_n \log^2 n \\ &\quad + 5674.5p_n^2 \log p_n \log^2 n + 39894.75p_n^2 \log^2 n \\ &> 2n^2 \log^8 p_n \log^2 n + 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n \\ &\quad + 8\omega(n)n^2 \log^8 p_n + 10np_n \log^6 p_n \log^2 n + 18.097336np_n \log^5 p_n \log^2 n \\ &\quad + 70.048668np_n \log^4 p_n \log^2 n + 409.206np_n \log^3 p_n \log^2 n. \quad (52)\end{aligned}$$

We have

$$\begin{aligned}6p_n^2 \log^6 p_n \log^2 n &= 6p_n \log^6 p_n \log^2 n \cdot p_n \\ &> 6p_n \log^6 p_n \log^2 n \cdot n(\log p_n - 1 \\ &\quad - \tfrac{1}{\log p_n} - \tfrac{3.024334}{\log^2 p_n} - \tfrac{14.893}{\log^3 p_n}) \quad \text{(by Inequality (8))} \\ &= 6np_n \log^7 p_n \log^2 n - 6np_n \log^6 p_n \log^2 n - 6np_n \log^5 p_n \log^2 n \\ &\quad - 18.146004np_n \log^4 p_n \log^2 n - 89.358np_n \log^3 p_n \log^2 n. \quad (53)\end{aligned}$$

Subtracting $2np_n \log^7 p_n \log^2 n$ from both sides of Inequality (52) and adding Inequality (53), we derive

$$\begin{aligned}&6p_n^2 \log^6 p_n \log^2 n + 14p_n^2 \log^5 p_n \log^2 n + 44.4p_n^2 \log^4 p_n \log^2 n \\ &\quad + 184.2p_n^2 \log^3 p_n \log^2 n + 937.5p_n^2 \log^2 p_n \log^2 n \\ &\quad + 5674.5p_n^2 \log p_n \log^2 n + 39894.75p_n^2 \log^2 n \\ &> 2n^2 \log^8 p_n \log^2 n + 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n \\ &\quad + 8\omega(n)n^2 \log^8 p_n + 4np_n \log^7 p_n \log^2 n \\ &\quad + 4np_n \log^6 p_n \log^2 n + 12.097336np_n \log^5 p_n \log^2 n \\ &\quad + 51.902664np_n \log^4 p_n \log^2 n + 319.848np_n \log^3 p_n \log^2 n. \quad (54)\end{aligned}$$

Finally, from Inequality(9), it follows that

$$\begin{aligned}4p_n^2\log^7 p_n\log^2 n &= 4p_n\log^7 p_n\log^2 n\cdot p_n\\ &> 4p_n\log^7 p_n\log^2 n\cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{3.024334}{\log^2 p_n}\\ &\quad - \tfrac{12.975666}{\log^3 p_n} - \tfrac{79.962}{\log^4 p_n})\\ &= 4np_n\log^8 p_n\log^2 n - 4np_n\log^7 p_n\log^2 n - 4np_n\log^6 p_n\log^2 n\\ &\quad - 12.097336np_n\log^5 p_n\log^2 n - 51.902664np_n\log^4 p_n\log^2 n\\ &\quad - 319.848np_n\log^3 p_n\log^2 n. \qquad (55)\end{aligned}$$

Then add Inequality (54) to Inequality (55) to get

$$\begin{aligned}&4p_n^2\log^7 p_n\log^2 n + 6p_n^2\log^6 p_n\log^2 n + 14p_n^2\log^5 p_n\log^2 n\\ &\quad + 44.4p_n^2\log^4 p_n\log^2 n + 184.2p_n^2\log^3 p_n\log^2 n + 937.5p_n^2\log^2 p_n\log^2 n\\ &\quad + 5674.5p_n^2\log p_n\log^2 n + 39894.75p_n^2\log^2 n\\ &> 4np_n\log^8 p_n\log^2 n + 2n^2\log^8 p_n\log^2 n + 2n^2\log^8 p_n\log n\\ &\quad - 2n^2\log^8 p_n\log\log n + 8\omega(n)n^2\log^8 p_n.\end{aligned}$$

We divide the last inequality by $8\log^8 p_n\log^2 n$ to obtain Inequality (36). Then, by Inequality (18), Proposition 2 is proved. □

Proof of Proposition 4. We consider the case where $n \ge 64,497,259,289$. To prove the statement, we first prove the following inequality:

$$\begin{aligned}&\frac{np_n}{2} + \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2\log\log n}{4\log^2 n} + \frac{\Omega(n)n^2}{\log^2 n}\\ &> \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + \frac{45.6p_n^2}{8\log^4 p_n}\\ &\quad + \frac{93.9p_n^2}{4\log^5 p_n} + \frac{952.5p_n^2}{8\log^6 p_n} + \frac{5755.5p_n^2}{8\log^7 p_n} + \frac{116371p_n^2}{16\log^8 p_n}. \qquad (56)\end{aligned}$$

According to the expression for $\Omega(n)$ and $\tau(x)$ in Proposition 4 , we obtain

$$\begin{aligned}&2n^2\log^8 p_n\log n - 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n\\ &= 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\ &\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n\\ &\quad + 28890.293784512888n^2\log^2 p_n\log^2 n.\end{aligned}$$

Then

$$\begin{aligned}&2n^2\log^8 p_n\log n - 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n\\ &\quad + 3901.214098574224n^2\log^2 p_n\log^2 n\end{aligned}$$

$$
\begin{aligned}
&= 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n\\
&\quad + 28890.293784512888n^2\log^2 p_n\log^2 n\\
&\quad + 3901.214098574224n^2\log^2 p_n\log^2 n\\
&= 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n\\
&\quad + 32791.507883087112n^2\log^2 p_n\log^2 n\\
&= 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n\\
&\quad + 32791.507883087112n\log p_n\log^2 n\cdot(n\log p_n)\\
&> 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n\\
&\quad + 32791.507883087112np_n\log p_n\log^2 n. \qquad \text{(by Inequality (2))}
\end{aligned}
$$

