#A61 INTEGERS 25 (2025)

EXTENSIONS OF SOME COMBINATORIAL IDENTITIES AND VARIOUS RELATED RESULTS

Horst Alzer[1]
Morsbacher Straße 10, Waldbröl, Germany
h.alzer@gmx.de

Kendall C. Richards
Southwestern University, Georgetown, Texas
richards@southwestern.edu

Received: 6/19/24, Revised: 1/20/25, Accepted: 6/4/25, Published: 6/27/25

Abstract

We present extensions of combinatorial identites published by Muthumalai, Sabar, Gould, Rathie and Lim, and we provide various related results. Among others, we prove the following formula which involves the Stirling numbers of the second kind and the Bernoulli numbers:

$$\sum_{k=0}^{n}(-1)^k \frac{\binom{n+1}{k+1}}{\binom{r+k}{k}} S(r+k,k) = \begin{cases} B_r & \text{if } 0 \le r \le n, \\ B_{n+1} - (\frac{-1}{2})^{n+1}(n+1)! & \text{if } r = n+1. \end{cases}$$

The special case $r = n$ is due to Muthumalai.

1. Introduction

Our work has been inspired by four papers published by Muthumalai [9], Sabar [14], Gould [6], and Rathie and Lim [13]. In these papers, the authors present remarkable identities for certain combinatorial sums. The aim of this paper is to extend these results and to deduce various related formulas. Some of these formulas include classical integer sequences, such as Stirling numbers and Euler numbers. Moreover, we show that an identity which was originally proved for integers is also valid for complex numbers, so that by differentiation we obtain new identities.

Throughout, we use the following notations. The *rising* and *falling factorials* are defined by

$$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad x^{\underline{n}} = x(x-1)\cdots(x-n+1),$$

DOI: 10.5281/zenodo.15756056

[1]corresponding author

respectively. *Kronecker's delta function* is given by

$$\delta_{nk} = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

The *harmonic numbers* are defined by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad (n \geq 1).$$

Moreover, we apply the *hypergeometric functions* ${}_3F_2$ and ${}_2F_1$ and *Euler's gamma function* Γ; see Abramowitz and Stegun [1, Sections 6, 15], NIST [10, Sections 5, 15, 16].

2. Main Results

2.1. Stirling Numbers of the Second Kind

The *Stirling numbers of the second kind* $S(n,k)$ denote the number of ways to partition a set of n elements into k nonempty subsets. They are given by the summation formula

$$S(n,k) = \frac{1}{k!} \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} (k-\nu)^n;$$

see NIST [10, p. 624]. In 2013, Muthumalai [9] offered a new connection between the Stirling numbers of the second kind and the classical *Bernoulli numbers* B_n,

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{\binom{n+1}{k+1}}{\binom{n+k}{k}} S(n+k,k). \tag{1}$$

The attempt to generalize Equation (1) led us to the following identities.

Theorem 1. *Let $j \geq 0$, $n \geq 0$ and $r \geq 0$ and let f be an r-times differentiable function. Then*

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n+1}{k+1} \left(f(x)^{k+1+j}\right)^{(r)} = \sum_{k=0}^{r} \binom{r}{k} \left(f(x)^j\right)^{(k)} \left[(1-f(x))^{n+1} - 1\right]^{(r-k)}, \tag{2}$$

and if f has no zero, then

$$\sum_{k=0}^{n} \binom{n+1}{k+1} \left(f(x)^{k+1-j}\right)^{(r)} = \sum_{k=0}^{r} \binom{r}{k} \left(\frac{1}{f(x)^j}\right)^{(k)} \left[(1+f(x))^{n+1} - 1\right]^{(r-k)}. \tag{3}$$

We show that Equations (2) and (3) can be applied to deduce new combinatorial identities involving well-known integer sequences. Our second theorem presents an extension of (1).

Theorem 2. *Let $n \geq 0$. Then*

$$\sum_{k=0}^{n}(-1)^k \frac{\binom{n+1}{k+1}}{\binom{r+k}{k}} S(r+k,k) = \begin{cases} B_r & \text{if } 0 \leq r \leq n, \\ B_{n+1} - (\frac{-1}{2})^{n+1}(n+1)! & \text{if } r = n+1. \end{cases} \tag{4}$$

The special case $r = n$ gives (1). Using (4) and Pascal's rule we obtain a combinatorial sum which is equal to 0.

Theorem 3. *Let $m \geq 1$ and $r \geq 0$. Then*

$$\sum_{k=0}^{r+m}(-1)^k \frac{\binom{r+m}{k}}{\binom{r+k}{k}} S(r+k,k) = 0. \tag{5}$$

An application of (3) yields the following counterpart of (4).

Theorem 4. *Let $n \geq 1$ and $r \geq 1$. Then*

$$\begin{aligned} &\sum_{k=0}^{n-1}(-1)^k \frac{\binom{n+1}{k+2}}{\binom{r+k}{k}} S(r+k,k) \\ &\quad = \begin{cases} rB_{r-1} + (n+r)B_r & \text{if } 1 \leq r \leq n, \\ (\frac{-1}{2})^{n+1}(n+1)! + (n+1)B_n + (2n+1)B_{n+1} & \text{if } r = n+1. \end{cases} \end{aligned} \tag{6}$$

2.2. Stirling Numbers of the First Kind

The *Stirling numbers of the first kind* $s(n,k)$ are given by the generating function

$$\sum_{k=0}^{n} s(n,k)x^k = x^{\underline{n}}.$$

Here, $(-1)^{n-k}s(n,k)$ is the number of permutations of n elements which contain exactly k cycles. The Stirling numbers of the first and second kind are connected by the elegant identities

$$\sum_{j=0}^{n} s(n,j)S(j,k) = \sum_{j=0}^{n} s(j,k)S(n,j) = \delta_{n,k};$$

see Quaintance and Gould [12, p. 171]. Applying (2) leads to our next result.

