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DIAGONAL SEQUENCES OF INTEGER PARTITIONS

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Abstract

Let $\mathcal{P}(n)$ be the set of partitions of the positive integer n. For $\alpha = (\alpha_1, ..., \alpha_t) \in \mathcal{P}(n)$ define the diagonal sequence $\delta(\alpha) = (d_k(\alpha))_{k\geq 1}$ via $d_k(\alpha) = \big|\{i \,|\, 1 \leq i \leq k \text{ and } \alpha_i + i - 1 \geq k\}\big|$. We show that the set of all partitions in $\mathcal{P}(n)$ with the same diagonal sequence is a partially ordered set under majorization with unique maximal and minimal elements and we give an explicit formula for the number of partitions with the same diagonal sequence.

1. Introduction

Let $\mathcal{P}(n)$ be the set of partitions of the natural number n, that is, for $\alpha = (\alpha_1, ..., \alpha_l) \in \mathcal{P}(n)$ we assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l \geq 1$, $\alpha_i \in \mathbb{N}$, and $\sum_{i=1}^{l} \alpha_i = n$. Let $\mathcal{P}(n, k)$ be the set of partitions of n with exactly k non-zero parts and let $\mathcal{P}(n, k)^*$ be the set of conjugates of $\mathcal{P}(n, k)$, that is, $\mathcal{P}(n, k)^*$ is the set of all partitions of n that have largest part equal to k.

Definition 1. For a partition $\alpha = (\alpha_1, \cdots, \alpha_l) \in \mathcal{P}(n)$, define its *diagonal sequence* $\delta(\alpha) = (d_k)_{k\geq 1}$ via

$$d_k = d_k(\alpha) = \big|\{i \,|\, 1 \leq i \leq k \text{ and } \alpha_i + i - 1 \geq k\}\big|.$$

We note that $d_k(\alpha)$ is the number of boxes in the kth downward diagonal, from right to left, of the Young diagram of the partition α. Since only finitely many $d_k(\alpha)$ are positive, we may omit writing trailing zeros for $\delta(\alpha)$. Using Young diagrams, we can visualize diagonal sequences in two ways. For example, the partition $\alpha = (7, 7, 4, 1, 1, 1) \in \mathcal{P}(21)$ has diagonal sequence $\delta(\alpha) = (1, 2, 3, 4, 4, 4, 2, 1)$. In the

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Young diagram on the left in Figure 1 the number in a box indicates the position of that box on the respective diagonal. In the Young diagram on the right in Figure 1 all the boxes on the same diagonal are marked with the same letter.

α

1	1	1	1	1	1	1
2	2	2	2	2	2	1
3	3	3	3			
4						
4						
4						

α

a	b	c	d	e	f	g
b	c	d	e	f	g	h
c	d	e	f			
d						
e						
f						

Figure 1: Young diagrams of the partition $\alpha = (7, 7, 4, 1, 1, 1)$

Definition 2. *Let* $\Delta(n) = \{\delta(\alpha) \mid \alpha \in \mathcal{P}(n)\}$.

This defines δ as a map from $\mathcal{P}(n)$ to $\Delta(n)$ with $\alpha \to \delta(\alpha)$. The map δ is surjective by definition and is not injective, as is easy to see. Different partitions may have the same diagonal sequence. If we move a box of the Young diagram on a diagonal in such a way that we end up with another Young diagram, then the two partitions will have the same diagonal sequence. For example, the partition $\beta = (8, 6, 4, 3) \in \mathcal{P}(21)$ in Figure 2 has the same diagonal sequence as the partition $\alpha = (7, 7, 4, 1, 1, 1)$ in Figure 1. The h-box in row 2 of α moved to the last position in row 1 of β, while the e-box in row 5 of α moved to the second position in row 4 of β, and the f-box in row 6 of α moved to the fourth box in row 4 of β. The reader may have noticed that there are more diagonal moves that yield different partitions with the same diagonal sequence. Theorem 1 in Section 4 gives an explicit expression for the number of partitions with the same diagonal sequence. The transition from α to β moved boxes up along diagonals. In fact, the boxes of the Young diagram of β are as high up along their respective diagonals as possible. We explore this idea in Section 3. We can define an equivalence relation in $\mathcal{P}(n)$ via $\alpha \sim \beta$ if and only

β

1	1	1	1	1	1	1	1
2	2	2	2	2	2		
3	3	3	3				
4	4	4					

β

a	b	c	d	e	f	g	h
b	c	d	e	f	g		
c	d	e	f				
d	e	f					

Figure 2: Young diagrams of the partition $\beta = (8, 6, 4, 3)$

if $\delta(\alpha) = \delta(\beta)$. The equivalence class of a partition $\alpha \in \mathcal{P}(n)$ is characterized by the invariant $\delta(\alpha) = d$. For $d \in \Delta(n)$, define $[\,d\,] = \delta^{-1}(d) \subseteq \mathcal{P}(n)$. In particular, the partition $\alpha \in \mathcal{P}(n)$ and its conjugate α^* have the same diagonal sequence and consequently belong to the same equivalence class, i.e., $\alpha \sim \alpha^*$ and $\delta(\alpha) = \delta(\alpha^*)$. The conjugate partition is obtained by reflecting the Young diagram about the downward diagonal through the first box, which maps each downward diagonal onto itself.

Diagonal sequences were used in [1] and in [5] to determine graphs and bipartite graphs with maximal sums of squares of the degrees.

In the next section, we prove properties of $\delta(\alpha)$ in preparation for the main results of Sections 3 and 4. In Section 3 we show that $[\,d\,]$ is a partially ordered set under majorization with unique maximal and unique minimal element. In Section 4 we give an explicit formula for the size of $[\,d\,]$. In Section 5 we list the 36 partitions in the equivalence class of $\alpha = (7, 7, 4, 1, 1, 1)$, that is, we list all $\alpha \in \mathcal{P}(21)$ with $\delta(\alpha) = (1, 2, 3, 4, 4, 4, 2, 1)$.

2. Properties of $\delta(\alpha)$

We first prove that the sequence $\delta(\alpha)$ increases from 1 to a positive integer q in increments of 1 and continues in a non-increasing order.

Lemma 1. *Let $\alpha = (\alpha_1, \cdots, \alpha_l) \in \mathcal{P}(n)$ with $\delta(\alpha) = (d_k)_{k\geq 1}$.*

1. *For all $1 \leq j \leq k+1$, $d_{k+1} - d_k \leq 1$ with equality if and only if $d_j = j$.*
2. *If $d_k \geq d_{k+1}$, then $d_{k+1} \geq d_{k+2}$.*

Proof. By definition, $d_{k+1} \leq d_k + 1$. If for some $1 \leq j \leq k$ we have $\alpha_j + j - 1 < k$, then

$$\alpha_{j+1} + (j+1) - 1 = \alpha_{j+1} + j \leq \alpha_j + j < k + 1.$$

Therefore, if $d_j < j$, then $d_j \geq d_{j+1}$. □

We use the notation $x^{(t)}$ to indicate that x is repeated t times in a sequence.