We have

$$
\begin{aligned}
&2n^2\log^8 p_n\log n - 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n\\
&\quad + 3901.214098574224n^2\log^2 p_n\log^2 n\\
&\quad + 25393.992116912888np_n\log p_n\log^2 n\\
&> 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n\\
&\quad + 32791.507883087112np_n\log p_n\log^2 n\\
&\quad + 25393.992116912888np_n\log p_n\log^2 n.
\end{aligned}
$$

Then

$$
\begin{aligned}
&2n^2\log^8 p_n\log n - 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n\\
&\quad + 3901.214098574224n^2\log^2 p_n\log^2 n\\
&\quad + 25393.992116912888np_n\log p_n\log^2 n\\
&> 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n\\
&\quad + 58185.5np_n\log p_n\log^2 n\\
&= 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n\\
&\quad + 58185.5p_n\log^2 n\cdot(n\log p_n)\\
&> 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&\quad + 205.592008n^2\log^4 p_n\log^2 n + 1420.426661948448n^2\log^3 p_n\log^2 n
\end{aligned}
$$

$$+ 58185.5p_n^2 \log^2 n. \qquad \text{(by Inequality (2))}$$

We have

$$\begin{aligned}
2n^2 \log^8 p_n \log n &- 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n \\
&+ 3901.214098574224 n^2 \log^2 p_n \log^2 n \\
&+ 25393.992116912888 n p_n \log p_n \log^2 n \\
&+ 608.496709025776 n^2 \log^3 p_n \log^2 n \\
&- 2028.923370974224 n^2 \log^2 p_n \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336 n^2 \log^6 p_n \log^2 n \\
&+ 34.6 n^2 \log^5 p_n \log^2 n + 205.592008 n^2 \log^4 p_n \log^2 n \\
&+ 1420.426661948448 n^2 \log^3 p_n \log^2 n \\
&+ 58185.5 p_n^2 \log^2 n + 608.496709025776 n^2 \log^3 p_n \log^2 n \\
&- 2028.923370974224 n^2 \log^2 p_n \log^2 n.
\end{aligned}$$

Then

$$\begin{aligned}
2n^2 \log^8 p_n \log n &- 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n \\
&+ 608.496709025776 n^2 \log^3 p_n \log^2 n + 1872.2907276 n^2 \log^2 p_n \log^2 n \\
&+ 25393.992116912888 n p_n \log p_n \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336 n^2 \log^6 p_n \log^2 n + 34.6 n^2 \log^5 p_n \log^2 n \\
&+ 205.592008 n^2 \log^4 p_n \log^2 n + 2028.923370974224 n^2 \log^3 p_n \log^2 n \\
&- 2028.923370974224 n^2 \log^2 p_n \log^2 n + 58185.5 p_n^2 \log^2 n \\
&= 2n^2 \log^7 p_n \log^2 n + 7.697336 n^2 \log^6 p_n \log^2 n + 34.6 n^2 \log^5 p_n \log^2 n \\
&+ 205.592008 n^2 \log^4 p_n \log^2 n \\
&+ 2028.923370974224 n \log^2 p_n \log^2 n \cdot n(\log p_n - 1) \\
&+ 58185.5 p_n^2 \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336 n^2 \log^6 p_n \log^2 n + 34.6 n^2 \log^5 p_n \log^2 n \\
&+ 205.592008 n^2 \log^4 p_n \log^2 n \\
&+ 2028.923370974224 n p_n \log^2 p_n \log^2 n \qquad \text{(by Inequality (4))} \\
&+ 58185.5 p_n^2 \log^2 n.
\end{aligned}$$

We have

$$\begin{aligned}
2n^2 \log^8 p_n \log n &- 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n \\
&+ 608.496709025776 n^2 \log^3 p_n \log^2 n + 1872.2907276 n^2 \log^2 p_n \log^2 n \\
&+ 25393.992116912888 n p_n \log p_n \log^2 n \\
&+ 3726.576629025776 n p_n \log^2 p_n \log^2 n - 5755.5 n p_n \log p_n \log^2 n
\end{aligned}$$

$$> 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n$$
$$+ 34.6n^2 \log^5 p_n \log^2 n + 205.592008n^2 \log^4 p_n \log^2 n$$
$$+ 2028.923370974224np_n \log^2 p_n \log^2 n$$
$$+ 58185.5p_n^2 \log^2 n$$
$$+ 3726.576629025776np_n \log^2 p_n \log^2 n - 5755.5np_n \log p_n \log^2 n.$$

Then

$$\begin{aligned}
2n^2 \log^8 p_n \log n &- 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n \\
&+ 608.496709025776n^2 \log^3 p_n \log^2 n + 1872.2907276n^2 \log^2 p_n \log^2 n \\
&+ 3726.576629025776np_n \log^2 p_n \log^2 n \\
&+ 19638.492116912888np_n \log p_n \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n + 34.6n^2 \log^5 p_n \log^2 n \\
&+ 205.592008n^2 \log^4 p_n \log^2 n + 5755.5np_n \log^2 p_n \log^2 n \\
&- 5755.5np_n \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&= 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n + 34.6n^2 \log^5 p_n \log^2 n \\
&+ 205.592008n^2 \log^4 p_n \log^2 n + 5755.5p_n \log p_n \log^2 n \cdot n(\log p_n - 1) \\
&+ 58185.5p_n^2 \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n + 34.6n^2 \log^5 p_n \log^2 n \\
&+ 205.592008n^2 \log^4 p_n \log^2 n \\
&+ 5755.5p_n^2 \log p_n \log^2 n \qquad \text{(by Inequality (4))} \\
&+ 58185.5p_n^2 \log^2 n.
\end{aligned}$$

