COTANGENT POWER SUMS AND CHARACTER COORDINATES

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Abstract

We show that certain sums studied in two recent papers are basically character coordinates (as they are called in the literature). These sums involve values of Dirichlet characters and powers of $\cot(\pi k/n)$, $1 \leq k \leq n-1$. We also show that a basic tool for the study of these sums was already given in 1987, in the form of the character coordinates of so-called cotangent numbers. By means of this tool, we obtain the results of the said papers in a simple and lucid way. We also show that the coefficients of the linear combinations used in the said papers are essentially the same.

1. Introduction

Let $n \geq 2, r \geq 1$ be integers. Let $\zeta_n = e^{2\pi i/n}$ and $k \in \mathbb{Z}$, $(k,n) = 1$. Since

$$i\cot(\pi k/n) = (1+\zeta_n^k)/(1-\zeta_n^k), \tag{1}$$

the numbers $i^r \cot^r(\pi k/n)$ lie in the nth cyclotomic field $\mathbb{Q}(\zeta_n)$. Let χ be a Dirichlet character mod n. In two recent papers the character sums

$$\sum_{k=1}^{n} \chi(k) i^r \cot^r(\pi k/n) \tag{2}$$

formed with these numbers have been expressed in terms of generalized Bernoulli numbers and Gauss sums; see [8, Cor. 13], [3, Cor. 2.19].

However, the authors of these papers do not refer to the fact that (2) is basically a *character coordinate*, more precisely, the $\overline{\chi}$-coordinate of $i^r \cot^r(\pi/n)$. Character coordinates have useful properties; see Sections 2 and 4. They have been known since 1959; see [9] and Section 2. In addition, they contain information about Galois modules; see [4].

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Moreover, the said authors make no use of the fact that the number $i^r \cot(\pi/n)$ has *natural components* with respect to character coordinates, namely, the so-called *cotangent numbers* $i^j \cot_{j-1}(\pi/n)$, $1 \le j \le r$, where $\cot_l$ is the lth derivative of the function cot (in particular, $\cot_0 = \cot$). Indeed, for these components the character coordinates were given by the present author already in 1987; see [4]. This fact has the following consequence. If we express the function $i^r \cot^r$ as a rational linear combination of the functions $i^j \cot_{j-1}$, $1 \le j \le r$, $j \equiv r \bmod 2$ (if r is even, one must include the constant function 1), we immediately obtain the character coordinates of $i^r \cot(\pi/n)$; see Theorem 1. This theorem is given in [8], whereas the paper [3] has an equivalent result, but only for primitive characters (see Section 4). We think that our approach to this theorem is the simplest one known so far.

Our plan is as follows. In Section 2 we recall some basic facts about character coordinates and exhibit the χ-coordinates of the cotangent numbers. In Section 3 we express $i^r \cot^r$ as a linear combination of the functions $i^j \cot_{j-1}$, as described above (see Proposition 1), and obtain the said Theorem 1. In Section 4 we show that the result of [3] gives the representation of $i^r \cot^r$ as a linear combination of the said functions in a different form, see Theorem 2. Properties of character coordinates play a decisive role in this connection.

The cotangent numbers $i \cot(\pi k/n)$, $1 \le k \le n$, $(k, n) = 1$, or, more generally, certain sums of these, give rise to relative class number formulas for abelian number fields; see [5, 10]. Certain $\mathbb{Q}$-linear combinations of these cotangent numbers play an important role in connection with various questions of number theory, even with the Riemann Hypothesis; see, for instance, [1, 2, 11, 12].