Corollary 1. *For all partitions $\alpha \in \mathcal{P}(n)$, there exist unique integers $q > 0$ and $s_1, \cdots, s_q \geq 0$ such that*

$$\delta(\alpha) = \left(1, 2, \cdots, q-1, q, q^{(s_q)}, (q-1)^{(s_{q-1})}, \cdots, 1^{(s_1)}\right).$$

with

$$\frac{q(q+1)}{2} + \sum_{k=1}^{q} k\, s_k = n.$$

Let $d = (d_1, d_2, \cdots, d_L) \in \Delta(n)$. We characterize two special elements of $[\,d\,]$. The notations $\overline{\alpha}$ and $\underline{\alpha}$ will become clear in Section 3.

Proposition 1. *Let* $s_1, \cdots, s_q \geq 0$ *be integers such that* $\frac{q(q+1)}{2} + \sum_{k=1}^{q} k\, s_k = n$. *Let* $\overline{\alpha}_i = q - i + 1 + \sum_{k=i}^{q} s_k$ *for* $1 \leq i \leq q$. *Then*

1. $\overline{\alpha} = (\overline{\alpha}_1, \cdots, \overline{\alpha}_q) \in \mathcal{P}(n)$,
2. $\overline{\alpha}_1 > \overline{\alpha}_2 > \cdots > \overline{\alpha}_q$,
3. $\delta(\overline{\alpha}) = \big(1, 2, \cdots, q-1, q, q^{(s_q)}, (q-1)^{(s_{q-1})}, \cdots, 1^{(s_1)}\big)$,
4. $s_i = \overline{\alpha}_i - \overline{\alpha}_{i+1} - 1$, $1 \leq i \leq q-1$, *and* $s_q = \overline{\alpha}_q - 1$.

Proof. By definition $\overline{\alpha}_1 > \overline{\alpha}_2 > \cdots > \overline{\alpha}_q$ and since

$$\sum_{i=1}^{q} \overline{\alpha}_i = \sum_{i=1}^{q} (q-i) + \sum_{i=1}^{q} \sum_{k=i}^{q} s_k = \frac{q(q+1)}{2} + \sum_{k=1}^{q} k\, s_k = n,$$

$\overline{\alpha} = (\overline{\alpha}_1, \cdots, \overline{\alpha}_q) \in \mathcal{P}(n)$.

Next, we will show that

$$\delta(\overline{\alpha}) = \left(1, 2, \cdots, q-1, q, q^{(s_q)}, (q-1)^{(s_{q-1})}, \cdots, 1^{(s_1)}\right).$$

If $1 \leq k \leq q$, then $\overline{\alpha}_i + i - 1 = q + \sum_{j=i}^{q} s_j \geq q \geq k$ for $1 \leq i \leq k$. Hence, $d_k = k$ for $1 \leq k \leq q$.

If $q < k \leq q + s_q$, then

$$\overline{\alpha}_i + q - 1 = q + \sum_{j=i}^{q} s_j \geq q + s_q \geq k \text{ for } 1 \leq i \leq q.$$

Hence, $d_k = q$ for $q < k \leq q + s_q$.

Similarly, if $q + \sum_{j=i}^{q} s_j < k \leq q + \sum_{j=i-1}^{q} s_j$, $i \geq 2$, then

$$\overline{\alpha}_j + j - 1 = q + \sum_{l=j}^{q} s_l < k \text{ for } j \geq i \text{ and}$$

$$\overline{\alpha}_j + j - 1 = q + \sum_{l=j}^{q} s_l \geq k \text{ for } j < i.$$

Hence, $d_k = i - 1$ for $q + \sum_{j=i}^{q} s_j < k \leq q + \sum_{j=i-1}^{q} s_j$.

Lastly, we show how to compute the values of s_i, $1 \leq i \leq q$, given $\overline{\alpha}$. Let $s = (s_1, s_2, , \cdots, s_q)$, $v = (q, q-1, \cdots, 1)$, and

$$T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

It follows that $\overline{\alpha} = v + Ts$ and $s = T^{-1}(\overline{\alpha} - v)$. Since

$$T^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

we get $s_i = \overline{\alpha}_i - \overline{\alpha}_{i+1} - 1$ for $1 \le i < q$ and $s_q = \overline{\alpha}_q - 1$. □

There are some consequences of Proposition 1 that are worth pointing out.

Corollary 2. *For all positive integers n, $|\Delta(n)|$ is equal to the number of partitions of n with distinct parts which is equal to the number of partitions of n with odd parts.*

Proof. By Proposition 1, for each diagonal sequence $\delta \in \Delta(n)$ there is a unique partition $\overline{\alpha}$ with $\overline{\alpha}_1 > \overline{\alpha}_2 > \cdots > \overline{\alpha}_q$. The second part follows from a well-known result that the number of partitions in $\mathcal{P}(n)$ with distinct parts is equal to the number of partitions in $\mathcal{P}(n)$ with odd parts. □

Let $\underline{\alpha} = \overline{\alpha}^*$ denote the conjugate partition of $\overline{\alpha}$.

Proposition 2. *If $\alpha \in \mathcal{P}(n)$ with*

$$\delta(\alpha) = \left(1, 2, \cdots, q-1, q, q^{(s_q)}, (q-1)^{(s_{q-1})}, \cdots, 1^{(s_1)}\right),$$

then

$$\underline{\alpha} = (q^{(s_q+1)}, (q-1)^{(s_{q-1}+1)}, \cdots, 1^{(s_1+1)}).$$

Proof. Let $\overline{\alpha}^* = (\overline{\alpha}^*_1, \overline{\alpha}^*_2, \cdots, \overline{\alpha}^*_t)$. Since $\overline{\alpha}_1 > \overline{\alpha}_2 > \cdots > \overline{\alpha}_q$, $t = \overline{\alpha}_1$ and for $\overline{\alpha}_{i+1} < k \le \overline{\alpha}_i$ we have $\overline{\alpha}^*_k = \underline{\alpha}_k = i$. By Proposition 1, $\overline{\alpha}_i - \overline{\alpha}_{i+1} = s_i + 1$ and the result follows. □

Figure 3 shows the Young diagrams of the partitions α, α^*, $\overline{\alpha}$ and $\underline{\alpha}$.

