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A NOTE ON DELETIONS FROM COMPOUND SEQUENCES THAT LEAVE THE FROBENIUS NUMBER INVARIANT

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Abstract

For a numerical semigroup S, we denote by $\mathrm{F}(S)$ the Frobenius number of S. For numerical semigroups generated by compound sequences C, we determine subsequences $C^{\star}$ of C for which $\mathrm{F}(\langle C^{\star}\rangle) = \mathrm{F}(\langle C\rangle)$.

1. Introduction

A *numerical semigroup* S is a submonoid of $\mathbb{Z}_{\geq 0}$ that has a finite complement $\mathrm{G}(S)$ in $\mathbb{Z}_{\geq 0}$. Given a set $A = \{a_1, \ldots, a_n\}$ of positive integers with $\gcd(a_1, \ldots, a_n) = 1$, let

$$\langle A\rangle = \{a_1x_1 + \cdots + a_kx_n : x_i \geq 0\}$$

denote the numerical semigroup generated by A. Every numerical semigroup S is of the form $\langle A\rangle$ for some finite set A. Moreover, every S has a unique minimal generating set, the cardinality of which is called the *embedding dimension* of the semigroup and denoted by $\mathbf{e}(S)$.

The *Frobenius number* of S is defined as the largest element in $\mathrm{G}(S)$:

$$\mathrm{F}(S) = \max \mathrm{G}(S).$$

A very useful tool in the study of numerical semigroups is the determination of *Apéry sets* of the semigroup. Given a numerical semigroup S generated by A and $a \in S \setminus \{0\}$, the Apéry set $\mathrm{Ap}(S, a)$ of S corresponding to a is given by

$$\mathrm{Ap}\big(S, a\big) = \{\mathbf{m}_x : 0 \leq x \leq a - 1\},$$

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where $\mathbf{m}_x$ denotes the least positive integer in S congruent to x modulo a. The following result, due to Brauer and Shockley [1], shows how $\mathrm{F}(S)$ can be determined from the Apéry set of S corresponding to any $a \in S \setminus \{0\}$.

Lemma 1 ([1])**.** *Let S be the numerical semigroup generated by A, and let $a \in A$. Then*

$$F(S) = \left(\max_{1 \le x \le a-1} \boldsymbol{m}_x \right) - a.$$

2. Main Results

Compound sequences were introduced and studied by Kiers, O'Neill, and Ponomarenko [2]. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two sequences of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for $i \ge j$. The *compound sequence* formed from these two sequences is the sequence $\{c_0, c_1, c_2, \ldots, c_n\}$, where

$$\begin{aligned} c_0 &= a_1 a_2 a_3 \cdots a_n, \\ c_1 &= b_1 a_2 a_3 \cdots a_n, \\ c_2 &= b_1 b_2 a_3 \cdots a_n, \\ \vdots &= \vdots \\ c_n &= b_1 b_2 b_3 \cdots b_n. \end{aligned} \tag{1}$$

Note that $\gcd(c_0, \ldots, c_n) = 1$. Two important special cases are:

(i) The compound sequence for $a_1 = \ldots = a_n = a$ and $b_1 = \ldots = b_n = b$, $\gcd(a, b) = 1$ is the geometric sequence

$$\{a^n, a^{n-1}b, a^{n-2}b^2, \ldots, b^n\}.$$

This was studied by Ong and Ponomarenko [3], and independently by Tripathi [5].

(ii) For pairwise coprime positive integers $a_1, \ldots, a_n$, the compound sequence for $\{a_2, a_3, \ldots, a_n\}$ and $\{a_1, a_2, \ldots, a_{n-1}\}$ is the sequence

$$\left\{ \frac{P}{a_1}, \frac{P}{a_2}, \ldots, \frac{P}{a_n} \right\},$$

where $P = a_1 a_2 \cdots a_n$. This was studied by Tripathi [4].

Kiers et. al. determine an Apéry set and the Frobenius number for such sequences; also see [6].