2. Character Coordinates

For a number $a \in \mathbb{Q}(\zeta_n)$ and a Dirichlet character χ mod n, the χ-coordinate $y(\chi|a)$ is defined by

$$y(\chi|a)\tau(\overline{\chi}_f) = \sum_{\substack{1\le k\le n\\(k,n)=1}} \overline{\chi}(k)\sigma_k(a); \tag{3}$$

see [9]. Here f is the conductor of χ, χ_f the character mod f attached to χ, $\overline{(\ )}$ the complex conjugation, and

$$\tau(\overline{\chi}_f) = \sum_{k=1}^{f} \overline{\chi}_f(k)\zeta_f^k$$

the (primitive) Gauss sum; furthermore, σ_k is the Galois automorphism of $\mathbb{Q}(\zeta_n)$ defined by $\zeta_n \mapsto \zeta_n^k$, $(k, n) = 1$.

Let $\mathbb{Q}(\chi)$ be the field of values of χ. Then $y(\chi|a) \in \mathbb{Q}(\chi)$ and the map

$$y(\chi|-) : \mathbb{Q}(\zeta_n) \to \mathbb{Q}(\chi) : a \mapsto y(\chi|a)$$

is $\mathbb{Q}$-linear and G-invariant, so

$$y(\chi|\sigma_k(a)) = \chi(k)y(\chi|a).$$

From Equation (1) we obtain

$$\sigma_k(i\cot(\pi/n)) = i\cot(\pi k/n). \tag{4}$$

In view of Equation (4), the sum given by Formula (2) has the form

$$\sum_{\substack{1\le k\le n\\(k,n)=1}} \chi(k)i^r\cot^r(\pi k/n) = y(\overline{\chi}|i^r\cot^r(\pi/n))\tau(\chi_f).$$

A number $a\in\mathbb{Q}(\zeta_n)$ is uniquely determined by its character coordinates, as the reconstruction formula

$$a = \frac{1}{\varphi(n)}\sum_{\chi\in\mathcal{X}} y(\chi|a)\tau(\overline{\chi}_f)$$

shows, where $\mathcal{X}$ is the set of all Dirichlet characters mod n; see [9].

Let $\mathcal{X}^+ = \{\chi\in\mathcal{X}; \chi(-1)=1\}$ and $\mathcal{X}^- = \{\chi\in\mathcal{X}; \chi(-1)=-1\}$. If $a\in\mathbb{Q}(\zeta_n)$ is real, then $y(\chi|a)=0$ for all $\chi\in\mathcal{X}^-$. If $a\in\mathbb{Q}(\zeta_n)$ is purely imaginary, then $y(\chi|a)=0$ for all $\chi\in\mathcal{X}^+$.

The cotangent number $i^r\cot_{r-1}(\pi/n)$ is real, if r is even, and purely imaginary, if r is odd. Accordingly, $y(\chi|i^r\cot_{r-1}(\pi/n))$ vanishes in the cases "r is even, $\chi\in\mathcal{X}^-$" and "r is odd, $\chi\in\mathcal{X}^+$". In the remaining cases we have

$$y(\chi|i^r\cot_{r-1}(\pi/n)) = \frac{\chi(-1)(2n)^r}{rf^r}\prod_{p\,|\,n}\left(1-\frac{\overline{\chi}_f(p)}{p^r}\right)B_{r,\chi_f}, \tag{5}$$

see [4, Thm. 2]. Here

$$B_{r,\chi_f} = f^{r-1}\sum_{k=1}^{f} B_r(k/f)\chi_f(k),$$

where $B_r(x)$ is the rth Bernoulli polynomial; see [13, Prop. 4.1].

3. Cotangent Powers and Cotangent Derivatives

Let $k\ge 1$ be an integer. Our main tool in this section is a special case of Lemma 4.1 in [6] for certain functions in t, namely,

$$\frac{1}{(1-e^t)^k} = \frac{1}{(k-1)!}\sum_{j=1}^{k} S(k,j)\frac{d^{j-1}}{dt^{j-1}}\left(\frac{1}{1-e^t}\right), \tag{6}$$

where $S(k,j)$ is the absolute value of the respective Stirling number of the first kind (i.e., the number of permutations of k objects with exactly j cycles). Since

$$\frac{1}{1-e^t} = \frac{i}{2}\cot(-it/2) + \frac{1}{2},$$

we have, for $j \geq 2$,

$$\frac{d^{j-1}}{dt^{j-1}}\left(\frac{1}{1-e^t}\right) = (-1)^{j-1}(i/2)^j \cot_{j-1}(-it/2).$$