We have

$$\begin{aligned}
2n^2 \log^8 p_n \log n &- 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n \\
&+ 608.496709025776n^2 \log^3 p_n \log^2 n + 1872.2907276n^2 \log^2 p_n \log^2 n \\
&+ 3726.576629025776np_n \log^2 p_n \log^2 n \\
&+ 19638.492116912888np_n \log p_n \log^2 n \\
&+ 109.897336n^2 \log^4 p_n \log^2 n - 315.489344n^2 \log^3 p_n \log^2 n \\
&- 315.489344n^2 \log^2 p_n \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n + 34.6n^2 \log^5 p_n \log^2 n \\
&+ 205.592008n^2 \log^4 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n \\
&+ 58185.5p_n^2 \log^2 n \\
&+ 109.897336n^2 \log^4 p_n \log^2 n - 315.489344n^2 \log^3 p_n \log^2 n \\
&- 315.489344n^2 \log^2 p_n \log^2 n.
\end{aligned}$$

Then

$$\begin{aligned}
2n^2\log^8 p_n\log n &- 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n\\
&+ 109.897336n^2\log^4 p_n\log^2 n + 293.007365025776n^2\log^3 p_n\log^2 n\\
&+ 1556.8013836n^2\log^2 p_n\log^2 n\\
&+ 3726.576629025776np_n\log^2 p_n\log^2 n\\
&+ 19638.492116912888np_n\log p_n\log^2 n\\
&> 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n\\
&+ 34.6n^2\log^5 p_n\log^2 n + 315.489344n^2\log^4 p_n\log^2 n\\
&- 315.489344n^2\log^3 p_n\log^2 n - 315.489344n^2\log^2 p_n\log^2 n\\
&+ 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
&= 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n\\
&+ 34.6n^2\log^5 p_n\log^2 n\\
&+ 315.489344n\log^3 p_n\log^2 n\cdot n(\log p_n - 1 - \tfrac{1}{\log p_n})\\
&+ 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
&> 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n\\
&+ 34.6n^2\log^5 p_n\log^2 n\\
&+ 315.489344np_n\log^3 p_n\log^2 n \qquad \text{(by Inequality (5))}\\
&+ 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n.
\end{aligned}$$

We have

$$\begin{aligned}
2n^2\log^8 p_n\log n &- 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n\\
&+ 109.897336n^2\log^4 p_n\log^2 n + 293.007365025776n^2\log^3 p_n\log^2 n\\
&+ 1556.8013836n^2\log^2 p_n\log^2 n + 3726.576629025776np_n\log^2 p_n\log^2 n\\
&+ 19638.492116912888np_n\log p_n\log^2 n\\
&+ 637.010656np_n\log^3 p_n\log^2 n - 952.5np_n\log^2 p_n\log^2 n\\
&- 952.5np_n\log p_n\log^2 n\\
&> 2n^2\log^7 p_n\log^2 n + 7.697336n^2\log^6 p_n\log^2 n + 34.6n^2\log^5 p_n\log^2 n\\
&+ 315.489344np_n\log^3 p_n\log^2 n\\
&+ 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
&+ 637.010656np_n\log^3 p_n\log^2 n - 952.5np_n\log^2 p_n\log^2 n\\
&- 952.5np_n\log p_n\log^2 n.
\end{aligned}$$

Then

$$2n^2\log^8 p_n\log n - 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n$$

$$
\begin{aligned}
&+ 109.897336n^2 \log^4 p_n \log^2 n + 293.00736502577 6n^2 \log^3 p_n \log^2 n \\
&+ 1556.8013836n^2 \log^2 p_n \log^2 n + 637.010656np_n \log^3 p_n \log^2 n \\
&+ 2774.07662902577 6np_n \log^2 p_n \log^2 n \\
&+ 18685.992116912888np_n \log p_n \log^2 n \\
> &\ 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n + 34.6n^2 \log^5 p_n \log^2 n \\
&+ 952.5np_n \log^3 p_n \log^2 n - 952.5np_n \log^2 p_n \log^2 n \\
&- 952.5np_n \log p_n \log^2 n \\
&+ 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
= &\ 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n + 34.6n^2 \log^5 p_n \log^2 n \\
&+ 952.5p_n \log^2 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1}{\log p_n}) \\
&+ 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
> &\ 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n + 34.6n^2 \log^5 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n \qquad \text{(by Inequality (5))} \\
&+ 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n.
\end{aligned}
$$

We have

$$
\begin{aligned}
&2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n \\
&\quad + 109.897336n^2 \log^4 p_n \log^2 n + 293.007365025776n^2 \log^3 p_n \log^2 n \\
&\quad + 1556.8013836n^2 \log^2 p_n \log^2 n + 637.010656np_n \log^3 p_n \log^2 n \\
&\quad + 2774.076629025776np_n \log^2 p_n \log^2 n \\
&\quad + 18685.992116912888np_n \log p_n \log^2 n \\
&\quad + 23.648668n^2 \log^5 p_n \log^2 n - 58.248668n^2 \log^4 p_n \log^2 n \\
&\quad - 58.248668n^2 \log^3 p_n \log^2 n - 173.328580912888n^2 \log^2 p_n \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n + 34.6n^2 \log^5 p_n \log^2 n \\
&\quad + 952.5p_n^2 \log^2 p_n \log^2 n \\
&\quad + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&\quad + 23.648668n^2 \log^5 p_n \log^2 n - 58.248668n^2 \log^4 p_n \log^2 n \\
&\quad - 58.248668n^2 \log^3 p_n \log^2 n - 173.328580912888n^2 \log^2 p_n \log^2 n.
\end{aligned}
$$