Theorem 5. *Let $j \geq 0$, $n \geq 0$, and $r \geq 0$. Then*

$$\sum_{k=0}^{n+1} (-1)^k \frac{\binom{n+1}{k}}{\binom{k+j+r}{r}} s(k+j+r, k+j) = \begin{cases} 0 & \text{if } 0 \leq r \leq n, \\ \frac{(n+1)!}{2^{n+1}} & \text{if } r = n+1. \end{cases} \tag{7}$$

2.3. Euler Numbers

The *Euler numbers* are defined by the Taylor series expansion

$$\frac{1}{\cosh(x)} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

A connection between E_n and $S(n,k)$ is given by the representation

$$E_{2n} = -4^{2n} \sum_{k=1}^{2n} (-1)^k \frac{S(2n,k)}{k+1} (3/4)^{\overline{k}};$$

see Jha [8]. Another application of (3) yields the following formula.

Theorem 6. *Let $n \geq 1$. Then*

$$\sum_{k=0}^{n-1} \binom{n}{k} E_k \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} T(\nu, n-k) + (2^{n+1} - 1) E_n = \sum_{k=0}^{n} \binom{n+1}{k+1} T(k,n),$$

where

$$T(k,n) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} (k-2j)^n.$$

2.4. Ordered Bell Numbers

The *ordered Bell numbers*, denoted by $a(n)$, count the weak orderings on a set of n elements. They are given by the generating function

$$\frac{1}{2-e^x} = \sum_{k=0}^{\infty} a(k) \frac{x^k}{k!}$$

and, explicitly, by the summation formula

$$a(n) = \sum_{k=0}^{n} k! S(n,k);$$

see Can and Joyce [2], Sprugnoli [15]. We apply (3) to obtain a counterpart of Theorem 6.

Theorem 7. *Let $n \geq 1$. Then*

$$\sum_{k=0}^{n-1} \binom{n}{k} a(k) \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} A(n-k,\nu) + (2^{n+1}-1)a(n) = \sum_{k=0}^{n} \binom{n+1}{k+1} A(n,k), \quad (8)$$

where

$$A(r,k) = \sum_{j=0}^{r} (-1)^j \binom{k}{j} j! S(r,j).$$

2.5. Powers and Falling Powers

In 2021, Sabar [14] presented an elegant formula for the difference of powers and falling powers,

$$n^p - n^{\underline{p}} = \sum_{k=0}^{p-1} (-1)^{p-k-1} k^p \binom{n}{k} \binom{n-k-1}{n-p} \quad (p = 1, 2, ..., n).$$

We use induction to prove the following extension.

Theorem 8. *Let $p \geq 1$ and $z \in \mathbb{C}$. Then*

$$z^p - z^{\underline{p}} = \sum_{k=0}^{p-1} (-1)^{p-k-1} k^p \binom{z}{k} \binom{z-k-1}{p-k-1}. \quad (9)$$

By differentiation, from (9) we obtain the next formula.

Corollary 1. *Let $p \geq 1$ and $z \in \mathbb{C} \setminus \{0, 1, ..., p-1\}$. Then*

$$p z^{p-1} - z^p \sum_{k=0}^{p-1} \frac{1}{z-k} = \sum_{k=0}^{p-1} (-1)^{p-k} \frac{k^p}{z-k} \binom{z}{k} \binom{z-k-1}{p-k-1}. \quad (10)$$

Remark 1. If we set $z = n \in \mathbb{N}$ with $n \geq p$, then (10) leads to a representation for the difference of two harmonic numbers,

$$H_n - H_{n-p} = \frac{p}{n} + \frac{1}{n^p} \sum_{k=0}^{p-1} (-1)^{p-k-1} \frac{k^p}{n-k} \binom{n}{k} \binom{n-k-1}{n-p}.$$

The special case $p = n$ gives

$$H_n = 1 + \frac{1}{n^n} \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{k^n}{n-k} \binom{n}{k}. \tag{11}$$

Remark 2. The following companion to (9) is given in Prudnikov et al. [11, 4.2.5.47] (see also Vassilev-Missana [16]):

$$z^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{(n-k)z}{n} \quad (n \geq 0; z \in \mathbb{C}).$$

By differentiation, we get

$$(-1)^n n z^{n-1} = \sum_{k=1}^{n} (-1)^k k \binom{n}{k} \binom{kz}{n} \sum_{j=0}^{n-1} \frac{1}{kz-j}. \tag{12}$$

We set $z = -1$ and $z = n \in \mathbb{N}$ in (12), respectively. This gives

$$1 = \sum_{k=1}^{n} (-1)^{n+k} \binom{n-1}{k-1} \binom{n+k-1}{n} (H_{n+k-1} - H_{k-1})$$

and

$$(-n)^n = \sum_{k=1}^{n} (-1)^k k \binom{n}{k} \binom{kn}{n} (H_{nk} - H_{n(k-1)}).$$

Remark 3. Since

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k} H_k = 1 + \frac{1}{2} \log^2(2) - \frac{1}{12} \pi^2,$$

see de Doelder [4], we obtain from (11) the series formula

$$\sum_{n=2}^{\infty} \frac{1}{n^{n+1}} \sum_{k=1}^{n-1} (-1)^{k-1} \frac{k^n}{n-k} \binom{n}{k} = \log(2) + \frac{1}{2} \log^2(2) - \frac{1}{12} \pi^2.$$

2.6. Hypergeometric Functions

The identity

$$\sum_{k=0}^{[m/2]} 2^{-2k} \binom{m-k}{k} \binom{m+x}{m-k} = 2^{-m} \binom{2m+2x}{m} \tag{13}$$

was published by Gould [6] in 1977. In 2023, Rathie and Lim [13] presented an interesting counterpart of (13). They used properties of hypergeometric functions to prove

$$\sum_{k=0}^{[m/2]} 2^{-2k} \frac{\binom{m}{2k}\binom{2k}{k}}{\binom{k+p}{k}} = 2^m \frac{(p+1/2)^{\overline{m}}}{(2p+1)^{\overline{m}}}. \tag{14}$$

Does there exist an identity which includes both formulas, (13) and (14), as special cases? The next theorem gives an affirmative answer.

