MODULAR GENERALIZATIONS OF SCHUR NUMBERS

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Abstract

Let $k, l, m \in \mathbb{Z}^+$. The generalized Schur number modulo m, $S_m(k, l)$, is the greatest $N \in \mathbb{Z}^+$ such that there exists a partition of $S = \{1, \ldots, N\}$ into k subsets, where in each subset S_i, if $x_1, \ldots, x_l \in S_i$ and $x_1 + \cdots + x_l \equiv y \pmod{m}$, then $y \notin S_i$. Note that the x_j's need not be distinct. The values of $S_m(k, l)$ for $m = 1, 2, 3$ were explicitly determined by Chappelon et al. in 2013. In this paper, we obtain new general results for $S_m(k, l)$ and $S_{p^i}(k, l)$ where p is a prime and $i \geq 1$ is an integer. Using these results, we determine the exact values of $S_m(k, l)$ for $m = 4, 5, 6, 7$.

1. Introduction

In 1916, Schur [11] showed that in every partition of the set of integers from 1 up to $\lfloor k!e \rfloor$ into k subsets, there exists a subset containing the numbers x, y, z satisfying $x + y = z$.

The *Schur number*, denoted by $S(k)$, is the greatest $N \in \mathbb{Z}^+$ such that the discrete interval $[1, N]$ admits a partition into k subsets, none of which contains numbers x, y, z satisfying the equation $x + y = z$. Schur's result [11] shows that

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$S(k)$ is always finite for all positive integers k. The problem of determining the values of $S(k)$ was posed by Guy [6, Problem E11]. This problem is difficult in general, and after more than a hundred years since Schur's original work, we only know the values of $S(k)$ for integers $1 \leq k \leq 5$.

Guy also posed a version of this problem in modular arithmetic [6, Problem E12]. This idea was originally conceived by Abbott and Wang [1] in 1977 with the intention to study new problems on sum-free sets of integers. They defined and studied the numbers $T(k)$, which is the greatest $N \in \mathbb{Z}^+$ such that the discrete interval $[1, N]$ admits a partition into k subsets, none of which contains numbers x, y, z satisfying

$$x + y \equiv z \pmod{N+1}.$$

They were interested in the relationship between $T(k)$ and $S(k)$ and demonstrated that $T(k) = S(k)$ for integers $1 \leq k \leq 4$. This led them to conjecture that equality is true for all $k \in \mathbb{Z}^+$; the recent work of Heule [7, p. 6599] confirms this for $k = 5$.

Inspired by the problem of Guy and Abbott and Wang, a formal study of modular generalizations of the Schur numbers was initiated by Chappelon, Marchena, and Domínguez [4]. The purpose of this paper is to further investigate such generalizations and obtain bounds as well as explicit values.

1.1. Definitions and Recent Developments

Let $k \in \mathbb{Z}^+$. Partition a set of integers S into k subsets $S_1, \ldots, S_k$ and set $P = \{S_1, \ldots, S_k\}$, where $S_i \cap S_j = \emptyset$ for all $i \neq j$. We call P a *k-partition* of S. Let $l \in \mathbb{Z}^+$. Then S is called *l-sum-free* if $x_1, \ldots, x_l \in S$ and $x_1 + \cdots + x_l = y$ imply $y \notin S$. The variables $x_1, \ldots, x_l$ need not be distinct and some of the S_i could be empty sets.

The *generalized Schur number*, $S(k, l)$ is the greatest $N \in \mathbb{Z}^+$ such that $[1, N]$ has a k-partition into l-sum-free sets. Clearly, $S(k, 2) = S(k)$. The following table [2] summarizes the latest values and lower bounds of the Schur numbers (see also [5, 7, 9, 10]).

k	1	2	3	4	5	6	7	8	9	10	11	12
$S(k,2)$	1	4	13	44	160	≥ 536	$\geq 1\,696$	$\geq 5\,286$	$\geq 17\,803$	$\geq 60\,948$	$\geq 203\,828$	$\geq 644\,628$

Table 1: Latest values and lower bounds of $S(k, 2)$

For a set of integers S and $m \in \mathbb{Z}^+$, if $x_1, \ldots, x_l \in S$ and $x_1 + \cdots + x_l \equiv y \pmod{m}$ imply $y \notin S$, then we say that the set S is *l-sum-free modulo m*. The elements $x_1, \ldots, x_l \in S$ need not be distinct. The *generalized Schur number modulo m*, $S_m(k, l)$ is the greatest $N \in \mathbb{Z}^+$ such that the discrete interval $[1, N]$ has a k-partition into l-sum-free sets modulo m.

The generalized Schur number and its modular counterpart are intrinsically connected by the bound

$$S_m(k,l) \leq S(k,l),$$

for a set that is l-sum-free modulo m is also l-sum-free; the converse is not true. For example, the set $\{1,4,6\}$ is both 2-sum-free modulo 5 and 2-sum-free. In contrast, the set $\{1,4,13\}$ is 2-sum-free but not 2-sum-free modulo 5 because $1+13 \equiv 4 \pmod 5$. The numbers $S_m(k,l)$ also have a trivial (but useful) upper bound shown in [4]. Note that, for every positive integer m, the sum of l copies of m is congruent to m modulo m. It follows that, if a set X is l-sum-free modulo m, then m is not a member of X. For this reason,

$$S_m(k,l) \leq m-1. \tag{1}$$

It is worth mentioning that the modular Schur numbers defined here are related to the numbers $T(k)$ defined by Abbott and Wang. As pointed out in [4] we have the following equality from the definition of these numbers:

$$T(k) = \max\{N \in \mathbb{Z}^+ : S_{N+1}(k,2) = N\}.$$

So, the problem of determining $T(k)$ is the same as determining the maximum $m = N+1$ for which Inequality (1) holds for a fixed k. We do not pursue the problem of determining $T(k)$ in this paper.

Chappelon, Marchena, and Domíguez explicitly determined the values of $S_m(k,l)$ for $m = 1,2,3$. For $m=1$, the set $\{x\}$ is never l-sum-free modulo 1 because the sum of l copies of every integer x is congruent to x modulo 1. So $S_1(k,l) = 0$ for all $k,l \geq 1$. It is also clear that if $l = 1$, then $S_m(k,1) = 0$ for all positive integers k,m. The values of $S_m(k,l)$ for $m = 2,3$ are as follows.

Theorem 1 ([4]). *Let $k,l \in \mathbb{Z}^+$. Then $S_2(k,l) = 0$ when $l \equiv 1 \pmod 2$, and $S_2(k,l) = 1$ when $l \equiv 0 \pmod 2$.*

Theorem 2 ([4]). *Let $k,l \in \mathbb{Z}^+$. If $l \equiv 1 \pmod 3$, then $S_3(k,l) = 0$. Furthermore, for l not congruent to 1 modulo 3,*

1. *$S_3(k,l) = 1$ for $k = 1$ and $l \equiv 0,2 \pmod 3$,*
2. *$S_3(k,l) = 2$ for $k \geq 2$ and $l \equiv 0,2 \pmod 3$.*

It was mentioned at the end of [4] that it is difficult to determine the values of $S_m(k,l)$ when m is large. In this paper, we obtain new exact values for $S_m(k,l)$. In Section 2, we obtain a condition for when a singleton is not l-sum-free modulo m and deduce general values of $S_m(k,l)$ as a consequence. Our most general result (Theorem 8) gives the explicit values of $S_{p^i}(k,l)$ where p is prime and $l \geq p^i - 1$. Using these results, we completely determine previously unknown values of $S_m(k,l)$ for $m = 4$, 5, 6, and 7 (see Theorems 6, 7, 9, and 10, respectively).