Then

$$
\begin{aligned}
&2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n \\
&\quad + 23.648668n^2 \log^5 p_n \log^2 n + 51.648668n^2 \log^4 p_n \log^2 n \\
&\quad + 234.758697025776n^2 \log^3 p_n \log^2 n \\
&\quad + 1383.472802687112n^2 \log^2 p_n \log^2 n
\end{aligned}
$$

$$
\begin{aligned}
&+ 637.010656np_n \log^3 p_n \log^2 n + 2774.07662902577 6np_n \log^2 p_n \log^2 n \\
&+ 18685.992116912888np_n \log p_n \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n \\
&+ 58.248668n^2 \log^5 p_n \log^2 n \\
&- 58.248668n^2 \log^4 p_n \log^2 n - 58.248668n^2 \log^3 p_n \log^2 n \\
&- 173.328580912888n^2 \log^2 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&= 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n \\
&+ 58.248668n \log^4 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n}) \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n \\
&+ 58.248668np_n \log^4 p_n \log^2 n \qquad \text{(by Inequality (10))} \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n.
\end{aligned}
$$

We have

$$
\begin{aligned}
2n^2 \log^8 &p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n \\
&+ 23.648668n^2 \log^5 p_n \log^2 n + 51.648668n^2 \log^4 p_n \log^2 n \\
&+ 234.758697025776n^2 \log^3 p_n \log^2 n \\
&+ 1383.472802687112n^2 \log^2 p_n \log^2 n \\
&+ 637.010656np_n \log^3 p_n \log^2 n \\
&+ 2774.076629025776np_n \log^2 p_n \log^2 n \\
&+ 18685.992116912888np_n \log p_n \log^2 n \\
&+ 129.551332np_n \log^4 p_n \log^2 n - 187.8np_n \log^3 p_n \log^2 n \\
&- 187.8np_n \log^2 p_n \log^2 n - 558.8300748np_n \log p_n \log^2 n \\
&> 2n^2 \log^7 p_n \log^2 n + 7.697336n^2 \log^6 p_n \log^2 n \\
&+ 58.248668np_n \log^4 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&+ 129.551332np_n \log^4 p_n \log^2 n - 187.8np_n \log^3 p_n \log^2 n \\
&- 187.8np_n \log^2 p_n \log^2 n - 558.8300748np_n \log p_n \log^2 n.
\end{aligned}
$$

Then

$$
\begin{aligned}
2n^2 \log^8 &p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n \\
&+ 23.648668n^2 \log^5 p_n \log^2 n + 51.648668n^2 \log^4 p_n \log^2 n \\
&+ 234.758697025776n^2 \log^3 p_n \log^2 n
\end{aligned}
$$

$$
\begin{aligned}
&+ 1383.472802687112 n^2 \log^2 p_n \log^2 n \\
&+ 129.551332 n p_n \log^4 p_n \log^2 n + 449.210656 n p_n \log^3 p_n \log^2 n \\
&+ 2586.27662902577 6 n p_n \log^2 p_n \log^2 n \\
&+ 18127.162042112888 n p_n \log p_n \log^2 n \\
> & \, 2n^2 \log^7 p_n \log^2 n + 7.697336 n^2 \log^6 p_n \log^2 n \\
&+ 187.8 n p_n \log^4 p_n \log^2 n - 187.8 n p_n \log^3 p_n \log^2 n \\
&- 187.8 n p_n \log^2 p_n \log^2 n - 558.8300748 n p_n \log p_n \log^2 n \\
&+ 952.5 p_n^2 \log^2 p_n \log^2 n + 5755.5 p_n^2 \log p_n \log^2 n + 58185.5 p_n^2 \log^2 n \\
= & \, 2n^2 \log^7 p_n \log^2 n + 7.697336 n^2 \log^6 p_n \log^2 n \\
&+ 187.8 p_n \log^3 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n}) \\
&+ 952.5 p_n^2 \log^2 p_n \log^2 n + 5755.5 p_n^2 \log p_n \log^2 n + 58185.5 p_n^2 \log^2 n \\
> & \, 2n^2 \log^7 p_n \log^2 n + 7.697336 n^2 \log^6 p_n \log^2 n \\
&+ 187.8 p_n^2 \log^3 p_n \log^2 n \qquad \text{(by Inequality (10))} \\
&+ 952.5 p_n^2 \log^2 p_n \log^2 n + 5755.5 p_n^2 \log p_n \log^2 n + 58185.5 p_n^2 \log^2 n.
\end{aligned}
$$

We have

$$
\begin{aligned}
2n^2 \log^8 & p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n \\
&+ 23.648668 n^2 \log^5 p_n \log^2 n + 51.648668 n^2 \log^4 p_n \log^2 n \\
&+ 234.758697025776 n^2 \log^3 p_n \log^2 n \\
&+ 1383.472802687112 n^2 \log^2 p_n \log^2 n \\
&+ 129.551332 n p_n \log^4 p_n \log^2 n + 449.210656 n p_n \log^3 p_n \log^2 n \\
&+ 2586.276629025776 n p_n \log^2 p_n \log^2 n \\
&+ 18127.162042112888 n p_n \log p_n \log^2 n \\
&+ 6n^2 \log^6 p_n \log^2 n - 13.697336 n^2 \log^5 p_n \log^2 n \\
&- 13.697336 n^2 \log^4 p_n \log^2 n - 40.758697025776 n^2 \log^3 p_n \log^2 n \\
&- 178.398678974224 n^2 \log^2 p_n \log^2 n \\
> & \, 2n^2 \log^7 p_n \log^2 n + 7.697336 n^2 \log^6 p_n \log^2 n \\
&+ 187.8 p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5 p_n^2 \log^2 p_n \log^2 n + 5755.5 p_n^2 \log p_n \log^2 n + 58185.5 p_n^2 \log^2 n \\
&+ 6n^2 \log^6 p_n \log^2 n - 13.697336 n^2 \log^5 p_n \log^2 n \\
&- 13.697336 n^2 \log^4 p_n \log^2 n - 40.758697025776 n^2 \log^3 p_n \log^2 n \\
&- 178.398678974224 n^2 \log^2 p_n \log^2 n.
\end{aligned}
$$