Theorem 2 ([2], Theorem 15). *Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ be two sequences of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for $i \geq j$. If $C = \{c_0, \dots, c_n\}$ is the compound sequence of $\{a_i\}$ and $\{b_i\}$, then the Apéry set for $S = \langle C \rangle$ corresponding to c_0 is given by*

$$Ap\big(S, c_0\big) = \left\{ \sum_{i=1}^{n} c_i x_i : 0 \leq x_i \leq a_i - 1, i = 1, \dots, n \right\}.$$

In particular,

$$F(S) = \sum_{i=1}^{n} (a_i - 1)c_i - c_0. \tag{2}$$

Let C denote the sequence compounded by two sequences, and let $C^\star$ denote the subsequence obtained from C by deletion of one element. Note that $C^\star$ need not generate a numerical semigroup. Proposition 3 provides a characterization of those subsequences $C^\star$ for which $\langle C^\star \rangle$ is a numerical semigroup.

Proposition 3. *Let $C = \{c_0, \dots, c_n\}$ be the sequence compounded by $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$. For a fixed $j \in \{0, 1, 2, \dots, n\}$, let $C^\star = \{c_0, \dots, c_{j-1}, c_{j+1}, \dots, c_n\}$. Then $\langle C^\star \rangle$ is a numerical semigroup if and only if either*

$$0 < j < n \text{ and } \gcd(a_j, b_{j+1}) = 1, \quad \text{or} \quad j = n \text{ and } a_j = 1.$$

Proof. If $j = 0$, then b_1 divides each element of $C^\star$. Since $b_1 > a_1$, $\gcd(C^\star) > 1$. If $j = n$, then a_n divides each element of $C^\star$. If $a_n > 1$, then $\gcd(C^\star) > 1$. If $a_n = 1$, then $C^\star$ is compounded by $\{a_1, \dots, a_{n-1}\}$ and $\{b_1, \dots, b_{n-1}\}$.

Suppose $0 < j < n$. Observe that $C^\star$ is the sequence compounded by $\{a'_1, \dots, a'_{n-1}\}$ and $\{b'_1, \dots, b'_{n-1}\}$, where

$$a'_i = \begin{cases} a_i & \text{if } i < j, \\ a_j a_{j+1} & \text{if } i = j, \\ a_{i+1} & \text{if } i > j, \end{cases} \quad \text{and} \quad b'_i = \begin{cases} b_i & \text{if } i < j, \\ b_j b_{j+1} & \text{if } i = j, \\ b_{i+1} & \text{if } i > j, \end{cases}$$

provided $\gcd(a'_r, b'_s) = 1$ for $r \geq s$.

If neither r nor s equals j, then $\gcd(a'_r, b'_s) = 1$ since C is a compound sequence. Thus, we need to only consider the case where at least one of r, s equals j. If $r = j$ and $s \neq j$, then $\gcd(a'_r, b'_s) = \gcd(a_j a_{j+1}, b_{s+1}) = 1$ since $\gcd(a_j, b_{s+1}) = \gcd(a_{j+1}, b_{s+1}) = 1$. A similar argument applies to the case where $s = j$ and $r \neq j$. If $r = s = j$, then $\gcd(a'_r, b'_s) = \gcd(a_j a_{j+1}, b_j b_{j+1}) = \gcd(a_j, b_{j+1})$.

We have shown that $\gcd(C^\star) = 1$ if and only if the two conditions in the statement of the proposition are met. Therefore, the same two conditions are necessary and sufficient for $\langle C^\star \rangle$ to be a numerical semigroup. □

In most cases, the deletion of a single element from the compound sequence of two sequences turns out to be the compound sequence of two other sequences, making the problem of invariance under a single-element deletion easy to resolve in those cases.

Theorem 4. *Let* $C = \{c_0, \dots, c_n\}$ *be the sequence compounded by* $\{a_1, \dots, a_n\}$ *and* $\{b_1, \dots, b_n\}$*. For a fixed* $j \in \{0, 1, 2, \dots, n\}$*, let* $C^\star = \{c_0, \dots, c_{j-1}, c_{j+1}, \dots, c_n\}$ *be such that* $\gcd(C^\star) = 1$*. Then* $F(\langle C^\star \rangle) = F(\langle C \rangle)$ *if and only if* $a_j = 1$*.*