Our results may have potential applications in theoretical physics.[1] In [8], Martin-Delgado proposed a *correlated exclusion principle*: if two particles occupy states corresponding to the quantum numbers x and y in the energy level $\mathcal{E}$, then a particle with quantum number $x + y$ cannot be in $\mathcal{E}$. This is mathematically equivalent to saying that $\mathcal{E}$ is 2-sum-free. For example, say we have particles with quantum numbers 1 to 13, and we want to fill the energy levels $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ while obeying the correlated exclusion principle. This is the same as constructing a 3-partition of $[1, 13]$ into 2-sum-free sets; such a construction is possible because $S(3, 2) = 13$, e.g.,

$$\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\} = \{\{1, 4, 10, 13\}, \{2, 3, 7, 11, 12\}, \{5, 6, 8, 9\}\}.$$

According to the paper, physically realizing such quantum systems require a lot of resources due to how $S(k, 2)$ grows. Thus, the construction of these quantum systems with modular Schur numbers and modular sum-free sets were suggested because modular arithmetic has a natural "cut-off" effect.

2. Some General Results for $S_m(k, l)$

To determine $S_m(k, l)$ for general moduli m, it is beneficial to know for what values of l is a set l-sum-free modulo m. Since we are allowed to repeatedly sum a single number, we begin by determining when a singleton $\{a\}$ is not l-sum-free modulo m using the following well-known result from the theory of congruences.

Theorem 3 ([3])**.** *Let a, b, m, x be integers such that $m > 0$. Let $d = \gcd(a, m)$. Then the linear congruence $ax \equiv b \pmod m$ has a solution if and only if $d \mid b$. If $d \mid b$, then it has exactly d mutually incongruent solutions modulo m. If x_0 is a solution, then all the d mutually incongruent solutions are given by $x \equiv x_0 + u(m/d) \pmod m$, where $0 \leq u \leq d - 1$.*

Theorem 4. *Let a, l, m be positive integers such that $a \not\equiv 0 \pmod m$. Suppose $d = \gcd(a, m)$. Then the following hold.*

1. *If $d = 1$, then $\{a\}$ is not l-sum-free modulo m if and only if $l \equiv 1 \pmod m$.*
2. *If $d > 1$, then $\{a\}$ is not l-sum-free modulo m if and only if $l \equiv 1 \pmod{m/d}$.*

Proof. Let $d = 1$. Suppose $\{a\}$ is not l-sum-free modulo m. Then

$$\sum_{i=1}^{l} a = \overbrace{a + a + \cdots + a}^{l \text{ times}} = al \equiv a \pmod m.$$

[1] We will need to borrow some terminology from physics but we will not give a rigorous physical explanation.

By Theorem 3, since $d = 1$, the linear congruence $al \equiv a \pmod{m}$ has a unique solution $l \equiv 1 \pmod{m}$. The converse is clear by the definition of an l-sum-free set modulo m.

Now, let $d > 1$. If $\{a\}$ is not l-sum-free modulo m, then $\sum_{i=1}^{l} a = al \equiv a \pmod{m}$. By Theorem 3, since $d > 1$, $al \equiv a \pmod{m}$ has exactly d incongruent solutions modulo m. Clearly $l \equiv 1 \pmod{m}$ is a solution and all the d incongruent solutions modulo m are given by $l \equiv 1 + u(m/d) \pmod{m}, 0 \leq u \leq d-1$. Hence, $l \equiv 1 \pmod{m/d}$. The converse is clear by the definition of an l-sum-free set modulo m. □

When $a = 1$, Theorem 4 gives the following corollary which simplifies our argument for the case where $l \equiv 1 \pmod{m}$.

Corollary 1. *Let $m > 0$ be an integer. The set $\{1\}$ is not l-sum-free modulo m if and only if $l \equiv 1 \pmod{m}$.*

Since $0 \leq S_m(k,l) \leq m-1$, by Corollary 1, if $l \equiv 1 \pmod{m}$, the set $\{1\}$ is not l-sum-free modulo m. Hence, the next corollary follows.

Corollary 2. *Let k, l, m be positive integers. Then $S_m(k,l) = 0$ if and only if $l \equiv 1 \pmod{m}$.*

Corollary 3. *Let k, l, m, a be positive integers and $d = \gcd(a, m)$. If $1 \leq a \leq m-1$ and $l \equiv 1 \pmod{m/d}$, then $S_m(k,l) < a$.*

Proof. By Theorem 4, the set $\{a\}$ is not l-sum-free modulo m. Therefore, every set containing a is not l-sum-free modulo m. □

Now the values of $S_m(k,l)$ are known for $m = 1, 2, 3$, as stated in Theorems 1 and 2. So we consider those cases where m is greater than 3. If m is an even modulus, we have the following.

Corollary 4. *Let k, l, n be positive integers such that $k, n \geq 2$ and $l \not\equiv 1 \pmod{2n}$. Then $S_{2n}(k,l) = 1$ when $l \equiv n+1 \pmod{2n}$. Furthermore, $S_{2n}(k,l) \geq 2$ when $l \not\equiv n+1 \pmod{2n}$.*

Proof. Note that $\gcd(2, 2n) = 2$. If $l \equiv n+1 \pmod{2n}$, then $l \equiv 1 \pmod{n}$. By Corollary 3, $S_{2n}(k,l) < 2$. Thus, $S_{2n}(k,l) = 1$ follows from Corollary 2, because $l \equiv n+1 \not\equiv 1 \pmod{2n}$.

Now for the second assertion. Suppose $l \not\equiv n+1 \pmod{2n}$. Then by part 2 of Theorem 4, $\{2\}$ is l-sum-free modulo $2n$. This implies that $\{\{1\}, \{2\}\}$ is a 2-partition of $[1, 2]$ into l-sum-free-sets modulo $2n$. Therefore, $S_{2n}(k,l) \geq 2$. □

Corollary 5. *Let k, l, m be positive integers such that $l, m \geq 2$. Suppose $l \not\equiv 1 \pmod{d}$ for all positive divisors d of m and $d \neq 1$. If $k \geq m-1$, then $S_m(k,l) = m-1$.*

Proof. Let $1 \leq a \leq m-1$ and $\gcd(a,m) = d_1$. By our assumption, $l \not\equiv 1 \pmod{m/d_1}$. So, by part 2 of Theorem 4, $\{a\}$ is l-sum-free modulo m. Therefore, $\{\{1\},\{2\},\ldots,\{m-1\}\}$ is a $(m-1)$-partition of $[1,m-1]$ into l-sum-free-sets modulo n. So, $S_m(k,l) \geq m-1$. The corollary then follows from Inequality (1). □

For $k = 1$, we have the following.

Theorem 5. *Let l and m be positive integers such that $l, m \geq 2$. Suppose $l \not\equiv 1 \pmod{m}$. Then $S_m(1,l) = 1$ if and only if $l = 2$ or $l \geq \frac{m+1}{2}$.*

Proof. By Corollary 2, $S_m(1,l) \geq 1$.

Suppose $l = 2$ or $l \geq \frac{m+1}{2}$. If $l = 2$, clearly, $1 + 1 \equiv 2 \pmod{m}$. Thus, $S_m(1,l) < 2$ and we conclude that $S_m(1,l) = 1$. If $\frac{m+1}{2} \leq l \leq m+1$, then

$$\overbrace{1+\cdots+1}^{2l-m-1}+\overbrace{2+\cdots+2}^{m+1-l} \equiv 1 \pmod{m}.$$

If $l \geq m+1$, by letting $l = am + b$ where $a \geq 1$ and $0 \leq b < m$, we have

$$\overbrace{1+\cdots+1}^{l-m+b-1}+\overbrace{2+\cdots+2}^{m-b+1} \equiv 1 \pmod{m}.$$

Therefore, $S_m(1,l) < 2$ for $l \geq \frac{m+1}{2}$. Hence, $S_m(1,l) = 1$.