Then

$$
2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n + 6n^2 \log^6 p_n \log^2 n
$$

$$
\begin{aligned}
&+ 9.951332n^2\log^5 p_n\log^2 n + 37.951332n^2\log^4 p_n\log^2 n\\
&+ 194n^2\log^3 p_n\log^2 n + 1205.074123712888n^2\log^2 p_n\log^2 n\\
&+ 129.551332np_n\log^4 p_n\log^2 n + 449.210656np_n\log^3 p_n\log^2 n\\
&+ 2586.276629025776np_n\log^2 p_n\log^2 n\\
&+ 18127.162042112888np_n\log p_n\log^2 n\\
> {}&2n^2\log^7 p_n\log^2 n + 13.697336n^2\log^6 p_n\log^2 n\\
&- 13.697336n^2\log^5 p_n\log^2 n - 13.697336n^2\log^4 p_n\log^2 n\\
&- 40.758697025776n^2\log^3 p_n\log^2 n\\
&- 178.398678974224n^2\log^2 p_n\log^2 n + 187.8p_n^2\log^3 p_n\log^2 n\\
&+ 952.5p_n^2\log^2 p_n\log^2 n + 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
= {}&2n^2\log^7 p_n\log^2 n\\
&+ 13.697336n\log^5 p_n\log^2 n\cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n})\\
&+ 187.8p_n^2\log^3 p_n\log^2 n + 952.5p_n^2\log^2 p_n\log^2 n\\
&+ 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
> {}&2n^2\log^7 p_n\log^2 n\\
&+ 13.697336np_n\log^5 p_n\log^2 n \qquad \text{(by Inequality (11))}\\
&+ 187.8p_n^2\log^3 p_n\log^2 n + 952.5p_n^2\log^2 p_n\log^2 n\\
&+ 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n.
\end{aligned}
$$

We have

$$
\begin{aligned}
2n^2\log^8 p_n\log n &- 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n + 6n^2\log^6 p_n\log^2 n\\
&+ 9.951332n^2\log^5 p_n\log^2 n + 37.951332n^2\log^4 p_n\log^2 n\\
&+ 194n^2\log^3 p_n\log^2 n + 1205.074123712888n^2\log^2 p_n\log^2 n\\
&+ 129.551332np_n\log^4 p_n\log^2 n + 449.210656np_n\log^3 p_n\log^2 n\\
&+ 2586.276629025776np_n\log^2 p_n\log^2 n\\
&+ 18127.162042112888np_n\log p_n\log^2 n\\
&+ 31.902664np_n\log^5 p_n\log^2 n - 45.6np_n\log^4 p_n\log^2 n\\
&- 45.6np_n\log^3 p_n\log^2 n - 135.6903696np_n\log^2 p_n\log^2 n\\
&- 593.9096304np_n\log p_n\log^2 n\\
> {}&2n^2\log^7 p_n\log^2 n + 13.697336np_n\log^5 p_n\log^2 n\\
&+ 187.8p_n^2\log^3 p_n\log^2 n + 952.5p_n^2\log^2 p_n\log^2 n\\
&+ 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
&+ 31.902664np_n\log^5 p_n\log^2 n - 45.6np_n\log^4 p_n\log^2 n\\
&- 45.6np_n\log^3 p_n\log^2 n - 135.6903696np_n\log^2 p_n\log^2 n
\end{aligned}
$$

$$- 593.9096304np_n \log p_n \log^2 n.$$

Then

$$\begin{aligned}
2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n + 6n^2 \log^6 p_n \log^2 n \\
+ 9.951332n^2 \log^5 p_n \log^2 n + 37.951332n^2 \log^4 p_n \log^2 n \\
+ 194n^2 \log^3 p_n \log^2 n + 1205.074123712888n^2 \log^2 p_n \log^2 n \\
+ 31.902664np_n \log^5 p_n \log^2 n + 83.951332np_n \log^4 p_n \log^2 n \\
+ 403.610656np_n \log^3 p_n \log^2 n \\
+ 2450.586259425776np_n \log^2 p_n \log^2 n \\
+ 17533.252411712888np_n \log p_n \log^2 n \\
> 2n^2 \log^7 p_n \log^2 n + 45.6np_n \log^5 p_n \log^2 n - 45.6np_n \log^4 p_n \log^2 n \\
- 45.6np_n \log^3 p_n \log^2 n - 135.6903696np_n \log^2 p_n \log^2 n \\
- 593.9096304np_n \log p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
+ 952.5p_n^2 \log^2 p_n \log^2 n \\
+ 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
= 2n^2 \log^7 p_n \log^2 n \\
+ 45.6p_n \log^4 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n}) \\
+ 187.8p_n^2 \log^3 p_n \log^2 n + 952.5p_n^2 \log^2 p_n \log^2 n \\
+ 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
> 2n^2 \log^7 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n \qquad \text{(by Inequality (11))} \\
+ 187.8p_n^2 \log^3 p_n \log^2 n + 952.5p_n^2 \log^2 p_n \log^2 n \\
+ 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n.
\end{aligned}$$