Corollary 3. *The multiset of integers of $\delta(\overline{\alpha}) = \delta(\underline{\alpha})$ is equal to the multiset of integers of $\underline{\alpha}$.*

Corollary 4. *Assume $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_t) \in \mathcal{P}(n)$. If $\alpha \ne \overline{\alpha}$, then $\alpha_i = \alpha_{i+1}$ for some $1 \le i \le t$. Equivalently, if $\alpha \ne \underline{\alpha}$, then $\alpha_i - \alpha_{i+1} > 1$ for some $1 \le i \le t$.*

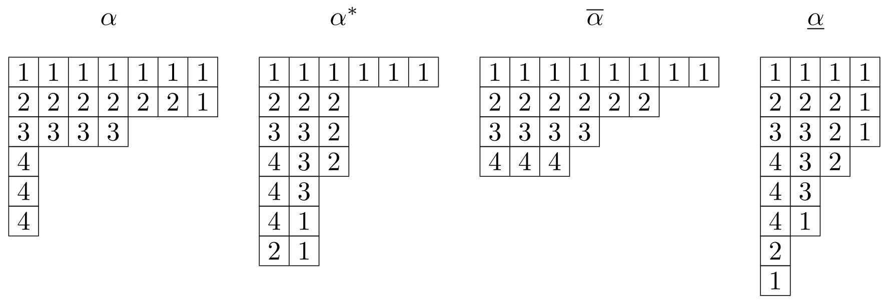

Figure 3: The Young diagrams of α, α^*, $\overline{\alpha}$ and $\underline{\alpha}$

In the above example, $\alpha = (7, 7, 4, 1, 1, 1) \in \mathcal{P}(21)$ has diagonal sequence

$$d(\alpha) = (1, 2, 3, 4, 4, 4, 2, 1) = (1, 2, 3, 4, 4^{(2)}, 3^{(0)}, 2^{(1)}, 1^{(1)}).$$

Therefore, $q = 4$, $s_4 = 2, s_3 = 0, s_2 = 1, s_1 = 1$. Proposition 1 implies that $\overline{\alpha} = (8, 6, 4, 3) \in \mathcal{P}(21)$ while Proposition 2 implies that $\underline{\alpha} = (4, 4, 4, 3, 2, 2, 1, 1) \in \mathcal{P}(21)$. The partition $\overline{\alpha}$ is obtained from the partition α by moving all boxes as far up along their respective diagonals as possible. The partition $\underline{\alpha} = (4, 4, 4, 3, 2, 2, 1, 1)$ is obtained from the partition α by moving all boxes as far down along their respective diagonals as possible.

Given a diagonal sequence $d \in \Delta(n)$ and $\alpha = (\alpha_1, \cdots, \alpha_t) \in [\,d\,]$, the values of α_1 are restricted to a certain set. Define

$$A_1 = \{q, q + s_q, q + s_q + s_{q-1}, \cdots, q + \sum_{i=1}^{q} s_i\}.$$

Proposition 3. *If* $\delta(\alpha) = \big(1, 2, \cdots, q-1, q, q^{(s_q)}, (q-1)^{(s_{q-1})}, \cdots, 1^{(s_1)}\big)$, *then* $\alpha_1 \in A_1$. *Equivalently, if* $\delta(\alpha) = \big(1, 2, \cdots, q-1, q, q^{(s_q)}, (q-1)^{(s_{q-1})}, \cdots, 1^{(s_1)}\big)$, *then* $\alpha \in \mathcal{P}(n, k)^*$ *for some* $k \in A_1$.

Proof. If $\alpha = (\alpha_1, \alpha_2, ...\alpha_t)$ with

$$\delta(\alpha) = \left(1, 2, \cdots, q-1, q, q^{(s_q)}, (q-1)^{(s_{q-1})}, \cdots, 1^{(s_1)}\right),$$

let $\alpha' = (\alpha_2, ...\alpha_t)$. The lengths of the first α_1 diagonals of α' decrease by 1 while the lengths of the other diagonals stay the same, that is, $\delta(\alpha') = (d'_k)_{k\geq 1}$ with $d'_i = d_i - 1$ for $1 \leq i \leq \alpha_1$ and $d'_i = d_i$ for $i > \alpha_1$. The constraints imposed by Lemma 1 imply that $\alpha_1 \in \{q, q + s_q, q + s_q + s_{q-1}, \cdots, q + \sum_{i=1}^{q} s_i\}$. □

For partitions with the same diagonal sequence, the sum of the squares of their parts plus the sum of the squares of the parts of their conjugates is an invariant.

For $\alpha = (\alpha_1, \cdots, \alpha_t) \in \mathcal{P}(n)$ define $s(\alpha) = \sum_{i=1}^{t} \alpha_i^2$. Let

$$\alpha^* = \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_r)$$

with $r = \alpha_1$. Since the Young diagram of α is contained in a t by r rectangle, it follows that $d_k = 0$ for all $k \geq t + r$. Let $h = t + r - 1$.

Proposition 4. *If $\alpha \in \mathcal{P}(n)$, then $s(\alpha)+s(\alpha^*) = (2, 4, \cdots, 2r)\cdot\delta(\alpha) = 2\sum_{k=1}^{h} k\, d_k$. As a consequence, if $\delta(\alpha) = \delta(\beta)$, then $s(\alpha) + s(\alpha^*) = s(\beta) + s(\beta^*)$.*

Proof. We use the well-known fact that n^2 is the sum of the first n odd integers. Consider the box in the i-th row and j-th column of the Young diagram of α. Such a box lies on the kth diagonal for $k = i + j - 1$. As such, it contributes $2j - 1$ to α_i^2 and contributes $2i - 1$ to β_j^2. Therefore, such a box contributes $2k$ to $s(\alpha) + s(\alpha^*)$. Every box on the k-th diagonal contributes the same amount, $2k$, to $s(\alpha) + s(\alpha^*)$. It follows that the total contributions of all the boxes on the k-th diagonal of the Young diagram is $2kd_k$. Summing over k yields the result. $\square$

We note that the converse of Proposition 4 is not true. For example, let $\alpha = (6, 2, 1)$ and $\beta = (5, 4)$, both of which are members of $\mathcal{P}(9)$. It is easy to see that $\delta(\alpha) = (1, 2, 3, 1, 1, 1) \neq (1, 2, 2, 2, 2) = \delta(\beta)$ and $s(\alpha) + s(\alpha^*) = s(\beta) + s(\beta^*) = 58$.

3. Majorization Order on $[\,d\,]$

The set $\mathcal{P}(n)$ is a partially ordered set under majorization. Recall that if $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s) \in \mathcal{P}(n)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_t) \in \mathcal{P}(n)$, we say α *majorizes* β if $\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \beta_i$ for all $1 \leq k \leq \min\{s, t\}$. If α majorizes β, we write $\alpha \succ \beta$. This partial order on $\mathcal{P}(n)$ induces a partial order on all subsets of $\mathcal{P}(n)$. In particular, we will show that if $d \in \Delta(n)$, then the partially ordered set $[\,d\,]$ has a unique maximal element $\overline{\alpha}$ and a unique minimal element $\underline{\alpha}$, where $\overline{\alpha}$ and $\underline{\alpha}$ are as defined in Section 2.

Proposition 5. *Let $d \in \Delta(n)$. If $\alpha \in [\,d\,] \subseteq \mathcal{P}(n)$, then $\overline{\alpha} \succ \alpha \succ \underline{\alpha}$.*

Proof. It is well-known ([4] Theorem 7.B.5) that for $\alpha, \beta \in \mathcal{P}(n)$, $\alpha \succ \beta$ if and only if $\beta^* \succ \alpha^*$. Since $\underline{\alpha} = \overline{\alpha}^*$, we only need to show that $\overline{\alpha} \succ \alpha$ for all $\alpha \in [\,d\,]$. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s) \in [\,d\,]$ and $\overline{\alpha} = (\overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_t) \in [\,d\,]$.