Suppose $S_m(1,l) = 1$. If $l = 2$, we are done. Suppose $l \neq 2$. There exist positive integers k_1, k_2 such that $k_1 + k_2 = l$ and

$$\overbrace{1+\cdots+1}^{k_1}+\overbrace{2+\cdots+2}^{k_2} \equiv 1 \text{ or } 2 \pmod{m},$$

i.e., $k_1 + 2k_2 = 1 + km$ or $2 + km$ for some $k \geq 0$. Since $l \geq 3$, we must have $k \geq 1$. So, $k_1 = 2(k_1+k_2) - (k_1+2k_2) = 2l - 1 - km$ or $2l - 2 - km$. In either case, $l \geq \frac{km+1}{2} \geq \frac{m+1}{2}$, for $k_1 \geq 0$. □

We will apply these results to determine the values of $S_4(k,l)$ and $S_6(k,l)$ in the next two sections.

3. Establishing $S_4(k,l)$

We will prove the following theorem.

Theorem 6. *Let $k, l \in \mathbb{Z}^+$. If $l \equiv 1 \pmod{4}$, then $S_4(k,l) = 0$. Furthermore, for l not congruent to 1 modulo 4,*

1. $S_4(k,l) = 1$ *for* $\begin{cases} k = 1, \\ k \geq 2 \text{ and } l \equiv 3 \pmod{4}, \end{cases}$

2. $S_4(k,l) = 3$ *for* $k \geq 2$ *and* $l \equiv 0, 2 \pmod 4$.

Proof. By Corollary 2, if $l \equiv 1 \pmod 4$, then $S_4(k,l) = 0$. From here onwards, we shall assume that $l \not\equiv 1 \pmod 4$.

Suppose $k = 1$. Now, either $l = 2$ or $l \geq 3 > \frac{4+1}{2} = \frac{5}{2}$. So, by Theorem 5, $S_4(1,l) = 1$. Additionally, by Corollary 4, $S_4(k,l)$ equals 1 when $k \geq 2, l \equiv 3 \pmod 4$.

Suppose $k = 2$. We claim that $\{\{2\},\{1,3\}\}$ is a 2-partition of $[1,3]$ into l-sum-free sets modulo 4 when $l \equiv 0, 2 \pmod 4$. Note that $\sum_{i=1}^{l} 2 = 2l \equiv 0 \pmod 4$ when $l \equiv 0, 2 \pmod 4$. So $\{2\}$ is l-sum-free modulo 4. Next, $1, 3 \equiv 1 \pmod 2$ implies that

$$\sum_{i=1}^{l-t} 1 + 3t = l + 2t \equiv l \equiv 0 \pmod 2.$$

So $\{1,3\}$ is l-sum-free modulo 4. Thus, $S_4(2,l) \geq 3$ when $l \equiv 0, 2 \pmod 4$. Equality follows from Inequality (1).

Let $k \geq 3$. Since $l \equiv 0, 2 \pmod 4$, we have $l \not\equiv 1 \pmod d$ for all positive divisors d of 4 and $d \neq 1$. By Corollary 5, $S_4(k,l)$ equals 3. □

4. Establishing $S_6(k,l)$

We will prove the following theorem.

Theorem 7. *Let* $k, l \in \mathbb{Z}^+$. *If* $l \equiv 1 \pmod 6$, *then* $S_6(k,l) = 0$. *Furthermore, for* l *not congruent to* 1 *modulo* 6,

1. $S_6(k,l) = 1$ *for* $\begin{cases} k = 1 \text{ and } l \neq 3, \\ k \geq 2 \text{ and } l \equiv 4 \pmod 6, \end{cases}$
2. $S_6(k,l) = 2$ *for* $\begin{cases} k = 1 \text{ and } l = 3, \\ k \geq 2 \text{ and } l \equiv 3, 5 \pmod 6, \end{cases}$
3. $S_6(k,l) = 3$ *for* $k = 2$ *and* $l \equiv 0, 2 \pmod 6, l \geq 6$,
4. $S_6(k,l) = 4$ *for* $k = 2$ *and* $l = 2$,
5. $S_6(k,l) = 5$ *for* $k \geq 3$ *and* $l \equiv 0, 2 \pmod 6$.

Proof. When $l \equiv 1 \pmod 6$, Corollary 2 gives $S_6(k,l) = 0$. So, we may assume that $l \not\equiv 1 \pmod 6$ from here onwards. The other cases will be considered below.

`Case 1:` $k = 1$. By Theorem 5, $S_6(1,l) = 1$ for $l = 2$ or $l \geq \frac{6+1}{2} = 3.5$. Thus, $S_6(1,l) = 1$ for $l \neq 3$. Suppose $l = 3$. Note that the sum $\sum_{i=1}^{3-t} 1 + 2t = 3 + t$, with

$0 \leq t \leq 3$, is never congruent to 1 or 2 modulo 6. So, $\{1,2\}$ is 3-sum-free modulo 6 and $S_6(1,3) \geq 2$. On the other hand, $2+2+3 \equiv 1 \pmod 6$ implies that $\{1,2,3\}$ is not 3-sum-free modulo 6. Therefore, $S_6(1,3) = 2$.

`Case 2:` $k \geq 2$ and $l \equiv 3,4,5 \pmod 6$. By Corollary 4, if $k \geq 2$, we have $S_6(k,l) = 1$ when $l \equiv 4 \pmod 6$ and $S_6(k,l) \geq 2$ when $l \not\equiv 4 \pmod 6$. Now, $3 = \gcd(3,6)$, and $l \equiv 3,5 \pmod 6$ implies that $l \equiv 1 \pmod 2$. So, by Corollary 3, $S_6(k,l)$ is strictly less than 3. Hence, $S_6(k,l) = 2$ for $l \equiv 3,5 \pmod 6$.

From here onwards, we may assume that $l \not\equiv 1,3,4,5 \pmod 6$.

`Case 3:` $k = 2$ and $l \equiv 0,2 \pmod 6$. First, we will show that $S_6(2,2) = 4$. We claim that $\{\{1,4\},\{2,3\}\}$ is the only 2-partition of $[1,4]$ into 2-sum-free sets modulo 6. Now, $\{1,2\}$ is not a 2-sum-free set modulo 6, because $1+1 \equiv 2 \pmod 6$. Next, $4+4 \equiv 2 \pmod 6$ and $3+4 \equiv 1 \pmod 6$ imply that $\{2,4\}$ and $\{1,3,4\}$ are not 2-sum-free sets modulo 6, respectively. By eliminating all the 2-partitions of $[1,4]$ that contain any one of $\{1,2\},\{2,4\}$, or $\{1,3,4\}$, we are left with $\{\{1,4\},\{2,3\}\}$. Note that, for all $0 \leq t \leq 2$, the sum $\sum_{i=1}^{2-t} 1+4t = 2+3t \not\equiv 1$ or $4 \pmod 6$ and $\sum_{i=1}^{2-t} 2+3t = 4+t \not\equiv 2$ or $3 \pmod 6$. Thus, $\{\{1,4\},\{2,3\}\}$ is the only 2-partition of $[1,4]$ into 2-sum-free sets modulo 6.

From $5+5 \equiv 4 \pmod 6$ and $2+3 \equiv 5 \pmod 6$, we see that $\{4,5\}$ and $\{2,3,5\}$ are not 2-sum-free sets modulo 6. Since no 2-partitions of $[1,5]$ can exist, we conclude that $S_6(2,2) = 4$.

Now, we proceed to show that $S_6(2,l) = 3$ for $l \equiv 0,2 \pmod 6$ and $l \geq 6$. In fact, we will show that $\{\{2\},\{1,3\}\}$ is the only 2-partition of $[1,3]$ into l-sum-free sets modulo 6.