Theorem 9. *Let $m \geq 1$ and $p \in \mathbb{C} \setminus \{-1,-2,...,-[m/2]\}$.*
(i) Let $x \in \mathbb{C} \setminus \{-1,-2,...,-[m/2]\}$. Then

$$\sum_{k=0}^{[m/2]} 2^{-2k} \frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k+p}{k}} = \frac{(x+1)^{\overline{m}}}{m!} {}_3F_2\Big(-\frac{m}{2}, \frac{1-m}{2}, 1; p+1, x+1; 1\Big). \tag{15}$$

(ii) Let $\nu \in \{1,2,...,[m/2]\}$. Then

$$\sum_{k=0}^{[m/2]} 2^{-2k} \frac{\binom{m-k}{k}\binom{m-\nu}{m-k}}{\binom{k+p}{k}} = 2^{-2\nu} \frac{(m-\nu)!}{m!} \frac{(-m)^{\overline{2\nu}}}{(p+1)^{\overline{\nu}}} {}_2F_1\Big(\nu-\frac{m}{2}, \nu-\frac{m-1}{2}; \nu+p+1; 1\Big). \tag{16}$$

Since

$${}_3F_2(a,b,1;c,1;1) = {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{17}$$

see Abramowitz and Stegun [1, (15.1.20)], we obtain

$${}_3F_2\Big(-\frac{m}{2}, \frac{1-m}{2}, 1; t+1, 1; 1\Big) = \frac{m!}{2^m (t+1)^{\overline{m}}}\binom{2m+2t}{m} = 2^m \frac{(t+1/2)^{\overline{m}}}{(2t+1)^{\overline{m}}}. \tag{18}$$

Using (18) and

$$\binom{m-k}{k}\binom{m}{m-k} = \binom{m}{2k}\binom{2k}{k}$$

we conclude that (15) with $p=0$ yields (13) and (15) with $x=0$ gives (14). The following corollary collects some special cases of (15).

Corollary 2. *Let $x \in \mathbb{C}$.*
(i) Let $m \geq 2$ be even. Then

$$\sum_{k=0}^{m/2} 2^{-2k} \frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k-m/2-1/2}{k}} = \begin{cases} \frac{x}{x+m/2}\binom{x+m}{m} & \text{if } x \neq -m/2, \\ 2^{-m}(-1)^{m/2}\frac{1}{\binom{(m-1)/2}{m/2}} & \text{if } x = -m/2. \end{cases} \tag{19}$$

(ii) Let $m \geq 1$ be odd. Then

$$\sum_{k=0}^{(m-1)/2} 2^{-2k} \frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k-m/2-1}{k}} = \begin{cases} \frac{x}{x+(m-1)/2}\binom{x+m}{m} & \text{if } x \neq (1-m)/2, \\ 2^{-m}(-1)^{(m-1)/2} \frac{m+1}{\binom{m/2}{(m-1)/2}} & \text{if } x = (1-m)/2. \end{cases} \tag{20}$$

(iii) Let $m \geq 1$. Then

$$\begin{aligned} &\sum_{k=0}^{[m/2]} 2^{-2k} \frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k-m+1/2-x}{k}} \\ &\quad = \begin{cases} \frac{x(x+m-1/2)}{(x+m/2)(x+(m-1)/2)}\binom{x+m}{m} & \text{if } x \neq -m/2, (1-m)/2, \\ 2^{-m}(-1)^{m/2} \frac{1}{\binom{(m-3)/2}{m/2}} & \text{if } m \text{ is even and } x = -m/2, \\ 2^{-m}(-1)^{(m-1)/2} \frac{m+1}{\binom{m/2-1}{(m-1)/2}} & \text{if } m \text{ is odd and } x = (1-m)/2. \end{cases} \end{aligned} \tag{21}$$

3. Proofs

Proof of Theorem 1. The proofs of (2) and (3) are similar, so that it suffices to consider (3). Let

$$F(x) = F_{j,n}(x) = \frac{1}{f(x)^j}\big[(1+f(x))^{n+1} - 1\big]. \tag{22}$$

The binomial formula gives for $y \neq 0$

$$\frac{1}{y^j}\big[(1+y)^{n+1} - 1\big] = \sum_{k=0}^{n} \binom{n+1}{k+1} y^{k+1-j}. \tag{23}$$

We use (23) with $y = f(x)$ and differentiate F r times. Then we conclude from (22):

$$F^{(r)}(x) = \sum_{k=0}^{n} \binom{n+1}{k+1} (f(x)^{k+1-j})^{(r)}. \tag{24}$$

Next, we apply (23) and the Leibniz formula. It follows that

$$F^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} \Big(\frac{1}{f(x)^j}\Big)^{(k)} \big[(1+f(x))^{n+1} - 1\big]^{(r-k)}. \tag{25}$$

From (24) and (25) we obtain (3). $\square$

In what follows, we set

$$f(x)=\frac{1-e^x}{x}\quad (x\neq 0),\quad f(0)=-1.$$

We need four lemmas. The following formula is given in Gradshteyn and Ryzhik [7, p. 1047].