In general, a box in the i-th row of the Young diagram of $\alpha \in \mathcal{P}(n)$ is the j-th box of its diagonal with $j \leq i$. Since $\overline{\alpha}_i > \overline{\alpha}_{i+1}$, it follows that every box in the i-th row of the Young diagram of $\overline{\alpha}$ is the i-th box of its respective diagonal. Therefore,

$$\sum_{i=1}^{k} \overline{\alpha}_i \geq \sum_{i=1}^{k} \alpha_i \text{ for } 1 \leq k \leq \min\{s, t\},$$

which is what we had to show. $\square$

We can further stratify the set $[\,d\,]$ by the number of (non-zero) parts of the partitions and, alternatively, by the size of the largest part of $\alpha \in [\,d\,]$. Define

$$[\,d\,]_k = [\,d\,] \cap \mathcal{P}(n,k) \text{ and } [\,d\,]_k^* = [\,d\,] \cap \mathcal{P}(n,k)^*.$$

By Proposition 3, $[\,d\,]_k = [\,d\,]_k^* = \emptyset$, unless $k \in A_1$. If $d = (d_1, d_2, \cdots, d_L)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) \in [\,d\,]_k$, then $\alpha' = \alpha_1 - 1, \alpha_2 - 1, \cdots, \alpha_k - 1 \in \mathcal{P}(n-k)$ with diagonal sequence

$$d' = (d_2 - 1, \cdots, d_k - 1, d_{k+1}, \cdots, d_L).$$

Let $\overline{\alpha}' = (\overline{\alpha}'_1, \overline{\alpha}'_2, \cdots, \overline{\alpha}'_k)$ be the maximal element in $[\,d'\,]$. Then $\overline{\alpha}(k) = (\overline{\alpha}'_1 + 1, \overline{\alpha}'_2 + 1, \cdots, \overline{\alpha}'_k + 1)$ is the maximal element in $[\,d\,]_k$. It follows that $\overline{\alpha}(k)^*$ is the minimal element in $[\,d\,]_k^*$.

Now we describe how to construct the maximal element $\overline{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_t) \in [\,d\,]_k^*$ for $k \in A_1$. Let $\gamma_1 = k$. Then $d' = (d_2 - 1, \cdots, d_k - 1, d_{k+1}, \cdots, d_L)$ is the diagonal sequence of some partition $\gamma' \in \mathcal{P}(n-k)$. By Proposition 3, largest part of γ' is restricted to a set A'_1. Let γ_2 the largest element in A'_1 less than or equal to k. This process continues in the obvious way and leads to the maximal element $\overline{\gamma} \in [\,d\,]_k^*$. It follows that $\overline{\gamma}^* = \underline{\alpha}(k)$ is the minimal element in $[\,d\,]_k$.

We illustrate this construction with an example. Let $d = (1,2,3,4,4,4,2,1)$ and $k = 6$.

Step 1: Set $\gamma_1 = 6$. Then $d' = (1,2,3,3,3,2,1)$ which implies $A'_1 = \{3,5,6,7\}$.

Step 2: Set $\gamma_2 = 6$. Then $d'' = (1,2,2,2,1,1)$ which implies $A^{(3)} = \{1,4,6\}$.

Step 3: Set $\gamma_3 = 6$. Then $d^{(3)} = (1,1,1)$ which implies $A_1^{(4)} = \{3\}$.

Step 4: Set $\gamma_4 = 3$ and the process ends.

We get $\overline{\gamma} = (6,6,6,3)$ and $\underline{\alpha}(6) = (4,4,4,3,3,3)$.

4. The Cardinality of $[\,d\,]$

Definition 3. Let $M = \{0^{(b_0)}, 1^{(b_1)}, \cdots, t^{(b_t)}\}$ be a multiset with b_i elements equal to i. Let $b = b_0 + b_1 + \cdots + b_t$. A *vn-arrangement* of the elements of M is a sequence $(v_1, v_2, \cdots, v_b)$ such that $v_{i+1} - v_i \le 1$ for $0 \le i < b$.

Proposition 6. *The number of vn-arrangements of M is $\prod_{i=0}^{t-1} \binom{b_i + b_{i+1}}{b_i}$.*

Proof. For $t = 0$, there is nothing to prove. When $t = 1$, any arrangement of the b_0 elements equal to 0 and the b_1 elements equal to 1 are vn-arrangements. There are $\binom{b_0+b_1}{b_0}$ such sequences. So assume the result holds for $t \ge 1$. Let $M = \{0^{(b_0)}, 1^{(b_1)}, \cdots, t^{(b_t)}, (t+1)^{(b_{t+1})}\}$. Any vn-arrangement of M arises from a vn-arrangement of $M' = \{0^{(b_0)}, 1^{(b_1)}, \cdots, t^{(b_t)}\}$ by adding any number of elements equal to $t+1$ at the very beginning of the arrangement or after an element equal to t. Since there are b_{t+1} elements equal to $t+1$, which can be placed in b_t+1 slots, there are $\binom{b_t+b_{t+1}}{b_t}$ vn-arrangements of M for every vn-arrangement of M'. By induction, the result holds. $\square$

The result generalizes and holds for any finite multiset $\{a^{(b_0)},(a+1)^{(b_1)},\cdots,(a+t)^{(b_t)}\}=a+M$, $a\in\mathbb{Z}$. It is worth pointing out the special case when $b_i=1$ for all $1\le i\le t$.

Corollary 5. *If M is any set of t consecutive integers, then the number of vn-arrangements of M is 2^{t-1}.*

The idea of vn-arrangements is related to the beautiful theory of counting sequences according to rises and falls, which was developed by Carlitz and others. For example, see [3] and [2]. As an aside, we can extend the definition of vn-arrangement as follows.

Definition 4. Let $M=\{0^{(b_0)},1^{(b_1)},2^{(b_2)},\cdots,t^{(b_t)}\}$ be a multiset with b_i elements equal to i. Let $b=b_0+b_1+\cdots+b_t$. A *k-vn-arrangement*, $0\le k\le t$, of the elements of M is a sequence $(v_1,v_2,\cdots,v_b)$ such that $v_{i+1}-v_i\le k$ for $1\le i<b$.

There is an explicit formula for the number of k-vn-arrangments of a given multiset. We adopt the convention that an empty product has value 1.

Proposition 7. *Let $M=\{0^{(b_0)},1^{(b_1)},2^{(b_2)},\cdots,t^{(b_t)}\}$ be a multiset with b_i elements equal to i. The number of k-vn-arrangements of M is*

$$\binom{b_0+b_1+\cdots+b_k}{b_0,b_1,\cdots,b_k}\prod_{i=1}^{t-k}\binom{b_i+b_{i+1}+\cdots+b_{i+k}}{b_{i+k}}.$$

Proof. For $t<k$, there are no constraints on the arrangements. The number of arrangements is the multinomial coefficient. So assume now $t\ge k$.