If $l \equiv 0 \pmod 6$, then $\sum_{i=1}^{l-3} 2+3(3) = 2l+3 \equiv 3 \pmod 6$. If $l \equiv 2 \pmod 6$ and $l \geq 8$, then $\sum_{i=1}^{l-4} 2+3(4) = 2l+4 \equiv 2 \pmod 6$. Hence, $\{2,3\}$ is not l-sum-free modulo 6. Next, $\{1,2\}$ is not l-sum-free modulo 6 for both cases, as $\sum_{i=1}^{l-1} 1+2 \equiv 1 \pmod 6$ when $l \equiv 0 \pmod 6$ and $\sum_{i=1}^{l-5} 1+2(5) \equiv 1 \pmod 6$ when $l \equiv 2 \pmod 6$ and $l \geq 8$. This implies that the only possible 2-partition of $[1,3]$ into l-sum-free sets modulo 6 is $\{\{2\},\{1,3\}\}$.

Clearly, $2l \not\equiv 2 \pmod 6$ for both cases. So, $\{2\}$ is l-sum-free modulo 6. It is left to show that $\{1,3\}$ is l-sum-free modulo 6. Consider the sum $\sum_{i=1}^{l-t} 1+3t = l+2t$, with $0 \leq t \leq l$. Since $l \equiv 0,2 \pmod 6$, l is even. So, regardless of the parity of t, the sum $l+2t$ is even and is never congruent to 1 or 3 modulo 6. Hence, $\{1,3\}$ is l-sum-free modulo 6. We have shown that $S_6(2,l) \geq 3$.

Finally, we will show that $S_6(2,l) < 4$. It is sufficient to show that it is not possible to extend $\{\{2\},\{1,3\}\}$ to include 4. If $l \equiv 0 \pmod 6$, then $\sum_{i=1}^{l-1} 2+4(1) = 2l+2 \equiv 2 \pmod 6$ and $\sum_{i=1}^{l-2} 1+3(2) = l+4 \equiv 4 \pmod 6$. If $l \equiv 2 \pmod 6$ and $l \geq 8$, then $2l \equiv 4 \pmod 6$ and $\sum_{i=1}^{l-1} 1+3(1) = l+2 \equiv 4 \pmod 6$. In both cases, no 2-partitions of $[1,4]$ into l-sum-free sets modulo 6 can exist. This completes the proof when $k = 2$ and $l \equiv 0,2 \pmod 6$.

Case 4: $k \geq 3$ and $l \equiv 0, 2 \pmod 6$. Suppose $k \geq 5$. Since $l \equiv 0, 2 \pmod 6$, $l \not\equiv 1 \pmod d$ for all positive divisors d of 6 and $d \neq 1$. By Corollary 5, $S_6(k, l) = 5$. So, it is left to consider the cases $k = 3$ and 4.

By Inequality (1), $S_6(k, l) \leq 5$. So, to show $S_6(k, l) = 5$ for $k = 3, 4$, it is sufficient to exhibit a 3-partition of $[1, 5]$ into l-sum-free sets modulo 6 when $l \equiv 0, 2 \pmod 6$.

Assume $k = 3$. Suppose $l \equiv 0 \pmod 6$. We claim that $\{\{4\}, \{1, 3\}, \{2, 5\}\}$ is a 3-partition of $[1, 5]$ into l-sum-free sets modulo 6. This follows by noting that $4l \equiv 0 \not\equiv 4 \pmod 6$, the sum $\sum_{i=1}^{l-t} 2 + 5t = 2l + 3t \equiv 3t \not\equiv 2 \text{ or } 5 \pmod 6$ for all $0 \leq t \leq l$, and the sum $\sum_{i=1}^{l-t} 1 + 3t = l + 2t \equiv 2t \not\equiv 1 \text{ or } 3 \pmod 6$ for all $0 \leq t \leq l$.

Suppose $l \equiv 2 \pmod 6$. We claim that $\{\{2\}, \{3, 5\}, \{1, 4\}\}$ is a 3-partition of $[1, 5]$ into l-sum-free sets modulo 6. This follows by noting that $2l \equiv 4 \not\equiv 2 \pmod 6$, the sum $\sum_{i=1}^{l-t} 1 + 4t = l + 3t \equiv 2 + 3t \not\equiv 1 \text{ or } 4 \pmod 6$ for all $0 \leq t \leq l$, and the sum $\sum_{i=1}^{l-t} 3 + 5t = 3l + 2t \equiv 2t \not\equiv 3 \text{ or } 5 \pmod 6$ for all $0 \leq t \leq l$.

Assume $k = 4$. We claim that $\{\{2\}, \{3\}, \{4\}, \{1, 5\}\}$ is a 4-partition of $[1, 5]$ into l-sum-free sets modulo 6 when $l \equiv 0, 2 \pmod 6$. Note that $al \not\equiv a \pmod 6$ for all $2 \leq a \leq 4$ when $l \equiv 0, 2 \pmod 6$. Furthermore, the sum $\sum_{i=1}^{l-t} 1 + 5t = l + 4t \equiv 0 \pmod 2$, which cannot be congruent to 1 or 5 modulo 6.

Theorem 7 is thus proved. □

5. $S_m(k, l)$ when m is a Prime Power

We now consider the case where m is a prime power. We shall write $m = p^i$ where p is a prime and $i \geq 1$ is an integer. The following result is a direct consequence of part 2 of Theorem 4.

Lemma 1. *Let p be a prime and $a, i, l \in \mathbb{Z}^+$. If $a \neq 0 \in \mathbb{Z}/p^i\mathbb{Z}$ and $l \not\equiv 1 \pmod p$, then the singleton $\{a\}$ is a l-sum-free set modulo p^i.*

Proof. Let $p^j = \gcd(a, p^i)$. Then $0 \leq j < i$ because $a \not\equiv 0 \pmod{p^i}$. By part 2 of Theorem 4, $\{a\}$ is not l-sum-free modulo p^i if and only if $l \equiv 1 \pmod{p^{i-j}}$. Since $l \not\equiv 1 \pmod p$ and $i - j \geq 1$, we cannot have $l \equiv 1 \pmod{p^{i-j}}$. Thus, the set $\{a\}$ is l-sum-free modulo p^i. □

The above lemma gives rise to a nice lower bound for $S_{p^i}(k, l)$ when $k \leq p - 1$. The matching upper bound and the case where $k \geq p$ will be proven in Theorem 8.

Corollary 6. *Let p be a prime and $i, k, l \in \mathbb{Z}^+$. If $l \not\equiv 1 \pmod p$, then $S_{p^i}(k, l) \geq k$ for all $1 \leq k \leq p - 1$.*

Proof. According to Lemma 1, the set $\{a\}$ is l-sum-free modulo p for all integers a, $1 \leq a \leq p - 1$. Therefore, the set $\{\{1\}, \{2\}, \ldots, \{k\}\}$ is a k-partition of $[1, k]$ into l-sum-free sets modulo p for all $1 \leq k \leq p - 1$. □

The next result follows from Theorems 1 and 2, and Corollary 5.

Corollary 7. *Let p be a prime and $i, k, l \in \mathbb{Z}^+$. If $l \not\equiv 1 \pmod{p}$, then $S_{p^i}(k, l) = p^i - 1$ for all $k \geq p^i - 1$.*

Since the case for $k \geq p^i - 1$ is covered by Corollary 7, it would be ideal to find exact values for $S_{p^i}(k, l)$ for $1 \leq k \leq p^i - 2$. We will be able to do so when l is large enough. First, we need the following lemmas.