Lemma 1. *Let $k\geq 0$ and $r\geq 0$. Then*

$$(f(x)^k)^{(r)}\Big|_{x=0}=(-1)^k\frac{S(r+k,k)}{\binom{r+k}{k}}.$$

Lemma 2. *Let $k\geq 0$. Then*

$$\Big(\frac{1}{f(x)}\Big)^{(k)}\Big|_{x=0}=-B_k.$$

Proof. We have

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!}.$$

It follows that

$$-\Big(\frac{1}{f(x)}\Big)^{(k)}\Big|_{x=0}=\Big(\frac{x}{e^x-1}\Big)^{(k)}\Big|_{x=0}=B_k.$$

□

Lemma 3. *Let $r\geq 0$. Then*

$$\big[(1+f(x))^{n+1}\big]^{(r)}\Big|_{x=0}=\begin{cases}0 & \text{if } 0\leq r\leq n,\\ (-1/2)^{n+1}(n+1)! & \text{if } r=n+1.\end{cases}\tag{26}$$

Proof. We have

$$1+f(x)=x\Big(-\sum_{\nu=0}^{\infty}\frac{x^\nu}{(\nu+2)!}\Big).$$

It follows that

$$(1+f(x))^{n+1}=(-1/2)^{n+1}x^{n+1}+a_1x^{n+2}+\dots.$$

This leads to (26). □

The next formula is due to Euler; see Chu and Zhou [3].

Lemma 4. *Let $r \geq 1$. Then*

$$\sum_{\nu=0}^{r} \binom{r}{\nu} B_\nu B_{r-\nu} = -rB_{r-1} - (r-1)B_r. \tag{27}$$

Proof of Theorem 2. We consider two cases.

`Case 1:` $0 \leq r \leq n$. Using Lemma 1 gives

$$\sum_{k=0}^{n} \binom{n+1}{k+1} (f(x)^k)^{(r)}\Big|_{x=0} = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{S(r+k,k)}{\binom{r+k}{k}}.$$

Next, we apply Lemmas 2 and 3. We obtain

$$\begin{aligned}
&\sum_{k=0}^{r} \binom{r}{k} \Big(\frac{1}{f(x)}\Big)^{(k)}\Big|_{x=0} \big[(1+f(x))^{n+1}-1\big]^{(r-k)}\Big|_{x=0} \\
&= \sum_{k=0}^{r-1} \binom{r}{k} \Big(\frac{1}{f(x)}\Big)^{(k)}\Big|_{x=0} \big[(1+f(x))^{n+1}-1\big]^{(r-k)}\Big|_{x=0} \\
&\qquad + \Big(\frac{1}{f(x)}\Big)^{(r)}\Big|_{x=0} \big[(1+f(x))^{n+1}-1\big]\Big|_{x=0} \\
&= \sum_{k=0}^{r-1} \binom{r}{k} (-B_k) \cdot 0 + (-B_r) \cdot (-1) = B_r.
\end{aligned}$$

From (3) with $j = 1$ we conclude that (4) holds.

`Case 2:` $r = n+1$. Using Lemma 1 gives

$$\sum_{k=0}^{n} \binom{n+1}{k+1} (f(x)^k)^{(n+1)}\Big|_{x=0} = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k \frac{S(k+n+1,k)}{\binom{k+n+1}{k}}$$

and using Lemmas 2 and 3 yields

$$\begin{aligned}
&\sum_{k=0}^{n+1} \binom{n+1}{k} \Big(\frac{1}{f(x)}\Big)^{(k)}\Big|_{x=0} \big[(1+f(x))^{n+1}-1\big]^{(n+1-k)}\Big|_{x=0} \\
&= \frac{1}{f(0)} \big[(1+f(x))^{n+1}\big]^{(n+1)}\Big|_{x=0} \\
&\qquad + \sum_{k=1}^{n} \binom{n+1}{k} \Big(\frac{1}{f(x)}\Big)^{(k)}\Big|_{x=0} \big[(1+f(x))^{n+1}-1\big]^{(n+1-k)}\Big|_{x=0} \\
&\qquad + \Big(\frac{1}{f(x)}\Big)^{(n+1)}\Big|_{x=0} \big[(1+f(0))^{n+1}-1\big] \\
&= -\Big(\frac{-1}{2}\Big)^{n+1} (n+1)! + 0 + B_{n+1}.
\end{aligned}$$

Applying (3) with $j = 1$ leads to (4). □

Proof of Theorem 3. We define

$$U(r,m) = \sum_{k=0}^{r+m} \binom{r+m+1}{k+1} V_k(r)$$

with

$$V_k(r) = (-1)^k \frac{S(r+k,k)}{\binom{r+k}{k}}.$$

From (4) with $n = r+m$ we obtain

$$B_r = U(r,m) \quad (r \geq 0; m \geq 0). \tag{28}$$

Let $m \geq 1$. We apply

$$\binom{N+1}{\nu} = \binom{N}{\nu} + \binom{N}{\nu-1} \tag{29}$$

with $N = r+m$, $\nu = k+1$. Then

$$\begin{aligned} U(r,m) &= \sum_{k=0}^{r+m} \left(\binom{r+m}{k+1} + \binom{r+m}{k} \right) V_k(r) \\ &= \sum_{k=0}^{r+m-1} \binom{r+m}{k+1} V_k(r) + \sum_{k=0}^{r+m} \binom{r+m}{k} V_k(r) \\ &= U(r,m-1) + \sum_{k=0}^{r+m} \binom{r+m}{k} V_k(r). \end{aligned} \tag{30}$$

From (28) and (30) we get (5). □

Proof of Theorem 4. We consider two cases.

`Case 1:` $1 \leq r \leq n$. Applying Lemmas 1 and 2 yields

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n+1}{k+1} (f(x)^{k-1})^{(r)}\big|_{x=0} \\ &= (n+1)\Big(\frac{1}{f(x)}\Big)^{(r)}\Big|_{x=0} + \sum_{k=0}^{n-1} \binom{n+1}{k+2} (f(x)^k)^{(r)}\big|_{x=0} \\ &= (n+1)(-B_r) + \sum_{k=0}^{n-1} \binom{n+1}{k+2} (-1)^k \frac{S(r+k,k)}{\binom{r+k}{k}}. \end{aligned} \tag{31}$$

The Cauchy product formula gives

$$\frac{1}{f(x)^2} = \Big(\frac{x}{e^x - 1}\Big)^2 = \sum_{m=0}^{\infty}\sum_{\nu=0}^{m}\binom{m}{\nu}B_\nu B_{m-\nu}\frac{x^m}{m!}.$$