We have

$$\begin{aligned}
2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n + 6n^2 \log^6 p_n \log^2 n \\
+ 9.951332n^2 \log^5 p_n \log^2 n + 37.951332n^2 \log^4 p_n \log^2 n \\
+ 194n^2 \log^3 p_n \log^2 n + 1205.074123712888n^2 \log^2 p_n \log^2 n \\
+ 31.902664np_n \log^5 p_n \log^2 n + 83.951332np_n \log^4 p_n \log^2 n \\
+ 403.610656np_n \log^3 p_n \log^2 n \\
+ 2450.586259425776np_n \log^2 p_n \log^2 n \\
+ 17533.252411712888np_n \log p_n \log^2 n \\
+ 2n^2 \log^7 p_n \log^2 n - 4n^2 \log^6 p_n \log^2 n - 4n^2 \log^5 p_n \log^2 n \\
- 11.902664n^2 \log^4 p_n \log^2 n - 52.097336n^2 \log^3 p_n \log^2 n \\
- 283.805328n^2 \log^2 p_n \log^2 n
\end{aligned}$$

$$\begin{aligned}
&> 2n^2\log^7 p_n \log^2 n + 45.6p_n^2\log^4 p_n\log^2 n\\
&\quad + 187.8p_n^2\log^3 p_n\log^2 n + 952.5p_n^2\log^2 p_n\log^2 n\\
&\quad + 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
&\quad + 2n^2\log^7 p_n\log^2 n - 4n^2\log^6 p_n\log^2 n - 4n^2\log^5 p_n\log^2 n\\
&\quad - 11.902664n^2\log^4 p_n\log^2 n - 52.097336n^2\log^3 p_n\log^2 n\\
&\quad - 283.805328n^2\log^2 p_n\log^2 n.
\end{aligned}$$

Then

$$\begin{aligned}
2n^2\log^8 p_n\log n &- 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n + 2n^2\log^7 p_n\log^2 n\\
&+ 2n^2\log^6 p_n\log^2 n + 5.951332n^2\log^5 p_n\log^2 n\\
&+ 26.048668n^2\log^4 p_n\log^2 n\\
&+ 141.902664n^2\log^3 p_n\log^2 n + 921.268795712888n^2\log^2 p_n\log^2 n\\
&+ 31.902664np_n\log^5 p_n\log^2 n + 83.951332np_n\log^4 p_n\log^2 n\\
&+ 403.610656np_n\log^3 p_n\log^2 n\\
&+ 2450.586259425776np_n\log^2 p_n\log^2 n\\
&+ 17533.252411712888np_n\log p_n\log^2 n\\
> &\ 4n^2\log^7 p_n\log^2 n - 4n^2\log^6 p_n\log^2 n - 4n^2\log^5 p_n\log^2 n\\
&- 11.902664n^2\log^4 p_n\log^2 n - 52.097336n^2\log^3 p_n\log^2 n\\
&- 283.805328n^2\log^2 p_n\log^2 n + 45.6p_n^2\log^4 p_n\log^2 n\\
&+ 187.8p_n^2\log^3 p_n\log^2 n + 952.5p_n^2\log^2 p_n\log^2 n\\
&+ 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
= &\ 4n\log^6 p_n\log^2 n\cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n}\\
&- \tfrac{13.024334}{\log^3 p_n} - \tfrac{70.951332}{\log^4 p_n})\\
&+ 45.6p_n^2\log^4 p_n\log^2 n + 187.8p_n^2\log^3 p_n\log^2 n\\
&+ 952.5p_n^2\log^2 p_n\log^2 n + 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n\\
> &\ 4np_n\log^6 p_n\log^2 n \qquad \text{(by Inequality (12))}\\
&+ 45.6p_n^2\log^4 p_n\log^2 n + 187.8p_n^2\log^3 p_n\log^2 n\\
&+ 952.5p_n^2\log^2 p_n\log^2 n + 5755.5p_n^2\log p_n\log^2 n + 58185.5p_n^2\log^2 n.
\end{aligned}$$

We have

$$\begin{aligned}
2n^2\log^8 p_n\log n &- 2n^2\log^8 p_n\log\log n + 8\Omega(n)n^2\log^8 p_n + 2n^2\log^7 p_n\log^2 n\\
&+ 2n^2\log^6 p_n\log^2 n + 5.951332n^2\log^5 p_n\log^2 n\\
&+ 26.048668n^2\log^4 p_n\log^2 n + 141.902664n^2\log^3 p_n\log^2 n\\
&+ 921.268795712888n^2\log^2 p_n\log^2 n
\end{aligned}$$

$$
\begin{aligned}
&+ 31.902664np_n \log^5 p_n \log^2 n + 83.951332np_n \log^4 p_n \log^2 n \\
&+ 403.610656np_n \log^3 p_n \log^2 n \\
&+ 2450.586259425776np_n \log^2 p_n \log^2 n \\
&+ 17533.252411712888np_n \log p_n \log^2 n \\
&+ 10np_n \log^6 p_n \log^2 n - 14np_n \log^5 p_n \log^2 n - 14np_n \log^4 p_n \log^2 n \\
&- 41.659324np_n \log^3 p_n \log^2 n - 182.340676np_n \log^2 p_n \log^2 n \\
&- 993.318648np_n \log p_n \log^2 n \\
> {}& 4np_n \log^6 p_n \log^2 n \\
&+ 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&+ 10np_n \log^6 p_n \log^2 n - 14np_n \log^5 p_n \log^2 n - 14np_n \log^4 p_n \log^2 n \\
&- 41.659324np_n \log^3 p_n \log^2 n - 182.340676np_n \log^2 p_n \log^2 n \\
&- 993.318648np_n \log p_n \log^2 n.
\end{aligned}
$$