Lemma 2. *Let p be a prime and $a, b, i, l \in \mathbb{Z}^+$. Suppose $l \geq p^i - 1$ and $l \not\equiv 1 \pmod{p}$.*

1. *If a and b belong to the same l-sum-free set modulo p^i, then $a \equiv b \pmod{p}$.*
2. *If $\gcd(k_1, p) = 1 = \gcd(k_2, p)$, $a = k_1 p^{j_1}$, $b = k_2 p^{j_2}$ where $j_1, j_2 \geq 1$, and $1 \leq a < b \leq p^i$, then a and b do not belong to the same l-sum-free set modulo p^i.*

Proof. Consider the equation

$$\sum_{i=1}^{l-t} a + bt = a(l-t) + bt \equiv a \pmod{p^i},$$

which is equivalent to the congruence

$$(b-a)t \equiv a(1-l) \pmod{p^i}. \tag{2}$$

We prove part 1. Now, a and b belonging to the same l-sum-free set modulo p^i means that no integer t $(0 \leq t \leq l)$ can satisfy Congruence (2). On the other hand, if $b - a \not\equiv 0 \pmod{p}$, then $b - a \not\equiv 0 \pmod{p^i}$. So, $\gcd(b-a, p^i) = 1$, and by Theorem 3, an integer t that satisfies Congruence (2) can be found. Furthermore, the integer t can be chosen so that $0 \leq t \leq p^i - 1 \leq l$. This contradicts the fact that a and b belong to the same l-sum-free set modulo p^i. Hence, $a \equiv b \pmod{p}$.

Now we prove part 2. We may assume that $i \geq j_2 \geq j_1$. Now, Congruence (2) becomes

$$(k_2 p^{j_2} - k_1 p^{j_1})t \equiv k_1 p^{j_1}(1-l) \pmod{p^i},$$

which is equivalent to

$$(k_2 p^{j_2 - j_1} - k_1)t \equiv k_1(1-l) \pmod{p^{i-j_1}}.$$

Suppose $j_2 = i$. Then $k_2 = 1$ and from $a < b$, we see that $j_1 < j_2 = i$. So, $i - j_1 \geq 1$. If $j_2 < i$, then $i - j_1 \geq 1$. Thus, in either case $i - j_1 \geq 1$.

If $\gcd(k_2 p^{j_2 - j_1} - k_1, p) = p$, then we must have $k_1(l-1) \equiv 0 \pmod{p}$. This implies that $l \equiv 1 \pmod{p}$ for $\gcd(k_1, p) = 1$. So, this case cannot occur. This means

the only possibility is $\gcd(k_2p^{j_2-j_1}-k_1,p)=1$, i.e., $\gcd(k_2p^{j_2-j_1}-k_1,p^{i-j_1})=1$. By Theorem 3, an integer t $(0\le t\le p^i-1\le l)$ satisfying the above equation can be found. So, $\{a,b\}$ is not a l-sum-free set modulo p^i, and thus a and b do not belong to the same l-sum-free set modulo p^i. □

Theorem 8. *Let p be a prime and $i,l\in\mathbb{Z}^+$. Suppose $l\ge p^i-1$ and $l\not\equiv 1 \pmod p$. Then*

$$S_{p^i}(k,l)=\begin{cases}k, & \text{if } 1\le k\le p-1 \text{ and } i\ge 1,\\ up-1, & \text{if } k=p+(u-2),\ i\ge 2 \text{ and } 2\le u\le p^{i-1},\\ p^i-1, & \text{if } k\ge p+(p^{i-1}-2) \text{ and } i\ge 1.\end{cases}$$

Proof. We split the proof into three cases.

`Case 1:` $1\le k\le p-1$ and $i\ge 1$. By Corollary 6, $S_{p^i}(k,l)\ge k$ for all $1\le k\le p-1$. If $S_{p^i}(k,l)\ge k+1$, then there are two integers a,b with $1\le a<b\le k+1$ such that both a,b belong to the same l-sum-free set modulo p^i. By part 1 of Lemma 2, $a\equiv b \pmod p$. This is not possible as $1\le a<b\le k+1\le p$. Hence, $S_{p^i}(k,l)=k$ for all $1\le k\le p-1$.

`Case 2:` $k=p+(u-2)$, $i\ge 2$, and $2\le u\le p^{i-1}$. For each $1\le j\le p-1$, let

$$S_j=\{x\in[1,up-1]\ :\ x\equiv j \pmod p\}.$$

For each $d\in[1,up-1]$ and $\gcd(d,p)=p$, let $S_d=\{d\}$.

Suppose S_{j_0} is not a l-sum-free set modulo p^i for some $1\le j_0\le p-1$. Then there are $a_1,a_2,\ldots,a_l,a_{l+1}\in S_{j_0}$ such that

$$a_1+a_2+\cdots+a_l\equiv a_{l+1} \pmod{p^i}.$$

This implies that $lj_0\equiv j_0 \pmod p$ as $a_r\equiv j_0 \pmod p$ for all r $(1\le r\le l+1)$. Since $\gcd(j_0,p)=1$, we have $l\equiv 1 \pmod p$, a contradiction. Thus, S_j is l-sum-free for all $1\le j\le p-1$.

According to Lemma 1, S_d is l-sum-free for all $d\in[1,up-1]$ with $\gcd(d,p)=p$. Hence,

$$\{S_1,S_2,\ldots,S_{p-1},S_p,S_{2p},\ldots,S_{(u-1)p}\}$$

is a k-partition of $[1,up-1]$ into l-sum-free sets modulo p^i and $S_{p^i}(k,l)\ge up-1$.

Suppose $S_{p^i}(k,l)\ge up$. By part 2 of Lemma 2, any pair in $\{p,2p,\ldots,up\}$ cannot be in the same l-sum-free set modulo p^i. Together with part 1 of Lemma 2, we see that any pair in $\{1,2,\ldots,p-1,p,2p,\ldots,up\}$ cannot be in the same l-sum-free set modulo p^i. Therefore, $k\ge p-1+u$, but this is not possible as $k=p+(u-2)$. Hence, $S_{p^i}(k,l)=up-1$.

Case 3: $k \geq p + (p^{i-1} - 2)$ and $i \geq 1$. By Inequality (1), $S_{p^i}(k, l) \leq p^i - 1$. It is sufficient to find a k-partition of $[1, p^i - 1]$ into l-sum-free sets modulo p^i. Now, for each $1 \leq j \leq p - 1$, let

$$S_j^* = \left\{x \in [1, p^i - 1] \ : \ x \equiv j \pmod{p}\right\}.$$

Suppose $i \geq 2$. For each $d \in [1, p^i - 1]$ and $\gcd(d, p) = p$, let $S_d^* = \{d\}$. Clearly,

$$\left\{S_1^*, S_2^*, \ldots, S_{p-1}^*, S_p^*, S_{2p}^*, \ldots, S_{(p^{i-1}-1)p}^*\right\} \tag{3}$$

is a k-partition of $[1, p^i - 1]$ into l-sum-free sets modulo p^i. Note that, if $i = 1$, Set (3) becomes

$$\{S_1, S_2, \ldots, S_{p-1}\},$$

which is also a k-partition of $[1, p - 1]$ where each S_i is l-sum-free modulo p.

This completes the proof of the theorem. □

We end this section with some consequences of Theorem 8. Firstly, we can recover part of Theorem 6 (for even $l \geq 3$):

$$S_4(k, l) = \begin{cases} 1, & \text{if } k = 1, \\ 3, & \text{if } k \geq 2. \end{cases}$$

By taking $i = 1$ in Theorem 8, the second consequence is as follows.

Corollary 8. *Suppose $l \geq p - 1$ is a positive integer and $l \not\equiv 1 \pmod{p}$. Then*

$$S_p(k, l) = \begin{cases} k, & \text{if } 1 \leq k \leq p - 1, \\ p - 1, & \text{if } k \geq p - 1. \end{cases}$$

Furthermore, by Theorem 8, we may obtain some of the values of $S_8(k, l)$ and $S_9(k, l)$. For brevity, we give an example when $p = 2$, $i = 3$, $k = 4$, and $l = 8$.