Hence

$$\Big(\frac{1}{f(x)^2}\Big)^{(m)}\Big|_{x=0} = \sum_{\nu=0}^{m}\binom{m}{\nu}B_\nu B_{m-\nu}. \tag{32}$$

Using (32) and (26) gives

$$\begin{aligned}
&\sum_{k=0}^{r}\binom{r}{k}\Big(\frac{1}{f(x)^2}\Big)^{(k)}\Big|_{x=0}\big[(1+f(x))^{n+1}-1\big]^{(r-k)}\Big|_{x=0} \\
&= \sum_{k=0}^{r-1}\binom{r}{k}\Big(\frac{1}{f(x)^2}\Big)^{(k)}\Big|_{x=0}\big[(1+f(x))^{n+1}\big]^{(r-k)}\Big|_{x=0} \\
&\qquad + \Big(\frac{1}{f(x)^2}\Big)^{(r)}\Big|_{x=0}\big[(1+f(0))^{n+1}-1\big] \\
&= -\sum_{\nu=0}^{r}\binom{r}{\nu}B_\nu B_{r-\nu}. \qquad (33)
\end{aligned}$$

We apply (3) with $j = 2$. Then, from (31) and (33), we obtain

$$-(n+1)B_r + \sum_{k=0}^{n-1}(-1)^k\frac{\binom{n+1}{k+2}}{\binom{r+k}{k}}S(r+k,k) = -\sum_{\nu=0}^{r}\binom{r}{\nu}B_\nu B_{r-\nu}. \tag{34}$$

Using (27) and (34) we conclude that (6) holds.

`Case 2:` $r = n+1$. We have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n+1}{k+1}(f(x)^{k-1})^{(n+1)}\Big|_{x=0} \\
&= (n+1)\Big(\frac{1}{f(x)}\Big)^{(n+1)}\Big|_{x=0} + \sum_{k=1}^{n}\binom{n+1}{k+1}\Big|_{x=0}(f(x)^{k-1})^{(n+1)}\Big|_{x=0} \\
&= (n+1)(-B_{n+1}) + \sum_{k=1}^{n}\binom{n+1}{k+1}(-1)^{k-1}\frac{S(n+k,k-1)}{\binom{n+k}{k-1}}
\end{aligned}$$

and

$$\sum_{k=0}^{n+1}\binom{n+1}{k}\Big(\frac{1}{f(x)^2}\Big)^{(k)}\Big|_{x=0}\big[(1+f(x))^{n+1}-1\big]^{(n+1-k)}\Big|_{x=0}$$

$$\begin{aligned}
&= \frac{1}{f(0)^2}\left[(1+f(x))^{n+1}\right]^{(n+1)}\Big|_{x=0} \\
&\qquad + \sum_{k=1}^{n}\binom{n+1}{k}\Big(\frac{1}{f(x)^2}\Big)^{(k)}\Big|_{x=0}\left[(1+f(x))^{n+1}\right]^{(n+1-k)}\Big|_{x=0} \\
&\qquad + \Big(\frac{1}{f(x)^2}\Big)^{(n+1)}\Big|_{x=0}\left[(1+f(0))^{n+1}-1\right] \\
&= \Big(\frac{-1}{2}\Big)^{n+1}(n+1)! + 0 + \Big(\frac{1}{f(x)^2}\Big)^{(n+1)}\Big|_{x=0}\cdot(-1) \\
&= \Big(\frac{-1}{2}\Big)^{n+1}(n+1)! + (n+1)B_n + nB_{n+1}.
\end{aligned}$$

Applying (3) with $j=2$ we conclude that (6) is valid. □

Proof of Theorem 5. We define

$$g(x) = \frac{\log(1+x)}{x} \quad (-1 < x \neq 0), \quad g(0) = 1.$$

Then

$$g(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^k \quad (-1 < x \leq 1).$$

Let $m \geq 0$. Using the series representation

$$g(x)^m = m!\sum_{k=0}^{\infty}s(m+k,m)\frac{x^k}{(m+k)!},$$

(see Quaintance and Gould [12, (13.34)]) gives, for $r \geq 0$,

$$(g(x)^m)^{(r)}\Big|_{x=0} = \frac{s(m+r,m)}{\binom{m+r}{r}}.$$

From

$$(1-g(x))^{n+1} = x^{n+1}\big((1/2)^{n+1} + c_1x + \cdots\big),$$

we conclude that

$$[(1-g(x))^{n+1}-1]^{(r)}\Big|_{x=0} = \begin{cases} -1 & \text{if } r=0, \\ 0 & \text{if } 1 \leq r \leq n, \\ (n+1)!/2^{n+1} & \text{if } r = n+1. \end{cases}$$

Let $j \geq 0$ and $0 \leq r \leq n+1$. It follows that

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k+1}(g(x)^{k+1+j})^{(r)}\Big|_{x=0} \\
&\quad = \sum_{k=0}^{n}(-1)^{k+1}\frac{\binom{n+1}{k+1}}{\binom{k+1+j+r}{r}}s(k+1+j+r,k+1+j) \qquad (35)
\end{aligned}$$

and

$$\sum_{k=0}^{r}\binom{r}{k}\left(g(x)^j\right)^{(k)}\Big|_{x=0}\left[(1-g(x))^{n+1}-1\right]^{(r-k)}\Big|_{x=0}$$

$$=\begin{cases} -\frac{s(j+r,j)}{\binom{j+r}{r}} & \text{if } 0\le r\le n,\\ \frac{(n+1)!}{2^{n+1}}-\frac{s(j+n+1,j)}{\binom{j+n+1}{j}} & \text{if } r=n+1.\end{cases} \tag{36}$$

From (2), (35) and (36) we obtain (7). □

Proof of Theorem 6. Let $h(x)=\cosh(x)$. Then

$$\left(\frac{1}{h(x)}\right)^{(k)}\Big|_{x=0}=E_k.$$

From

$$h(x)^k=\left(\frac{e^{-x}+e^x}{2}\right)^k=\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}e^{(k-2\nu)x},$$

we obtain

$$(h(x)^k)^{(n)}\Big|_{x=0}=\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}(k-2\nu)^n=T(k,n).$$