Then

$$
\begin{aligned}
2n^2 \log^8 & p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n) n^2 \log^8 p_n + 2n^2 \log^7 p_n \log^2 n \\
&+ 2n^2 \log^6 p_n \log^2 n + 5.951332n^2 \log^5 p_n \log^2 n \\
&+ 26.048668n^2 \log^4 p_n \log^2 n + 141.902664n^2 \log^3 p_n \log^2 n \\
&+ 921.268795712888n^2 \log^2 p_n \log^2 n \\
&+ 10np_n \log^6 p_n \log^2 n + 17.902664np_n \log^5 p_n \log^2 n \\
&+ 69.951332np_n \log^4 p_n \log^2 n + 361.951332np_n \log^3 p_n \log^2 n \\
&+ 2268.245583425776np_n \log^2 p_n \log^2 n \\
&+ 16539.933763712888np_n \log p_n \log^2 n \\
> {}& 14np_n \log^6 p_n \log^2 n - 14np_n \log^5 p_n \log^2 n - 14np_n \log^4 p_n \log^2 n \\
&- 41.659324np_n \log^3 p_n \log^2 n - 182.340676np_n \log^2 p_n \log^2 n \\
&- 993.318648np_n \log p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n \\
&+ 187.8p_n^2 \log^3 p_n \log^2 n + 952.5p_n^2 \log^2 p_n \log^2 n \\
&+ 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
= {}& 14p_n \log^5 p_n \log^2 n \cdot n(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} \\
&- \tfrac{13.024334}{\log^3 p_n} - \tfrac{70.951332}{\log^4 p_n}) \\
&+ 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
> {}& 14p_n^2 \log^5 p_n \log^2 n \qquad \text{(by Inequality (12))} \\
&+ 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n
\end{aligned}
$$

$$+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n.$$

We have

$$\begin{aligned}
2n^2 \log^8 & p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n + 2n^2 \log^7 p_n \log^2 n \\
& + 2n^2 \log^6 p_n \log^2 n + 5.951332n^2 \log^5 p_n \log^2 n \\
& + 26.048668n^2 \log^4 p_n \log^2 n + 141.902664n^2 \log^3 p_n \log^2 n \\
& + 921.268795712888n^2 \log^2 p_n \log^2 n \\
& + 10np_n \log^6 p_n \log^2 n + 17.902664np_n \log^5 p_n \log^2 n \\
& + 69.951332np_n \log^4 p_n \log^2 n + 361.951332np_n \log^3 p_n \log^2 n \\
& + 2268.245583425776np_n \log^2 p_n \log^2 n \\
& + 16539.933763712888np_n \log p_n \log^2 n \\
& + 2n^2 \log^8 p_n \log^2 n - 2n^2 \log^7 p_n \log^2 n - 2n^2 \log^6 p_n \log^2 n \\
& - 5.951332n^2 \log^5 p_n \log^2 n - 26.048668n^2 \log^4 p_n \log^2 n \\
& - 141.902664n^2 \log^3 p_n \log^2 n - 921.268795712888n^2 \log^2 p_n \log^2 n \\
> & \ 14p_n^2 \log^5 p_n \log^2 n \\
& + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
& + 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
& + 2n^2 \log^8 p_n \log^2 n - 2n^2 \log^7 p_n \log^2 n - 2n^2 \log^6 p_n \log^2 n \\
& - 5.951332n^2 \log^5 p_n \log^2 n - 26.048668n^2 \log^4 p_n \log^2 n \\
& - 141.902664n^2 \log^3 p_n \log^2 n - 921.268795712888n^2 \log^2 p_n \log^2 n.
\end{aligned}$$

Then

$$\begin{aligned}
2n^2 \log^8 & p_n \log^2 n + 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n \\
& + 10np_n \log^6 p_n \log^2 n + 17.902664np_n \log^5 p_n \log^2 n \\
& + 69.951332np_n \log^4 p_n \log^2 n + 361.951332np_n \log^3 p_n \log^2 n \\
& + 2268.245583425776np_n \log^2 p_n \log^2 n \\
& + 16539.933763712888np_n \log p_n \log^2 n \\
> & \ 2n^2 \log^8 p_n \log^2 n - 2n^2 \log^7 p_n \log^2 n - 2n^2 \log^6 p_n \log^2 n \\
& - 5.951332n^2 \log^5 p_n \log^2 n - 26.048668n^2 \log^4 p_n \log^2 n \\
& - 141.902664n^2 \log^3 p_n \log^2 n - 921.268795712888n^2 \log^2 p_n \log^2 n \\
& + 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
& + 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
= & \ 2n \log^7 p_n \log^2 n \cdot n \cdot \left(\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n}\right. \\
& \left. - \tfrac{70.951332}{\log^4 p_n} - \tfrac{460.634397856444}{\log^5 p_n}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
> {}& 2np_n \log^7 p_n \log^2 n \qquad \text{(by Inequality (13))} \\
&+ 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n.
\end{aligned}
$$

We have

$$
\begin{aligned}
&2n^2 \log^8 p_n \log^2 n + 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n \\
&\quad + 10np_n \log^6 p_n \log^2 n + 17.902664np_n \log^5 p_n \log^2 n \\
&\quad + 69.951332np_n \log^4 p_n \log^2 n + 361.951332np_n \log^3 p_n \log^2 n \\
&\quad + 2268.245583425776np_n \log^2 p_n \log^2 n \\
&\quad + 16539.933763712888np_n \log p_n \log^2 n + 4np_n \log^7 p_n \log^2 n \\
&\quad - 6np_n \log^6 p_n \log^2 n - 6np_n \log^5 p_n \log^2 n \\
&\quad - 17.853996np_n \log^4 p_n \log^2 n - 78.146004np_n \log^3 p_n \log^2 n \\
&\quad - 425.707992np_n \log^2 p_n \log^2 n - 2763.806387138664np_n \log p_n \log^2 n \\
&> 2np_n \log^7 p_n \log^2 n \\
&\quad + 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&\quad + 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&\quad + 4np_n \log^7 p_n \log^2 n - 6np_n \log^6 p_n \log^2 n - 6np_n \log^5 p_n \log^2 n \\
&\quad - 17.853996np_n \log^4 p_n \log^2 n - 78.146004np_n \log^3 p_n \log^2 n \\
&\quad - 425.707992np_n \log^2 p_n \log^2 n - 2763.806387138664np_n \log p_n \log^2 n.
\end{aligned}
$$