Example 1. By Theorem 8, $S_8(4, 8) = 7$. To obtain the lower bound, we follow the proof of Theorem 8 and set

$$S_1 = \{x \in [1, 7] : x \equiv 1 \pmod{2}\} = \{1, 3, 5, 7\}.$$

Note that if S_1 is not an 8-sum-free set modulo 8, then there exists $a_1, \ldots, a_l, a_{l+1} \in S_1$ such that

$$a_1 + \cdots + a_l \equiv a_{l+1} \pmod{8}.$$

Since all the a_i's are congruent to 1 modulo 2, the above congruence implies that $l \equiv 1 \pmod{2}$, a contradiction. So S_1 is 8-sum-free modulo 8. Next we isolate each $d \in [1, 7]$ with $\gcd(d, 2) = 2$ as a singleton, $S_d = \{d\}$. That is, put

$$S_2 = \{2\}, S_4 = \{4\}, S_6 = \{6\}.$$

By Lemma 1, each of these S_d's are 8-sum-free modulo 8. Thus, the set

$$\{S_1, S_2, S_4, S_6\} = \{\{1,3,5,7\}, \{2\}, \{4\}, \{6\}\}$$

is a 4-partition of $[1,7]$ into 8-sum-free sets modulo 8, which implies that $S_8(4,8) \geq 7$.

To get the upper bound, we argue that $[1,8]$ has no 4-partition into 8-sum-free sets modulo 8. Notice that part 2 of Lemma 2 implies that no two numbers from $\{2,4,6,8\}$ can belong to the same 8-sum-free set modulo 8. Combining this with part 1 of Lemma 2, we infer that the only way to partition $[1,8]$ into sets that are 8-sum-free modulo 8 is to put each number in a singleton. That is, we would need at least an 8-partition of $[1,8]$. However, this is not possible because we are only considering 4-partitions of $[1,8]$. Therefore, $S_8(4,8) \leq 7$ and equality follows.

In the next two sections, we apply the above general results to determine the values of $S_5(k,l)$ and $S_7(k,l)$.

6. Establishing $S_5(k,l)$

We will prove the following theorem.

Theorem 9. *Let $k, l \in \mathbb{Z}^+$. If $l \equiv 1 \pmod 5$, then $S_5(k,l) = 0$. Furthermore, for l not congruent to 1 modulo 5,*

1. $S_5(k,l) = 1$ *for* $k = 1$,
2. $S_5(k,l) = k$ *for* $k = 2,3$ *and* $l \neq 2$,
3. $S_5(k,l) = 4$ *for* $\begin{cases} k = 2,3 \text{ and } l = 2, \\ k \geq 4. \end{cases}$

Proof. When $l \equiv 1 \pmod 5$, we have $S_5(k,l) = 0$, by Corollary 2. From here onwards, we assume $l \not\equiv 1 \pmod 5$. By Theorem 5, $S_5(1,l) = 1$ if and only if $l \geq 2$. By Corollary 8, for $l \geq 4$, $S_5(k,l) = k$ if $1 \leq k \leq 4$ and $S_5(k,l) = 4$ if $k \geq 4$. By Corollary 5, $S_5(k,l) = 4$ for all $k \geq 4$. So, it is left to show that $S_5(k,3) = k$ for $k = 2,3$ and $S_5(k,2) = 4$ for $k = 2,3$.

By Corollary 6, $S_5(2,3) \geq 2$. Now, $2+2+2 \equiv 1 \pmod 5$, $1+1+1 \equiv 3 \pmod 5$, and $2+3+3 \equiv 3 \pmod 5$ imply that $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ are not 3-sum-free, respectively. Thus, $S_5(2,3) = 2$.

By Corollary 6, $S_5(3,3) \geq 3$. Now, $1+4+4 \equiv 4 \pmod 5$, $4+4+4 \equiv 2 \pmod 5$, and $3+3+3 \equiv 4 \pmod 5$ imply that $\{1,4\}$, $\{2,4\}$, and $\{3,4\}$ are not 3-sum-free, respectively. Therefore, $S_5(3,3) = 3$.

Let us consider $S_5(2,2)$. The sum $\sum_{i=1}^{2-t} 2+3t = 4+t$, with $0 \leq t \leq 2$, is never congruent to 2 or 3 modulo 5. Furthermore, the sum $\sum_{i=1}^{2-t} 1+4t = 2+3t$, with $0 \leq t \leq 2$, is never congruent to 1 or 4 modulo 5. So $\{\{1,4\},\{2,3\}\}$ is a 2-partition of $[1,4]$ into 2-sum-free sets modulo 5. This implies that $S_5(2,2) \geq 4$. Together with Inequality (1), $S_5(2,2) = 4$.

Lastly, let us consider $S_5(3,2)$. Since $\sum_{i=1}^{2-t} 1+4t = 2+3t$, with $0 \leq t \leq 2$, is never congruent to 1 or 4 modulo 5, $\{\{1,4\},\{2\},\{3\}\}$ is a 3-partition of $[1,4]$ into 2-sum-free sets modulo 5. This implies that $S_5(3,2) \geq 4$. It follows from Inequality (1) that $S_5(3,2) = 4$.

This completes the proof of the theorem. □

7. Establishing $S_7(k,l)$

We will fully determine the values of $S_7(k,l)$ in this section. Since the proof is rather technical, we briefly outline it here. We start by using some of the general results we have proven thus far to immediately obtain certain values of $S_7(k,l)$. In particular, we will need Corollaries 2, 5, 8, and Theorem 5. Then we will deal with the remaining cases using Lemma 1 and Corollary 6. Moreover, anytime we need to show that $S_7(k,l) = 6$, we shall appeal to Inequality (1) for the upper bound and construct a suitable k-partition of $[1,6]$ for the lower bound.

Theorem 10. *Let $k,l \in \mathbb{Z}^+$. If $l \equiv 1 \pmod 7$, then $S_7(k,l) = 0$. Furthermore, for l not congruent to 1 modulo 7,*

1. $S_7(k,l) = 1$ *for* $k = 1$ *and* $l \neq 3$,
2. $S_7(k,l) = 2$ *for* $\begin{cases} k = 1 \text{ and } l = 3, \\ k = 2 \text{ and } l \geq 4, \end{cases}$
3. $S_7(k,l) = 3$ *for* $\begin{cases} k = 2 \text{ and } l = 2,3, \\ k = 3 \text{ and } l \geq 5, \end{cases}$
4. $S_7(k,l) = 4$ *for* $k = 4$ *and* $l \geq 5$,
5. $S_7(k,l) = 5$ *for* $\begin{cases} k = 3 \text{ and } l = 3, \\ k = 5 \text{ and } l \geq 5, \end{cases}$
6. $S_7(k,l) = 6$ *for* $\begin{cases} k = 3 \text{ and } l = 2,4, \\ k = 4 \text{ and } l = 2,3,4, \\ k = 5 \text{ and } l = 2,3,4, \\ k \geq 6. \end{cases}$

Proof. When $l \equiv 1 \pmod 7$, we have $S_7(k,l) = 0$ by Corollary 2. From here onwards, we shall assume that $l \not\equiv 1 \pmod 7$. By Theorem 5, $S_7(1,l) = 1$ if and only if $l = 2$ or $l \geq 4$. By Corollary 8, for $l \geq 6$, $S_7(k,l) = k$ for $1 \leq k \leq 6$ and $S_7(k,l) = 6$ if $k \geq 6$. By Corollary 5, $S_7(k,l) = 6$ for all $k \geq 6$. So, it is left to show that

(a) $S_7(1,3) = 2$,

(b) $S_7(2,l) = \begin{cases} 3, & \text{if } l = 2,3, \\ 2, & \text{if } l = 4,5, \end{cases}$

(c) $S_7(3,l) = \begin{cases} 6, & \text{if } l = 2,4, \\ 5, & \text{if } l = 3, \\ 3, & \text{if } l = 5, \end{cases}$

(d) $S_7(4,l) = \begin{cases} 6, & \text{if } l = 2,3,4, \\ 4, & \text{if } l = 5, \end{cases}$

(e) $S_7(5,l) = \begin{cases} 6, & \text{if } l = 2,3,4, \\ 5, & \text{if } l = 5. \end{cases}$

We shall consider each of these cases separately.