Proof. We have

$$\mathrm{Ap}\big(\langle C^\star\rangle, c_0\big) = \left\{ \sum_{i=1}^{n-1} c_i' x_i : 0 \le x_i \le a_i' - 1, i = 1, \dots, n-1 \right\},$$

where

$$c_i' = \begin{cases} c_i & \text{if } i < j, \\ c_{i+1} & \text{if } i \ge j, \end{cases}$$

by Theorem 2. Hence

$$\begin{aligned}
\mathrm{F}(\langle C^\star \rangle) &= \sum_{i=1}^{n-1} (a_i' - 1) c_i' - c_0 \\
&= \sum_{i=1}^{j-1} (a_i - 1) c_i + (a_j a_{j+1} - 1) c_{j+1} + \sum_{i=j+1}^{n-1} (a_{i+1} - 1) c_{i+1} - c_0 \\
&= \sum_{i=1}^{j-1} (a_i - 1) c_i + (a_j a_{j+1} - 1) c_{j+1} + \sum_{i=j+2}^{n} (a_i - 1) c_i - c_0 \\
&= \sum_{i=1}^{n} (a_i - 1) c_i + \big((a_j a_{j+1} - 1) c_{j+1} - (a_j - 1) c_j - (a_{j+1} - 1) c_{j+1}\big) - c_0 \\
&= \sum_{i=1}^{n} (a_i - 1) c_i + (a_j - 1)(a_{j+1} c_{j+1} - c_j) - c_0 \\
&= \mathrm{F}(\langle C \rangle) + (a_j - 1)(a_{j+1} c_{j+1} - c_j).
\end{aligned}$$

Since $a_{j+1} c_{j+1} \ge c_{j+1} > c_j$, we have $\mathrm{F}(\langle C^\star \rangle) = \mathrm{F}(\langle C \rangle)$ if and only if $a_j = 1$. □

Theorem 5 gives the embedding dimension of sequences compounded by two sequences. We use this and Theorem 4 in Theorem 6 to show that the minimal generating set of $\langle C \rangle$ is the subset of C of least cardinality among those with the same Frobenius number.

Theorem 5. *Let $C = \{c_0, \ldots, c_n\}$ be the sequence compounded by $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$. Then the embedding dimension*

$$\boldsymbol{e}(C) = n - k + 1,$$

where $k = |\{i : a_i = 1\}|$.

Proof. Let $\{a_{i_1}, \ldots, a_{i_{n-k}}\}$ be the subsequence of a_i's obtained by removing each $a_i = 1$. Define the subsequence $\{\overline{b}_{i_1}, \ldots, \overline{b}_{i_{n-k}}\}$ by

$$\overline{b}_{i_1} = b_1 \cdots b_{i_1}, \quad \overline{b}_{i_2} = b_{i_1+1} \cdots b_{i_2}, \quad \ldots \quad \overline{b}_{i_{n-k}} = b_{i_{n-k-1}+1} \cdots b_{i_{n-k}}.$$

We note that the sequence $\overline{C}$ compounded by $\{a_{i_1}, \ldots, a_{i_{n-k}}\}$ and $\{\overline{b}_{i_1}, \ldots, \overline{b}_{i_{n-k}}\}$ is the subsequence $\{c_0, c_{i_1}, \ldots, c_{i_{n-k}}\}$ of C. Moreover, $\mathrm{Ap}\big(\langle C\rangle, c_0\big) = \mathrm{Ap}\big(\langle \overline{C}\rangle, c_0\big)$ by Theorem 2, so that $\overline{C}$ is a generating set for $\langle C\rangle$. Further, the deletion of any element of $\overline{C}$ from $\overline{C}$ results in a change in the Frobenius number by Theorem 4, thereby proving that $\overline{C}$ is the minimal generating set for $\langle C\rangle$. □

Theorem 6. *Let $C = \{c_0, \ldots, c_n\}$ be the sequence compounded by $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$. Then the subset of C of least cardinality among those with the same Frobenius number is the minimal generating set for C.*

Proof. Let $\overline{C}$ denote the minimal generating set for $\langle C\rangle$, as in Theorem 5. Then $\langle \overline{C}\rangle$ and $\langle C\rangle$ have the same Frobenius number, whereas the deletion of any element from $\overline{C}$ results in a numerical semigroup with a larger Frobenius number by Theorem 4.

Now suppose C' is a subset of C of smaller cardinality than $\overline{C}$. Then there exists $c_j \in \overline{C} \setminus C'$, and the Frobenius number of $\langle C \setminus \{c_j\}\rangle$ is larger than the Frobenius number of $\langle C\rangle$. Therefore, $\langle C'\rangle$ also has a larger Frobenius number than that of $\langle C\rangle$.

This completes the proof. □

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