Let

$$L_{k,n}(x)=[(1+h(x))^{n+1}-1]^{(n-k)}.$$

Then

$$L_{n,n}(0)=2^{n+1}-1.$$

Let $0\le k\le n-1$. We have

$$L_{n,k}(x)=[(1+h(x))^{n+1}]^{(n-k)}=\sum_{\nu=0}^{n+1}\binom{n+1}{\nu}(h(x)^\nu)^{(n-k)}=\sum_{\nu=0}^{n+1}\binom{n+1}{\nu}T(\nu,n-k).$$

Using (3) with $j=1$ and $r=n$ gives

$$\sum_{k=0}^{n}\binom{n+1}{k+1}T(k,n)=\sum_{k=0}^{n-1}\binom{n}{k}E_k\sum_{\nu=0}^{n+1}\binom{n+1}{\nu}T(\nu,n-k)+(2^{n+1}-1)E_n.$$ □

Proof of Theorem 7. Let $w(x)=2-e^x$. We obtain

$$\left(\frac{1}{w(x)}\right)^{(k)}\Big|_{x=0}=a(k)$$

and

$$(w(x)^k)^{(n)}\big|_{x=0}=2^kZ(k,n)$$

with

$$Z(k,n)=\sum_{\nu=0}^{k}\Big(-\frac{1}{2}\Big)^{\nu}\binom{k}{\nu}\nu^{n}.$$

Let

$$R_{k,n}(x)=\big[(1+w(x))^{n+1}-1\big]^{(n-k)}.$$

We have

$$R_{n,n}(0)=2^{n+1}-1,$$

and if $0\le k\le n-1$, then

$$R_{k,n}(0)=\big[(1+w(x))^{n+1}\big]^{(n-k)}\big|_{x=0}=\sum_{\nu=0}^{n+1}\binom{n+1}{\nu}2^{\nu}Z(\nu,n-k).$$

Applying (3) with $j=1$ and $r=n$ yields

$$\sum_{k=0}^{n}\binom{n+1}{k+1}2^{k}Z(k,n)=\sum_{k=0}^{n-1}\binom{n}{k}a(k)\sum_{\nu=0}^{n+1}\binom{n+1}{\nu}2^{\nu}Z(\nu,n-k)+(2^{n+1}-1)a(n). \tag{37}$$

In Gould [5, (1.126)] we find

$$\sum_{\nu=0}^{n}\binom{n}{\nu}\nu^{r}x^{\nu}=(1+x)^{n}\sum_{j=0}^{r}(-1)^{j}\binom{n}{j}\Big(\frac{x}{x+1}\Big)^{j}\sum_{k=0}^{j}(-1)^{k}\binom{j}{k}k^{r}. \tag{38}$$

Using (38) with $x=-1/2$ and

$$S(r,j)=\frac{1}{j!}\sum_{k=0}^{j}(-1)^{j+k}\binom{j}{k}k^{r}$$

gives

$$2^{n}Z(n,r)=\sum_{j=0}^{r}\binom{n}{j}(-1)^{j}j!S(r,j). \tag{39}$$

From (37) and (39) we conclude that (8) holds. $\square$

Proof of Theorem 8. We use induction on p. If $p=1$, then both sides of (9) are equal to 0. Next, we assume that (9) is valid. We denote the sum on the right-hand side of (9) by $S_p(z)$. Applying the induction hypothesis gives

$$z^{p+1}-z^{\underline{p+1}}=z^{p+1}-(p+1)!\binom{z}{p+1}=pz^{p}+(z-p)S_p(z). \tag{40}$$

We have

$$(z-p)\binom{z-k-1}{p-k-1}=(p-k)\binom{z-k-1}{p-k}. \tag{41}$$

Using (40) and (41) yields

$$z^{p+1} - z^{\underline{p+1}} - S_{p+1}(z) = pT_p(z), \tag{42}$$

where

$$T_p(z) = z^p - p^p\binom{z}{p} - \sum_{k=0}^{p-1}(-1)^{p-k}k^p\binom{z}{k}\binom{z-k-1}{p-k}.$$

Next, we apply (9), (29), and the two formulas

$$\binom{z}{k}\binom{z-k}{p-k} = \binom{p}{k}\binom{z}{p}, \quad \sum_{k=0}^{n}(-1)^k k^n\binom{n}{k} = (-1)^n n!;$$

see Gould [5, (1.13)]. Then we obtain

$$\begin{aligned}
T_p(z) &= \big(p! - p^p\big)\binom{z}{p} - \sum_{k=0}^{p-1}(-1)^{p-k}k^p\binom{z}{k}\left[\binom{z-k-1}{p-k-1} + \binom{z-k-1}{p-k}\right] \\
&= \big(p! - p^p\big)\binom{z}{p} - \sum_{k=0}^{p-1}(-1)^{p-k}k^p\binom{z}{k}\binom{z-k}{p-k} \\
&= \big(p! - p^p\big)\binom{z}{p} - \binom{z}{p}\sum_{k=0}^{p-1}(-1)^{p-k}k^p\binom{p}{k} \\
&= \binom{z}{p}\left[p! - \sum_{k=0}^{p}(-1)^{p-k}k^p\binom{p}{k}\right] = 0. \qquad (43)
\end{aligned}$$

From (42) and (43) we conclude that (9) holds with $p+1$ instead of p. □

Proof of Corollary 1. Let $L_p(z)$ and $R_p(z)$ be the expressions given on the left-hand side and on the right-hand side of (9), respectively. We have

$$\frac{d}{dz}\binom{z}{p} = \binom{z}{p}\sum_{k=0}^{p-1}\frac{1}{z-k}. \tag{44}$$