For $k=0$, there is only one arrangement, $(t^{(b_t)},(t-1)^{(b_{t-1})},\cdots 1^{(b_1)},\cdots,0^{(b_0)})$, and the result holds. For $k=1$, we get the result of Proposition 6. Now assume $k\ge 2$. We proceed by induction on $t\ge k$. If $t=k$, the results hold. Now assume that the result holds for some $t\ge k$. Let $M=\{0^{(b_0)},1^{(b_1)},\cdots,t^{(b_t)},(t+1)^{(b_{t+1})}\}$. Any k-vn-arrangement of M arises from a k-vn-arrangement of $M'=\{0^{(b_0)},1^{(b_1)},\cdots,t^{(b_t)}\}$ by adding any number of elements equal to $t+1$ at the very beginning or after an element equal to $t+1-k,t+2-k,\cdots,t$. Since there are b_{t+1} elements equal to $t+1$, which can be placed in $b_{t+1-k}+b_{t+2-k}+\cdots+b_t$ slots, the result follows by induction. $\square$

For a partition $\alpha=(\alpha_1,...,\alpha_t)\in\mathcal{P}(n)$, define $v(\alpha)=(v_1,v_2,\cdots,v_t)$ where $v_i=\alpha_i+i-1$ for $1\le i\le t$. The definition of $\delta(\alpha)=(d_k)_{k\ge1}$ can now be restated as

$$d_k=\big|\{i\,|\,1\le i\le k \text{ and } v_i\ge k\}\big|. \tag{1}$$

Since $v_i = \alpha_i + i - 1$, we have $v_i \geq i$ for $1 \leq i \leq t$. Since α is a non-increasing sequence, we have $v_{i+1} - v_i \leq 1$ for $1 \leq i < L$. In the special case of $\overline{\alpha}$, we have $\overline{v}_i = q + \sum_{k=i}^{q} s_k$ for $1 \leq i \leq q$. In particular, $\overline{v_1} = q + \sum_{i=1}^{q} s_i = L$ and $\overline{v}_q = q + s_q = l$.

In what follows, we assume $\delta = (1, 2, \cdots, q, d_{q+1}, d_{q+2}, \cdots, d_L) \in \Delta(n)$, $q \geq d_{q+1}$. Set $b_i = d_i - d_{i+1}$ for $q \leq i < L$ and $b_L = d_L$. By Proposition 3, if $\alpha \in [\, d\,]$, then $\alpha \in \mathcal{P}(n,k)$ for some $k \in A_1$. Let $[\, d\,]_k = [\, d\,] \cap \mathcal{P}(n,k)$. Our first result counts the number of elements in $[\, d\,]_q$, that is, those partitions in $\mathcal{P}(n)$ that have exactly q parts.

Proposition 8. *The cardinality of $[d]_q$ is given by*

$$\big|\, [\, d\,]_q \big| = \big|\, [\, d\,] \cap \mathcal{P}(n,q) \,\big| = \prod_{i=q}^{L-1} \binom{b_i + b_{i+1}}{b_i}.$$

Proof. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_q) \in [d]_q$ and $v(\alpha) = (\alpha_1, \alpha_2 + 1, \cdots, \alpha_q + q - 1)$. We will show that $v(\alpha)$ is a vn-arrangement of the multiset $\{\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_q\}$. Since $\alpha_i \leq \alpha_{i+1}$ for all $1 \leq i < q$, it follows that $v_{i+1} - v_i \leq 1$ which implies that $v(\alpha)$ is a vn-arrangement. Since $\alpha, \overline{\alpha} \in [\, d\,]$, Equation (1) implies that

$$d_k = \Big|\{i \,|\, 1 \leq i \leq k \text{ and } v_i \geq k\}\Big| = \Big|\{j \,|\, 1 \leq j \leq k \text{ and } \overline{v_j} \geq k\}\Big| \text{ for all } q \leq k.$$

Since $d_k = 0$ for $k > L$ and $\sum_{i=1}^{q} v_i = \sum_{j=1}^{q} \overline{v}_j$, we conclude that $\{v_1, v_2, \cdots, v_q\} = \{\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_q\}$ as multisets.

Writing the elements of $\overline{v}$ as a multiset of consecutive integers, we have

$$M = \{q^{(b_q)}, \cdots, l^{(b_l)}, (l+1)^{(b_{l+1})}, \cdots, L^{(b_L)}\}.$$

By Proposition 3, $b_i \neq 0$ if and only if $i \in A_1$. The result now follows from Proposition 6. $\square$

The expression for the cardinalities of $\big|\, [\, d\,]_k \big|$ when $k > q$ are more complicated.

Proposition 9. *If $k > q$, then*

$$\big|\, [\, d\,]_k \,\big| = \big|\, [\, d\,] \cap \mathcal{P}(n,k) \,\big| = \binom{b_{k-1} + b_k}{b_{k-1} + 1} \prod_{i=q}^{k-2} \binom{b_i + b_{i+1} + 1}{b_i + 1} \prod_{i=k}^{L-1} \binom{b_i + b_{i+1}}{b_i}$$

When $b_k = 0$, the binomial coefficient $\binom{b_{k-1}+b_k}{b_{k-1}+1}$ equals 0. This implies $\big|\, [\, d\,]_k \,\big| \neq 0$ if and only if $k \in A_1$.

Proof. We proceed by induction on n. The result is straightforward to verify for small values of n. Let $d = (1, 2, \cdots q, d_{q+1} \cdots, d_L)$, be a diagonal sequence of some partition $\alpha \in \mathcal{P}(n)$ with $k \in A_1$. If $k \notin A_1$, there is nothing to prove.

For $k \in A_1$, there exists a partition $\alpha = (\alpha_1, \cdots \alpha_k) \in \mathcal{P}(n)$ with $\alpha_k > 0$ and $\delta(\alpha) = d$. Hence, the partition $\alpha' = (\alpha_1 - 1 \cdots \alpha_k - 1)$ belongs to $\mathcal{P}(n-k)$ and $\delta(\alpha') = d' = (d'_i)_{1 \leq i \leq L-1} = (1, 2, \cdots, q-1, d_q - 1, \cdots, d_{k-1} - 1, d_k, \cdots, d_{L-1})$.