`Case 1:` $k = 1$ and $l = 3$. Since $\sum_{i=1}^{3-t} 1 + 2t = 3 + t$, with $0 \leq t \leq 3$, is never congruent to 1 or 2 modulo 7, $\{1,2\}$ is 3-sum-free modulo 7. So $S_7(k,l) \geq 2$. Now, $1+1+1 \equiv 3 \pmod 7$ and $3+3+3 \equiv 2 \pmod 7$ imply that $\{1,3\}$ and $\{2,3\}$ are not 3-sum-free modulo 7, respectively. Thus, $S_7(1,3) = 2$.

`Case 2:` $k = 2$ and $l = 4,5$. We will show that $S_7(2,l) = 2$. By Corollary 6, $S_7(2,l) \geq 2$. Now, $2+2+2+2 \equiv 1 \pmod 7$, $2+2+3+3 \equiv 3 \pmod 7$, and $1+3+3+3 \equiv 3 \pmod 7$ imply that $\{1,2\}$, $\{2,3\}$, and $\{1,3\}$ are not 4-sum-free modulo 7, respectively. Thus, $S_7(2,4) = 2$.

Next, $1+1+2+2+2 \equiv 1 \pmod 7$, $3+3+3+3+3 \equiv 1 \pmod 7$, and $2+2+2+2+2 \equiv 3 \pmod 7$ imply that $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ are not 5-sum-free modulo 7, respectively. Hence, $S_7(2,5) = 2$.

`Case 3:` $k = 2$ and $l = 2,3$. We will show that $S_7(2,l) = 3$. Suppose $l = 2$. Note that, for all $0 \leq t \leq 2$, the sum $\sum_{i=1}^{2-t} 1 + 3t = 2 + 2t$ is never congruent to 1 or 3 modulo 7, and $\sum_{i=1}^{2-t} 2 + 3t = 4 + t$ is never congruent to 2 or 3 modulo 7. Now, $\{1,2\}$ is not 2-sum-free modulo 7 because $1+1 \equiv 2 \pmod 7$. So $\{\{2\},\{1,3\}\}$ and $\{\{1\},\{2,3\}\}$ are the only two possible 2-partitions of $[1,3]$ into 2-sum-free sets modulo 7. Thus, $S_7(2,2) \geq 3$. However, note that $4+4 \equiv 1 \pmod 7$ and $2+2 \equiv 4 \pmod 7$. We infer that 4 cannot belong to the same 2-sum-free set modulo 7 as 1 or 2. This implies that $S_7(2,2) = 3$.

Suppose $l = 3$. From the proof in Case 1, $\{1,3\}$ and $\{2,3\}$ are not 3-sum-free modulo 7 but $\{1,2\}$ is 3-sum-free modulo 7. Therefore, $\{\{3\},\{1,2\}\}$ is the only possible 2-partition of $[1,3]$ into 3-sum-free sets modulo 7, implying that $S_7(2,3) \geq 3$. However, $1+1+2 \equiv 4 \pmod 7$ and $3+3+4 \equiv 3 \pmod 7$ imply that $\{1,2,4\}$ and $\{3,4\}$ are not 3-sum-free modulo 7, respectively. Hence, $S_7(2,3) = 3$.

`Case 4:` $k = 3$ and $l = 5$. By Corollary 6, $S_7(3,5) \geq 3$. Now, $1+1+1+4+4 \equiv 4 \pmod 7$, $2+2+4+4+4 \equiv 2 \pmod 7$, and $3+3+3+4+4 \equiv 3 \pmod 7$ imply that $\{1,4\}$, $\{2,4\}$, and $\{3,4\}$ are not 5-sum-free modulo 7, respectively. From the proof in Case 2, $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ are not 5-sum-free modulo 7. So no 3-partition of $[1,4]$ into 5-sum-free sets modulo 7 can exist. Thus $S_7(3,5) = 3$.

`Case 5:` $k = 3$ and $l = 3$. We shall show that

$$\{\{1\},\{2,4\},\{3,5\}\}, \quad \{\{2\},\{1,4\},\{3,5\}\}, \quad \{\{4\},\{1,2\},\{3,5\}\} \tag{4}$$

are the only possible 3-partitions of $[1,5]$ into 3-sum-free sets modulo 7. From the proof in Case 3, $\{1,3\}$, $\{2,3\}$, $\{3,4\}$, $\{1,2,4\}$ are not 3-sum-free sets modulo 7. Note that $2+5+5 \equiv 5 \pmod 7$, $5+5+5 \equiv 1 \pmod 7$, and $4+4+4 \equiv 5 \pmod 7$. So $\{2,5\}$, $\{1,5\}$, and $\{4,5\}$ are not 3-sum-free modulo 7. Then eliminating all the 3-partitions of $[1,5]$ containing any one of $\{1,3\}$, $\{2,3\}$, $\{3,4\}$, $\{1,2,4\}$, $\{1,5\}$, $\{2,5\}$, or $\{4,5\}$, we are left with those shown in (4).

Note that, for all $0 \leq t \leq 3$, the sum $\sum_{i=1}^{3-t} 1+4t = 3+3t$ is never congruent to 1 or 4 modulo 7, $\sum_{i=1}^{3-t} 2+4t = 6+2t$ is never congruent to 2 or 4 modulo 7, and $\sum_{i=1}^{3-t} 3+5t = 9+2t$ is never congruent to 3 or 5 modulo 7. Moreover, $\{1,2\}$ is 3-sum-free modulo 7 from the proof in Case 1. So the sets in (4) are the only 3-partitions of $[1,5]$ into 3-sum-free sets modulo 7.

Finally, note that none of the 3-partitions in (4) can be extended to include 6 because $1+6+6 \equiv 6 \pmod 7$, $2+2+2 \equiv 6 \pmod 7$, $6+6+6 \equiv 4 \pmod 7$, and $3+5+5 \equiv 6 \pmod 7$. Since no 3-partitions of $[1,6]$ into 3-sum-free can exist, $S_7(3,3) = 5$.

`Case 6:` $k = 3$ and $l = 2, 4$. We will show that $S_7(3,l) = 6$. By Inequality (1), it is sufficient to find a 3-partition of $[1,6]$ into l-sum-free sets modulo 7.

Suppose $l = 2$. Note that, for all $0 \leq t \leq 2$, the sum $\sum_{i=1}^{2-t} 1+5t = 2+4t$ is never congruent to 1 or 5 modulo 7, $\sum_{i=1}^{2-t} 2+6t = 4+4t$ is never congruent to 2 or 6 modulo 7, and $\sum_{i=1}^{2-t} 3+4t = 6+t$ is never congruent to 3 or 4 modulo 7. So $\{\{1,5\},\{2,6\},\{3,4\}\}$ is a 3-partition of $[1,6]$ into 2-sum-free sets modulo 7, which implies that $S_7(3,2) = 6$.

Suppose $l = 4$. Note that, for all $0 \leq t \leq 4$, the sum $\sum_{i=1}^{4-t} 1+6t = 4+5t$ is never congruent to 1 or 6 modulo 7, $\sum_{i=1}^{4-t} 2+5t = 8+3t$ is never congruent to 2 or 5 modulo 7, and $\sum_{i=1}^{4-t} 3+4t = 12+t$ is never congruent to 3 or 4 modulo 7. So $\{\{1,6\},\{2,5\},\{3,4\}\}$ is a 3-partition of $[1,6]$ into 4-sum-free sets modulo 7 and so $S_7(3,4) = 6$.