Applying (44) gives

$$L_p'(z) = pz^{p-1} - p!\binom{z}{p}\sum_{k=0}^{p-1}\frac{1}{z-k} \tag{45}$$

and

$$\begin{aligned}
R_p'(z) &= \sum_{k=0}^{p-1}(-1)^{p-k-1}k^p\binom{z}{k}\binom{z-k-1}{p-k-1}\left[\sum_{j=0}^{p-1}\frac{1}{z-j} - \frac{1}{z-k}\right] \\
&= \left(z^p - p!\binom{z}{p}\right)\sum_{j=0}^{p-1}\frac{1}{z-j} + \sum_{k=0}^{p-1}(-1)^{p-k}\frac{k^p}{z-k}\binom{z}{k}\binom{z-k-1}{p-k-1}. \qquad (46)
\end{aligned}$$

Since $L_p'(z) = R_p'(z)$, we conclude from (45) and (46) that (10) holds. □

Proof of Theorem 9. (i) Let $0 \le k \le [m/2]$ and $p, x \notin \{-1, -2, ..., -[m/2]\}$. We have

$$\binom{m-k}{k}\binom{m+x}{m-k} = \frac{(-m)^{\overline{2k}}}{m!k!}\frac{(x+1)^{\overline{m}}}{(x+1)^{\overline{k}}}$$

and

$$\binom{k+p}{k} = \frac{(p+1)^{\overline{k}}}{k!}, \quad 2^{-2k}(-m)^{\overline{2k}} = \Big(\frac{-m}{2}\Big)^{\overline{k}}\Big(\frac{1-m}{2}\Big)^{\overline{k}}.$$

It follows that

$$\begin{aligned}\sum_{k=0}^{[m/2]} 2^{-2k}\frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k+p}{k}} &= \frac{(x+1)^{\overline{m}}}{m!}\sum_{k=0}^{[m/2]}\frac{(-m/2)^{\overline{k}}((1-m)/2)^{\overline{k}}}{(p+1)^{\overline{k}}(x+1)^{\overline{k}}} \\ &= \frac{(x+1)^{\overline{m}}}{m!}\,{}_3F_2\Big(-\frac{m}{2}, \frac{1-m}{2}, 1; p+1, x+1; 1\Big).\end{aligned}$$

(ii) Let $\nu \in \{1, ..., [m/2]\}$. We use the limit relation

$$\begin{aligned}\lim_{\sigma\to -n}\frac{1}{\Gamma(\sigma)}\,{}_3F_2(a,b,c;d,\sigma;z) &= \frac{z^{n+1}}{(n+1)!}\frac{a^{\overline{n+1}}b^{\overline{n+1}}c^{\overline{n+1}}}{d^{\overline{n+1}}} \\ &\quad \times {}_3F_2(a+n+1, b+n+1, c+n+1; d+n+1, n+2; z),\end{aligned}$$

where n is a nonnegative integer; see Prudnikov et al. [11, p. 438]. Then we obtain

$$\begin{aligned}&\lim_{x\to -\nu}\frac{(x+1)^{\overline{m}}}{m!}\,{}_3F_2\Big(-\frac{m}{2}, \frac{1-m}{2}, 1; p+1, x+1; 1\Big) \\ &= \lim_{x+1\to 1-\nu}\frac{\Gamma(x+1+m)}{m!}\cdot\frac{1}{\Gamma(x+1)}\,{}_3F_2\Big(-\frac{m}{2}, \frac{1-m}{2}, 1; p+1, x+1; 1\Big) \\ &= \frac{\Gamma(1-\nu+m)}{m!}\cdot\frac{(-m/2)^{\overline{\nu}}((1-m)/2)^{\overline{\nu}}1^{\overline{\nu}}}{\nu!(p+1)^{\overline{\nu}}} \\ &\qquad \times {}_3F_2\Big(-\frac{m}{2}+\nu, \frac{1-m}{2}+\nu, 1+\nu; p+1+\nu, \nu+1; 1\Big) \\ &= 2^{-2\nu}\frac{(m-\nu)!}{m!}\frac{(-m)^{\overline{2\nu}}}{(p+1)^{\overline{\nu}}}\,{}_2F_1\Big(\nu-\frac{m}{2}, \nu-\frac{m-1}{2}; \nu+p+1; 1\Big). \qquad (47)\end{aligned}$$

From (15) and (47) we conclude that (16) is valid. □

Proof of Corollary 2. (i) Let $m \ge 2$ be even. First, let $x \notin \{-1, ..., -m/2\}$. Using

(15) with $p = -(m+1)/2$ and the second formula in (17) gives

$$\begin{aligned}\sum_{k=0}^{m/2} 2^{-2k} \frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k-m/2-1/2}{k}} &= \frac{(x+1)^{\overline{m}}}{m!}\,{}_3F_2\Big(-\frac{m}{2}, \frac{1-m}{2}, 1; \frac{1-m}{2}, x+1; 1\Big)\\ &= \frac{(x+1)^{\overline{m}}}{m!}\,{}_2F_1\Big(-\frac{m}{2}, 1; x+1; 1\Big)\\ &= \frac{x}{x+m/2}\binom{x+m}{m}.\end{aligned}$$

By continuity, we conclude that (19) holds for all $x \neq -m/2$. Since

$$\lim_{x\to -m/2} \frac{x}{x+m/2}\binom{x+m}{m} = 2^{-m}(-1)^{m/2}\frac{1}{\binom{(m-1)/2}{m/2}},$$

we obtain (19) for $x = -m/2$.

(ii) Let $m \geq 1$ be odd and $x \notin \{-1, ..., -(m-1)/2\}$. We apply (15) with $p = -m/2 - 1$. This gives

$$\begin{aligned}\sum_{k=0}^{(m-1)/2} 2^{-2k} \frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k-m/2-1}{k}} &= \frac{(x+1)^{\overline{m}}}{m!}\,{}_2F_1\Big(\frac{1-m}{2}, 1; x+1; 1)\\ &= \frac{x}{x+(m-1)/2}\binom{x+m}{m}.\end{aligned}$$

By continuity, (20) holds for all $x \neq (1-m)/2$. Using the limit relation

$$\lim_{x\to(1-m)/2} \frac{x}{x+(m-1)/2}\binom{x+m}{m} = 2^{-m}(-1)^{(m-1)/2}\frac{m+1}{\binom{m/2}{(m-1)/2}},$$

we conclude that (20) is also valid for $x = (1-m)/2$.