If $\beta = (\beta_1, \cdots \beta_l) \in [\,d'\,]_l$ for some $l \leq k$, then $(\beta_1 + 1, \cdots \beta_l + 1, 1^{(k-l)}) \in [\,d\,]_k$ and vice versa. Hence,

$$\big|\,[\,d\,]_k\,\big| = \sum_{l=q-1}^{k} \big|\,[\,d'\,]_l\,\big|.$$

We observe that for $i \geq k-1$, $c_{k-1} = d'_k - d'_{k-1} = d_{k+1} - d_k - 1 = b_k - 1$ while for $i \neq k-1$, $c_i = d'_{i+1} - d'_i = d_{i+2} - d_{i+1} = b_{i+1}$. By induction,

$$\big|\,[\,d'\,]_q\,\big| = \prod_{i=q-1}^{L-2} \binom{c_i + c_{i+1}}{c_i}$$

and

$$\big|\,[\,d'\,]_l\,\big| = \binom{c_{l-1} + c_l}{c_{l-1} + 1} \prod_{i=q-1}^{l-3} \binom{c_i + c_{i+1} + 1}{c_i + 1} \prod_{i=l-1}^{L-2} \binom{c_i + c_{i+1}}{c_i}$$

for $l > q-1$.

Notice that

$$\begin{aligned}\big|\,[\,d'\,]_{q-1}\,\big| + \big|\,[\,d'\,]_q\,\big| = &\prod_{i=q-1}^{L-2} \binom{c_i + c_{i+1}}{c_i} \\ &+ \binom{c_{q-1} + c_q}{c_{q-1} + 1} \prod_{i=q-1}^{q-2} \binom{c_i + c_{i+1} + 1}{c_i + 1} \prod_{i=q}^{L-2} \binom{c_i + c_{i+1}}{c_i} \\ = &\binom{c_{q-1} + c_q + 1}{c_{q-1} + 1} \prod_{i=q}^{L-2} \binom{c_i + c_{i+1}}{c_i},\end{aligned}$$

and, by induction,

$$\begin{aligned}\big|\,[\,d\,]_k\,\big| = \sum_{l=q-1}^{k} \big|\,[\,d'\,]_l\,\big| &= \prod_{i=q}^{k} \binom{c_{i-1} + c_i + 1}{c_{i-1} + 1} \prod_{i=k}^{L-2} \binom{c_i + c_{i+1}}{c_i} \\ &= \binom{b_{k-1} + b_k}{b_{k-1} + 1} \prod_{i=q}^{k-2} \binom{b_i + b_{i+1} + 1}{b_i + 1} \prod_{i=k}^{L-1} \binom{b_i + b_{i+1}}{b_i}.\end{aligned}$$

□

The cardinality of $[\,d\,]$ now follows by addition of the cardinalities of $[\,d\,]_k$ for $q \leq k \leq L$ with a similar and somewhat simpler argument.

Theorem 1. *Assume $\delta = (1, 2, \cdots, q, d_{q+1}, d_{q+2}, \cdots, d_L) \in d(n)$. Set $b_i = d_i - d_{i+1}$ for $q \leq i < L$ and $b_L = d_L$. The number of partitions $\alpha \in \mathcal{P}(n)$ with $d(\alpha) = d$ is*

$$\big|\,[\,d\,]\,\big| = \prod_{i=q}^{L-1} \binom{b_i + b_{i+1} + 1}{b_i + 1}. \tag{2}$$

Proof. Clearly $\big|\,[\,d\,]\,\big| = \sum_{k=q}^{L} \big|\,[\,d\,]_k\,\big|$ and

$$\begin{aligned}\sum_{k=q}^{L} \big|\,[\,d\,]_k\,\big| = &\prod_{i=q}^{L-1} \binom{b_i + b_{i+1}}{b_i} \\ &+ \sum_{k=q+1}^{L} \binom{b_{k-1} + b_k}{b_{k-1} + 1} \prod_{i=q}^{k-2} \binom{b_i + b_{i+1} + 1}{b_i + 1} \prod_{i=k}^{L-1} \binom{b_i + b_{i+1}}{b_i}.\end{aligned}$$

Notice that

$$\begin{aligned}\big|\,[\,d\,]_q\,\big| + \big|\,[\,d\,]_{q+1}\,\big| = &\prod_{i=q}^{L-1} \binom{b_i + b_{i+1}}{b_i} \\ &+ \binom{b_q + b_{q+1}}{b_q + 1} \prod_{i=q}^{q-1} \binom{b_i + b_{i+1} + 1}{b_i + 1} \prod_{i=q+1}^{L-1} \binom{b_i + b_{i+1}}{b_i} \\ = &\binom{b_q + b_{q+1} + 1}{b_q + 1} \prod_{i=q+1}^{L-1} \binom{b_i + b_{i+1}}{b_i}\end{aligned}$$

and, by induction,

$$\big|\,[\,d\,]\,\big| = \sum_{k=q}^{L} \big|\,[\,d\,]_k\,\big| = \prod_{i=q}^{L-1} \binom{b_i + b_{i+1} + 1}{b_i + 1}.$$

□

We point out some corollaries to the main result.

Corollary 6.

1. *If $s_i \geq 2$ for all $1 \leq i < q$, then $\big|\,[\,d\,]_q\big| = 1$.*
2. *If $s_i \geq 2$ for all $1 \leq i < q$ and $s_q \geq 1$, then $\big|\,[\,d\,]\,\big| = \prod_{i=q}^{L}(b_i + 1)$.*
3. *If m is a natural number, then there exists a partition $\alpha \in \mathcal{P}(n)$ for some n such that $\delta(\alpha) = d$ and $|\,[\,d\,]\,| = m$.*
4. *Let $d = (1, 2, \cdots, q, k^{(s_k)}, (k-1)^{(s_{k-1})}, \cdots, 2^{(s_2)}, 1^{(s_1)})$ with $k < q$ and $s_k \geq 2$ and define $d' = (1, 2, \cdots, k, k^{(s_k)}, (k-1)^{(s_{k-1})}, \cdots, 2^{(s_2)}, 1^{(s_1)})$. Then $\big|\,[\,d\,]\,\big| = \big|\,[\,d'\,]\,\big|$.*

5. *Let* $d = (1, 2, \cdots, q, q^{(s_q)}, (q-1)^{(s_{q-1})}, \cdots, 2^{(s_2)}, 1^{(s_1)})$*. Let* $\sigma_i = \min\{s_i, 2\}$ *for* $1 \le i \le q$ *and define* $d' = (1, 2, \cdots, q, q^{(\sigma_q)}, (q-1)^{(\sigma_{q-1})}, \cdots, 2^{(\sigma_2)}, 1^{(\sigma_1)})$*. Then* $\big|\,[\,d\,]\,\big| = \big|\,[\,d'\,]\,\big|$*.*

Proof.

1. Part 1 follows from the fact that if $s_k \ge 2$ for all $1 \le k < q$, then $b_i = 0$ or $b_{i+1} = 0$ for all $q \le i < L$. It follows that $\binom{b_i+b_{i+1}}{b_i} = 1$ for all $q \le i < L$.

2. For the proof of part 2 assume $s_k \ge 2$ for $1 \le k \le q$ and $s_q \ge 1$. This implies that if $b_i \ne 0$, then $b_{i-1} = b_{i+1} = 0$. It follows that

$$\binom{b_{i-1}+b_i+1}{b_{i-1}+1}\binom{b_i+b_{i+1}+1}{b_i+1} = (b_i+1)\cdot 1 = b_i + 1.$$

3. Part 3 follows from part 2 by factoring m.

4. Part 4 follows from the fact that $s_k \ge 2$ implies $b_{q+1} = 0$ and $\binom{b_q+b_{q+1}}{b_q+1} = 1$ for any choice of b_q.