Case 7: $k = 4$ and $l = 5$. By Corollary 6, $S_7(4, 5) \geq 4$. Note that $1+1+1+4+4 \equiv 4 \pmod 7$, $2+2+4+4+4 \equiv 2 \pmod 7$, and $3+3+4+4+4 \equiv 4 \pmod 7$. So the set $\{a, 4\}$, with $1 \leq a \leq 3$, is not 5-sum-free modulo 7. Furthermore, $1+1+1+1+1 \equiv 5 \pmod 7$, $2 + 2 + 2 + 5 + 5 \equiv 2 \pmod 7$, $3 + 3 + 3 + 5 + 5 \equiv 5 \pmod 7$, and $5 + 5 + 5 + 5 + 5 \equiv 4 \pmod 7$. So the set $\{a, 5\}$, with $1 \leq a \leq 4$ is not 5-sum-free modulo 7. From the proof in Case 2, $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are not 5-sum-free modulo 7. Hence, no 4-partitions of $[1, 5]$ into 5-sum-free sets modulo 7 can exist. Thus, $S_7(4, 5) = 4$.

Case 8: $k = 4$ and $l = 2, 3, 4$. We will show that $S_7(4, l) = 6$. By Inequality (1), it is sufficient to find a 4-partition of $[1, 6]$ into l-sum-free sets modulo 7.

Suppose $l = 2$. Note that, for all $0 \leq t \leq 2$, the sum $\sum_{i=1}^{2-t} 1 + 6t = 2 + 5t$ is never congruent to 1 or 6 modulo 7, and $\sum_{i=1}^{2-t} 2 + 5t = 4 + 3t$ is never congruent to 2 or 5 modulo 7. This implies that $\{1, 6\}$ and $\{2, 5\}$ are 2-sum-free sets modulo 7. By Lemma 1, $\{3\}$ and $\{4\}$ are 2-sum-free sets modulo 7. Therefore, $\{\{3\}, \{4\}, \{1, 6\}, \{2, 5\}\}$ is a 4-partition of $[1, 6]$ into 2-sum-free sets modulo 7, implying that $S_7(4, 2) = 6$.

Suppose $l = 3$. Note that, for all $0 \leq t \leq 3$, the sum $\sum_{i=1}^{3-t} 3 + 6t = 9 + 3t$ is never congruent to 3 or 6 modulo 7. From the proof in Case 1, $\{1, 2\}$ is 3-sum-free modulo 7. It follows from Lemma 1 that $\{\{4\}, \{5\}, \{1, 2\}, \{3, 6\}\}$ is a 4-partition of $[1, 6]$ into 3-sum-free sets modulo 7. So, $S_7(4, 3) = 6$.

Suppose $l = 4$. From the proof in Case 6, $\{1, 6\}$ and $\{2, 5\}$ are 4-sum-free modulo 7. By Lemma 1, $\{\{3\}, \{4\}, \{1, 6\}, \{2, 5\}\}$ is a 4-partition of $[1, 6]$ into 4-sum-free sets modulo 7. Hence, $S_7(4, 3) = 6$.

Case 9: $k = 5$ and $l = 5$. By Corollary 6, $S_7(5, 5) \geq 5$. Observe that $1 + 1 + 6 + 6 + 6 \equiv 6 \pmod 7$, $6 + 6 + 6 + 6 + 6 \equiv 2 \pmod 7$, $3 + 3 + 6 + 6 + 6 \equiv 3 \pmod 7$, $4 + 4 + 4 + 4 + 4 \equiv 6 \pmod 7$, and $5 + 5 + 5 + 6 + 6 \equiv 6 \pmod 7$. So the set $\{a, 6\}, 1 \leq a \leq 5$, is not 5-sum-free modulo 7. From the proof in Case 7, the set $\{a, 4\}$, with $1 \leq a \leq 3$, is not 5-sum-free modulo 7, the set $\{a, 5\}$, with $1 \leq a \leq 4$ is not 5-sum-free modulo 7, and $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are not 5-sum-free modulo 7. Hence, no 5-partitions of $[1, 6]$ into 5-sum-free sets modulo 7 can exist. Thus, $S_7(5, 5) = 5$.

Case 10: $k = 5$ and $= 2, 3, 4$. We will show that $S_7(5, l) = 6$. By Inequality (1), it is sufficient to find a 5-partition of $[1, 6]$ into l-sum-free sets modulo 7.

Suppose $l = 2$. From the proof in Case 8, $\{1, 6\}$ is 2-sum-free modulo 7. It follows from Lemma 1, that $\{\{2\}, \{3\}, \{4\}, \{5\}, \{1, 6\}\}$ is a 5-partition of $[1, 6]$ into 2-sum-free sets modulo 7.

Suppose $l = 3$. From the proof in Case 8, $\{3, 6\}$ is 3-sum-free modulo 7. So, by Lemma 1, $\{\{1\}, \{2\}, \{4\}, \{5\}, \{3, 6\}\}$ is a 5-partition of $[1, 6]$ into 3-sum-free sets modulo 7.

Suppose $l = 4$. From the proof in Case 8, $\{1, 6\}$ is 4-sum-free modulo 7. Again,

by Lemma 1, $\{\{2\},\{3\},\{4\},\{5\},\{1,6\}\}$ is a 5-partition of $[1,6]$ into 4-sum-free sets modulo 7.

Theorem 10 is thus proved. □

8. Concluding Remarks

In this paper, we determined previously unknown values of $S_m(k,l)$. In particular, we determined the exact values of $S_m(k,l)$ when $l \equiv 1 \pmod{m}$ (Corollary 2). When $l \not\equiv 1 \pmod{m}$, we also obtained exact values when $m = 2n$ and $l \equiv n+1 \pmod{2n}$, when $k \geq m-1$, for all $l \not\equiv 1 \pmod{d}$, where $d > 1$ divides m, and when $l = 2$ or $l \geq (m+1)/2$ (Corollaries 4, 5, and Theorem 5). Our most comprehensive result is Theorem 8, which gives, for big enough l, the exact values of $S_{p^i}(k,l)$. Additionally, we completely established the values of $S_m(k,l)$ when $m = 4$, 5, 6, and 7 in Theorems 6, 7, 9, and 10, respectively. These results leave the following unresolved cases.

Problem 1. *Let $i,k,l,m,n \in \mathbb{Z}^+$ and p be a prime.*

1. *What are the values of $S_{2n}(k,l)$ when $l \not\equiv 1, n+1 \pmod{2n}, n \geq 4$?*
2. *For all $l \not\equiv 1 \pmod{d}$, where $d > 1$ divides $m \geq 8$, what are the values of $S_m(k,l)$ when $k < m-1$?*
3. *What are the values of $S_m(1,l)$ when $2 < l < \frac{m+1}{2}, m \geq 8$?*
4. *What are the values of $S_{p^i}(k,l)$ when $l < p^i - 1$?*
5. *What are the values of $S_m(k,l)$ when $m \geq 8$?*

Since we now fully know $S_m(k,l)$ for $1 \leq m \leq 7$, a natural step is to look at $m = 8$ and 9; this would make progress on parts 4 and 5 of Problem 1. As mentioned before, Theorem 8 gives some of the values of $S_8(k,l)$ and $S_9(k,l)$ whenever l is at least 7 or 8, respectively. Furthermore, the method in this paper can work to resolve these two cases. But if the proof of $S_7(k,l)$ (Theorem 10) is any indication, then there remains a nontrivial amount of case analysis to fully determine $S_8(k,l)$ and $S_9(k,l)$. This suggests that a different approach is needed to explicitly determine the values of $S_m(k,l)$ for large values of m.

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