(iii) Let $m \geq 1$, $x \in \mathbb{C}$, $p = 1/2 - m - x$. First, we assume that

$$p \notin \{-1, ..., -[m/2]\} \quad \text{and} \quad x \notin \{-1, ..., -[m/2]\}.$$

It follows that $x \neq -m/2$ and $x \neq (1-m)/2$. We have

$${}_3F_2(-n, a, b; c, a+b-c-n+1; 1) = \frac{(c-a)^{\overline{n}}(c-b)^{\overline{n}}}{c^{\overline{n}}(c-a-b)^{\overline{n}}}; \tag{48}$$

see Prudnikov et al. [11, (7.4.4.88)]. To prove (21) we consider two cases.

`Case 1:` $m \geq 2$ is even. We apply (48) with

$$n = \frac{m}{2}, \quad a = \frac{1-m}{2}, \quad b = 1, \quad c = x+1,$$

and (15) with $p = 1/2 - m - x$. This gives

$$\sum_{k=0}^{[m/2]} 2^{-2k} \frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k-m+1/2-x}{k}} = \frac{(x+1)^{\overline{m}}}{m!} \frac{(x+(m+1)/2))^{\overline{m/2}}}{(x+(m-1)/2)^{\overline{m/2}}} \frac{x^{\overline{m/2}}}{(x+1)^{\overline{m/2}}}$$
$$= \frac{x(x+m-1/2)}{(x+m/2)(x+(m-1)/2)}\binom{x+m}{m}. \tag{49}$$

`Case 2:` $m \geq 1$ is odd. Using (48) with

$$n = \frac{m-1}{2}, \quad a = -\frac{m}{2}, \quad b = 1, \quad c = x+1,$$

and (15) with $p = 1/2 - m - x$ leads to

$$\sum_{k=0}^{[m/2]} 2^{-2k} \frac{\binom{m-k}{k}\binom{m+x}{m-k}}{\binom{k-m+1/2-x}{k}} = \frac{(x+1)^{\overline{m}}}{m!} \frac{(x+m/2+1)^{\overline{(m-1)/2}}}{(x+m/2)^{\overline{(m-1)/2}}} \frac{x^{\overline{(m-1)/2}}}{(x+1)^{\overline{(m-1)/2}}}$$
$$= \frac{x(x+m-1/2)}{(x+m/2)(x+(m-1)/2)}\binom{x+m}{m}. \tag{50}$$

By continuity, we conclude from (49) and (50) that (21) is valid for all $x \neq -m/2$ and $x \neq (1-m)/2$.

The limit relation for m even

$$\lim_{x\to -m/2} \frac{x(x+m-1/2)}{(x+m/2)(x+(m-1)/2)}\binom{x+m}{m} = 2^{-m}(-1)^{m/2}\frac{1}{\binom{(m-3)/2}{m/2}},$$

and the limit relation for m odd

$$\lim_{x\to (1-m)/2} \frac{x(x+m-1/2)}{(x+m/2)(x+(m-1)/2)}\binom{x+m}{m} = 2^{-m}(-1)^{(m-1)/2}\frac{m+1}{\binom{m/2-1}{(m-1)/2}},$$

give that (21) also holds if m is even and $x = -m/2$ and if m is odd and $x = (1-m)/2$. This completes the proof of Corollary 2. $\square$

Acknowledgement. We thank Bruce Landman and the referee for helpful comments.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1964.

[2] M. B. Can and M. Joyce, Ordered Bell numbers, Hermite polynomials, skew Young tableaux, and Borel orbits, *J. Combin. Theory Ser. A* **119** (2012), 1798-1810.

[3] W. Chu and R. R. Zhou, Convolutions of Bernoulli and Euler polynomials, *Sarajevo J. Math.* **6** (2010), 147-163.

[4] P. J. de Doelder, On some series containing $\psi(x) - \psi(y)$ and $(\psi(x) - \psi(y))^2$ for certain values of x and y, *J. Comput. Appl. Math.* **37** (1991), 125-141.

[5] H. W. Gould, *Combinatorial Identities: A standardized set of tables listing 500 binomial coefficient summations*, Morgantown, W. Va., 1972.

[6] H. W. Gould, Generalizations of a formula of Touchard for Catalan numbers, *J. Combin. Theory Ser. A* **23** (1977), 351-353.

[7] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 7-th ed., Elsevier, Amsterdam, 2007.

[8] S. K. Jha, A new explicit formula for Bernoulli numbers involving the Euler number, *Mosc. J. Comb. Number Theory* **8** (2019), 385-387.

[9] R. K. Muthumalai, A note on Bernoulli numbers, *Notes Numb. Th. Disc. Math.* **19** (2013), 59-65.

[10] *NIST Handbook of Mathematical Functions*, F. W. J. Olver et al. (eds.), Camb. Univ. Press, 2010.

[11] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series, vol. 1: Elementary Functions*, Gordon and Breach Science Publishers, New York, 1986.

[12] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers: The unpublished Notes of H.W. Gould*, World Sci., New Jersey, 2016.

[13] A. K. Rathie and D. Lim, A note on generalization of combinatorial identities due to Gould and Touchard, *Axioms* **12** (2023), 268.

[14] T. Sabar, A note on the difference of powers and falling powers, *Int. J. Math. Math. Sci.* 2021, Article ID 5593780, 4 pp.

[15] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994), 267-290.

[16] M. V. Vassilev-Missana, Note on some identities related to binomial coefficients, *Notes Numb. Th. Disc. Math.* **11** (2005), 9-12.