5. Part 5 follows the fact that if $s_i > 2$, then $b_j = b_{j+1} = 0$ for some j and $\binom{b_j+b_{j+1}+1}{b_j+1} = 1$.

□

Given Equation (2), we can characterize diagonal sequences $d \in \Delta(n)$ and the sets $[\,d\,]$ for which $\big|\,[\,d\,]\,\big|$ is small or a prime number.

Corollary 7. *Assume* $d \in \Delta(n)$ *for some positive integer* n*.*

1. *If* $\big|\,[\,d\,]\,\big| = 1$*, then there exists a positive integer* q *such that* $n = \binom{q+1}{2}$*,* $d = (1, 2, \cdots, q-1, q)$*, and* $[\,d\,] = \{(q, q-1, \cdots, 2, 1)\}$*.*

2. *If* $\big|\,[\,d\,]\,\big| = 2$*, then there exist integers* $q \ge 1$*,* $k \ge 2$ *such that* $n = \binom{q+1}{2} + k$*,* $d = (1, 2, \cdots, q, 1^{(k)})$*, and* $[\,d\,] = \{(q+k, q-1, \cdots, 2, 1), (q, q-1, \cdots, 2, 1, 1^{(k)})\}$*, or* $n = 2$*,* $d = (1, 1)$*, and* $[\,d\,] = \{(2), (1, 1)\}$*.*

3. *If* $\big|\,[\,d\,]\,\big| = 3$*, then*

   (a) *there exist integers* $q \ge 2$*,* $k \ge 2$ *such that* $n = \binom{q+1}{2} + 2k$*,* $d = (1, 2, \cdots, q, 2^k)$*, and*

   $[\,d\,] = \{(q+k, q-1+k, q-2, \cdots, 2, 1), (q, q-1, \cdots, 2^{(k+1)}, 1), (q+k, q-1, \cdots, 2, 1^{(k+1)}\}$*; or*

   (b) $d = (1, 2, 1)$ *and* $[\,d\,] = \{(3, 1), (2, 2), (2, 1, 1)\}$*; or*

   (c) $d = (1, 2, 2)$ *and* $[\,d\,] = \{(3, 2), (3, 1, 1), (2, 2, 1)\}$*.*

4. *If* $\big|\,[\,d\,]\,\big| = 4$*, then*

(a) *there exist integers* $q \geq 3$, $k \geq 2$, *such that* $n = \binom{q+1}{2} + 3k$, $d = (1, 2, \cdots, q, 3^{(k)})$, *and*

$[\,d\,] = \{(q+k, q-1+k, q-2+k, q-3, \cdots, 2, 1), (q+k, q-1+k, q-2, \cdots, 2, 1^{(k+1)}), (q+k, q-1, \cdots, 3, 2^{(k+1)}, 1), (q, q-1, \cdots, 3^{(k+1)}, 2, 1)\}$*; or*

(b) *there exist integers* $q, k, l \geq 2$, *such that* $n = \binom{q+1}{2} + 2k + l$, $d = (1, 2, \cdots, q, 2^{(k)}, 1^{(l)})$, *and*

$[\,d\,] = \{(q+k+l, q+k, q-2, \cdots, 1), (q+k+l, q-1, \cdots, 1^{(k+1)}), (q+k, q-1, \cdots, 1^{(k+l+1)}), (q, q-1, \cdots, 2^{(k+1)}, 1^{(l+1)})\}$*; or*

(c) *there exist an integer* $k \geq 2$, *such that* $n = 5 + k$, $d = (1, 2, 2, 1^{(k)})$, *and* $[\,d\,] = \{(3+k, 2), (3+k, 1, 1), (3, 1^{(k+2)}), (2, 2, 1^{(k+1)})\}$*; or*

(d) $d = (1, 2, 3, 3)$ *and* $[\,d\,] = \{(4,3,2), (4,3,1,1), (4,2,2,1), (3,3,2,1)\}$.

We leave it to the reader to generalize parts (2) and (3) and find all partitions α and $d = \delta(\alpha)$ such that $\big|[\,d\,]\big| = p$, where p is a prime.

Proof. 1. If $\big|[\,d\,]\big| = 1$, then $\binom{b_i+b_{i+1}+1}{b_i+1} = 1$ for $q \leq i < L$, which implies $b_i = 0$ for $q < i \leq L$ and $b_q = q$. Hence, $d = (1, 2, \cdots, q)$ and $[\,d\,] = \{(q, q-1, \cdots, 2, 1)\}$.

2. If $\big|[\,d\,]\big| = 2$, then there exists a $q \leq j < L$ with $\binom{b_j+b_{j+1}+1}{b_j+1} = 2$ while $\binom{b_i+b_{i+1}+1}{b_i+1} = 1$ for all $q \leq i \neq j < L$. Hence, $b_L = 1$ while $b_i = 0$ for all $q < i < L$, which implies $d = (1, 2, \cdots, q, 1, 1, \cdots, 1)$ and $n = \binom{q+1}{2} + k$ for some positive integer k. It follows that $[\,d\,] = \{(q+k, q-1, \cdots, 2, 1), (q, q-1, \cdots, 2, 1, 1^{(k)})\}$.

3. If $\big|[\,d\,]\big| = 3$, then $\binom{b_{L-1}+b_L+1}{b_{L-1}+1} = 3$, while $\binom{b_j+b_{j+1}+1}{b_j+1} = 1$ for all $q \leq j < L-1$, which implies there exist integers $q \geq 2$, $k > 1$ such that $d = (1, 2, \cdots, q, 2^{(k)})$ or $d = (1, 2, 1)$. The result follows.

4. If $\big|[\,d\,]\big| = 4$, then $\binom{b_{L-1}+b_L+1}{b_{L-1}+1} = 4$, $L \geq q+2$, while $\binom{b_i+b_{i+1}+1}{b_i+1} = 1$ for $q < i < L-1$, or there exist integers $k \geq q+2$, $L \geq k = 2$ with $\binom{b_k+b_{k+1}+1}{b_k+1} = 2$ and $\binom{b_{L-1}+b_L+1}{b_{L-1}+1} = 2$ while $\binom{b_i+b_{i+1}+1}{b_i+1} = 1$ for all $i > q$, $i \neq k, L-1$. The result follows.