Then

$$
\begin{aligned}
&2n^2 \log^8 p_n \log^2 n + 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n \\
&\quad + 4np_n \log^7 p_n \log^2 n + 4np_n \log^6 p_n \log^2 n \\
&\quad + 11.902664np_n \log^5 p_n \log^2 n + 52.097336np_n \log^4 p_n \log^2 n \\
&\quad + 283.805328np_n \log^3 p_n \log^2 n + 1842.537591425776np_n \log^2 p_n \log^2 n \\
&\quad + 13776.127376574224np_n \log p_n \log^2 n \\
&> 6np_n \log^7 p_n \log^2 n - 6np_n \log^6 p_n \log^2 n - 6np_n \log^5 p_n \log^2 n \\
&\quad - 17.853996np_n \log^4 p_n \log^2 n - 78.146004np_n \log^3 p_n \log^2 n \\
&\quad - 425.707992np_n \log^2 p_n \log^2 n - 2763.806387138664np_n \log p_n \log^2 n \\
&\quad + 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&\quad + 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&= 6p_n \log^6 p_n \log^2 n \cdot n \cdot (\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n}
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{70.951332}{\log^4 p_n} - \tfrac{460.634397856444}{\log^5 p_n}) \\
&+ 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
> &\ 6p_n^2 \log^6 p_n \log^2 n \qquad \text{(by Inequality (13))} \\
&+ 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n.
\end{aligned}
$$

We have

$$
\begin{aligned}
2n^2 \log^8 &p_n \log^2 n + 2n^2 \log^8 p_n \log n - 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n \\
&+ 4np_n \log^7 p_n \log^2 n + 4np_n \log^6 p_n \log^2 n \\
&+ 11.902664np_n \log^5 p_n \log^2 n + 52.097336np_n \log^4 p_n \log^2 n \\
&+ 283.805328np_n \log^3 p_n \log^2 n + 1842.537591425776np_n \log^2 p_n \log^2 n \\
&+ 13776.127376574224np_n \log p_n \log^2 n \\
&+ 4np_n \log^8 p_n \log^2 n - 4np_n \log^7 p_n \log^2 n - 4np_n \log^6 p_n \log^2 n \\
&- 11.902664np_n \log^5 p_n \log^2 n - 52.097336np_n \log^4 p_n \log^2 n \\
&- 283.805328np_n \log^3 p_n \log^2 n - 1842.537591425776np_n \log^2 p_n \log^2 n \\
&- 13776.127376574224np_n \log p_n \log^2 n \\
> &\ 6p_n^2 \log^6 p_n \log^2 n \\
&+ 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n \\
&+ 4np_n \log^8 p_n \log^2 n - 4np_n \log^7 p_n \log^2 n - 4np_n \log^6 p_n \log^2 n \\
&- 11.902664np_n \log^5 p_n \log^2 n - 52.097336np_n \log^4 p_n \log^2 n \\
&- 283.805328np_n \log^3 p_n \log^2 n - 1842.537591425776np_n \log^2 p_n \log^2 n \\
&- 13776.127376574224np_n \log p_n \log^2 n.
\end{aligned}
$$

Then

$$
\begin{aligned}
4np_n \log^8 p_n &\log^2 n + 2n^2 \log^8 p_n \log^2 n + 2n^2 \log^8 p_n \log n \\
&- 2n^2 \log^8 p_n \log\log n + 8\Omega(n)n^2 \log^8 p_n \\
> &\ 4np_n \log^8 p_n \log^2 n - 4np_n \log^7 p_n \log^2 n - 4np_n \log^6 p_n \log^2 n \\
&- 11.902664np_n \log^5 p_n \log^2 n - 52.097336np_n \log^4 p_n \log^2 n \\
&- 283.805328np_n \log^3 p_n \log^2 n - 1842.537591425776np_n \log^2 p_n \log^2 n \\
&- 13776.127376574224np_n \log p_n \log^2 n + 6p_n^2 \log^6 p_n \log^2 n \\
&+ 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n + 187.8p_n^2 \log^3 p_n \log^2 n \\
&+ 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n + 58185.5p_n^2 \log^2 n
\end{aligned}
$$

$$= 4p_n \log^7 p_n \log^2 n \cdot n \cdot (\log p_n - 1 - \tfrac{1}{\log p_n} - \tfrac{2.975666}{\log^2 p_n} - \tfrac{13.024334}{\log^3 p_n}$$
$$- \tfrac{70.951332}{\log^4 p_n} - \tfrac{460.634397856444}{\log^5 p_n} - \tfrac{3444.031844143556}{\log^6 p_n})$$
$$+ 6p_n^2 \log^6 p_n \log^2 n + 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n$$
$$+ 187.8p_n^2 \log^3 p_n \log^2 n + 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n$$
$$+ 58185.5p_n^2 \log^2 n$$
$$> 4p_n^2 \log^7 p_n \log^2 n \qquad \text{(by Inequality (14))}$$
$$+ 6p_n^2 \log^6 p_n \log^2 n + 14p_n^2 \log^5 p_n \log^2 n + 45.6p_n^2 \log^4 p_n \log^2 n$$
$$+ 187.8p_n^2 \log^3 p_n \log^2 n + 952.5p_n^2 \log^2 p_n \log^2 n + 5755.5p_n^2 \log p_n \log^2 n$$
$$+ 58185.5p_n^2 \log^2 n.$$

We divide the last inequality by $8 \log^8 p_n \log^2 n$ to obtain Inequality (56). Then, by Inequality (19), Proposition 4 is proved. □