□

5. Examples

We illustrate the results using our example $\alpha = (7, 7, 4, 1, 1, 1) \in \mathcal{P}(21)$ with diagonal sequence $\delta(\alpha) = (1, 2, 3, 4, 4, 4, 2, 1)$. It follows that $q = 4$ and $b_4 = b_5 = 0, b_6 =$

$2, b_7 = b_8 = 1$. By Proposition 3, if $\alpha \in [\,d\,]$, then $\alpha \in [\,d\,]_4 \cup [\,d\,]_6 \cup [\,d\,]_7 \cup [\,d\,]_8$. By Proposition 8, Proposition 9, and Theorem 1 we get

$$\big|\,[\,d\,]_4\,\big| = \binom{0+0}{0}\binom{0+2}{0}\binom{2+1}{2}\binom{1+1}{1} = 1 \cdot 1 \cdot 3 \cdot 2 = 6,$$

$$\big|\,[\,d\,]_6\,\big| = \binom{2}{1}\binom{1}{1}\binom{3}{1}\binom{2}{1} = 2 \cdot 1 \cdot 3 \cdot 2 = 6,$$

$$\big|\,[\,d\,]_7\,\big| = \binom{3}{3}\binom{1}{1}\binom{3}{1}\binom{2}{1} = 1 \cdot 1 \cdot 3 \cdot 2 = 6,$$

$$\big|\,[\,d\,]_8\,\big| = \binom{2}{2}\binom{1}{1}\binom{3}{1}\binom{4}{3} = 1 \cdot 1 \cdot 3 \cdot 4 = 12, \text{ and}$$

$$\big|\,[\,d\,]\,\big| = \binom{0+0+1}{0+1}\binom{0+2+1}{0+1}\binom{2+1+1}{2+1}\binom{1+1+1}{1+1} = 1 \cdot 3 \cdot 4 \cdot 3 = 36.$$

The 36 partitions $\alpha \in [\,d\,] = [\,d\,]_4 \cup [\,d\,]_6 \cup [\,d\,]_7 \cup [\,d\,]_8$ are listed in Table 1.

$4A$	$(8,6,4,3)$	$7A$	$(8,5,4,1,1,1,1)$
$4B$	$(8,5,5,3)$	$7B$	$(8,5,2,2,2,1,1)$
$4C$	$(8,5,4,4)$	$7C$	$(8,3,3,3,2,1,1)$
$4D$	$(7,7,4,3)$	$7D$	$(6,5,2,2,2,2,2)$
$4E$	$(6,6,6,3)$	$7E$	$(6,3,3,3,2,2,2)$
$4F$	$(6,5,5,5)$	$7F$	$(4,4,4,3,2,2,2)$
$6A$	$(8,6,4,1,1,1)$	$8A$	$(7,5,4,1,1,1,1,1)$
$6B$	$(8,6,2,2,2,1)$	$8B$	$(7,5,2,2,2,1,1,1)$
$6C$	$(8,5,5,1,1,1)$	$8C$	$(7,3,3,3,2,1,1,1)$
$6D$	$(8,5,2,2,2,2)$	$8D$	$(6,6,4,1,1,1,1,1)$
$6E$	$(8,3,3,3,3,1)$	$8E$	$(6,6,2,2,2,1,1,1)$
$6F$	$(8,3,3,3,2,2)$	$8F$	$(6,5,5,1,1,1,1,1)$
$6G$	$(7,7,4,1,1,1)$	$8G$	$(6,5,2,2,2,2,1,1)$
$6H$	$(7,7,2,2,2,1)$	$8H$	$(6,3,3,3,3,1,1,1)$
$6I$	$(6,6,6,1,1,1)$	$8I$	$(6,3,3,3,2,2,1,1)$
$6J$	$(6,3,3,3,3,3)$	$8J$	$(4,4,4,4,2,1,1,1)$
$6K$	$(4,4,4,4,4,1)$	$8K$	$(4,4,4,3,3,1,1,1)$
$6L$	$(4,4,4,3,3,3)$	$8L$	$(4,4,4,3,2,2,1,1)$

Table 1: Partitions with diagonal sequence $(1,2,3,4,4,4,2,1)$ ordered by length.

Alternatively, we can collect the elements of $[\,d\,]^* = [\,d\,]^*_4 \cup [\,d\,]^*_6 \cup [\,d\,]^*_7 \cup [\,d\,]^*_8$ by the size of $\alpha_1 \in A_1$ as in Table 2.

$4A^*$	$(4,4,4,3,2,2,1,1)$	$7A^*$	$(7,3,3,3,2,1,1,1)$
$4B^*$	$(4,4,4,3,3,1,1,1)$	$7B^*$	$(7,5,2,2,2,1,1,1)$
$4C^*$	$(4,4,4,4,2,1,1,1)$	$7C^*$	$(7,5,4,1,1,1,1,1)$
$4D^*$	$(4,4,4,3,2,2,2)$	$7D^*$	$(7,7,2,2,2,1)$
$4E^*$	$(4,4,4,3,3,3)$	$7E^*$	$(7,7,4,1,1,1)$
$4F^*$	$(4,4,4,4,4,1)$	$7F^*$	$(7,7,4,3)$
$6A^*$	$(6,3,3,3,2,2,1,1)$	$8A^*$	$(8,3,3,3,2,1,1)$
$6B^*$	$(6,5,2,2,2,2,1,1)$	$8B^*$	$(8,5,2,2,2,1,1)$
$6C^*$	$(6,3,3,3,3,1,1,1)$	$8C^*$	$(8,5,4,1,1,1,1)$
$6D^*$	$(6,6,2,2,2,1,1,1)$	$8D^*$	$(8,3,3,3,2,2)$
$6E^*$	$(6,5,5,1,1,1,1,1)$	$8E^*$	$(8,5,2,2,2,2)$
$6F^*$	$(6,6,4,1,1,1,1,1)$	$8F^*$	$(8,3,3,3,3,1)$
$6G^*$	$(6,3,3,3,2,2,2)$	$8G^*$	$(8,6,2,2,2,1)$
$6H^*$	$(6,5,2,2,2,2,2)$	$8H^*$	$(8,5,5,1,1,1)$
$6I^*$	$(6,3,3,3,3,3)$	$8I^*$	$(8,6,4,1,1,1)$
$6J^*$	$(6,6,6,1,1,1)$	$8J^*$	$(8,5,4,4)$
$6K^*$	$(6,5,5,5)$	$8K^*$	$(8,5,5,3)$
$6L^*$	$(6,6,6,3)$	$8L^*$	$(8,6,4,3)$

Table 2: Partitions with diagonal sequence $(1,2,3,4,4,4,2,1)$ ordered by largest part.

Furthermore,

$$\begin{array}{ll}
\overline{\alpha}(4) = \overline{\alpha} = (8,6,4,3) & \underline{\alpha}(4) = (6,5,5,5) \\
\overline{\alpha}(6) = (8,6,4,1,1,1) & \underline{\alpha}(6) = (4,4,4,3,3,3) \\
\overline{\alpha}(7) = (8,5,4,1,1,1,1) & \underline{\alpha}(7) = (4,4,4,3,2,2,2) \\
\overline{\alpha}(8) = (7,5,4,1,1,1,1,1) & \underline{\alpha}(8) = \underline{\alpha} = (4,4,4,3,2,2,1,1).
\end{array}$$

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