If we insert this into Equation (6), we obtain

$$\frac{1}{(1-e^t)^k} = \frac{1}{2} + \frac{1}{(k-1)!}\sum_{j=1}^{k} S(k,j)(-1)^{j-1}(i/2)^j \cot_{j-1}(-it/2). \tag{7}$$

On the other hand,

$$i\cot(-it/2) = \frac{2}{1-e^t} - 1.$$

Therefore, the binomial formula yields

$$(i\cot(-it/2))^r = (-1)^r + \sum_{k=1}^{r}\binom{r}{k}\frac{2^k}{(1-e^t)^k}(-1)^{r-k}.$$

In this identity, we replace $1/(1-e^t)^k$ by the right-hand side of Equation (7) and change the order of summation in the resulting double sum. This gives

$$(i\cot(-it/2))^r = C + \sum_{j=1}^{r} i^j \cot_{j-1}(-it/2)\frac{(-1)^{j-1}}{2^j}\sum_{k=j}^{r}\frac{(-1)^{r-k}2^k}{(k-1)!}\binom{r}{k}S(k,j),$$

with

$$C = (-1)^r + \frac{1}{2}\sum_{k=1}^{r}\binom{r}{k}2^k(-1)^{r-k} = \frac{(-1)^r+1}{2}.$$

Thus, we may write

$$(i\cot(-it/2))^r = \frac{(-1)^r+1}{2} + \sum_{j=1}^{r} c_{r,j}\, i^j \cot_{j-1}(-it/2),$$

with

$$c_{r,j} = (-1)^{r-1}\sum_{k=j}^{r}\frac{(-2)^{k-j}}{(k-1)!}\binom{r}{k}S(k,j). \tag{8}$$

The change of variables $(-it/2) \mapsto x$ yields the following proposition.

Proposition 1. *For $r \geq 1$, we have*

$$i^r \cot^r = \frac{(-1)^r + 1}{2} + \sum_{j=1}^{r} c_{r,j}\, i^j \cot_{j-1}$$

with $c_{r,j}$ as in Equation (8).

Since the functions $i^j \cot_{j-1}$, $j \geq 1$, are $\mathbb{Q}$-linearly independent, we have $c_{r,j} = 0$ if $r \not\equiv j \bmod 2$. If r is odd, we may write, therefore,

$$i^r \cot^r = \sum_{\substack{1 \leq j \leq r \\ j \equiv 1 \bmod 2}} c_{r,j}\, i^j \cot_{j-1}\,. \tag{9}$$

If r is even, we obtain

$$i^r \cot^r = 1 + \sum_{\substack{1 \leq j \leq r \\ j \equiv 0 \bmod 2}} c_{r,j}\, i^j \cot_{j-1}\,. \tag{10}$$

Let χ_0 denote the principal character mod n. We have, for $\chi \in \mathcal{X}$,

$$y(\chi|1) = \begin{cases} 0, & \text{if } \chi \neq \chi_0; \\ \varphi(n), & \text{if } \chi = \chi_0. \end{cases} \tag{11}$$

Since the χ-coordinate is $\mathbb{Q}$-linear, Equation (5) gives the following result.

Theorem 1. *If $r \geq 1$ is odd and $\chi \in \mathcal{X}^-$ has the conductor f, then*

$$y(\chi|i^r \cot^r(\pi/n)) = -\sum_{\substack{1 \leq j \leq r \\ j \equiv 1 \bmod 2}} c_{r,j}\, \frac{(2n)^j}{jf^j} \prod_{p\,|\,n} \left(1 - \frac{\overline{\chi}_f(p)}{p^j}\right) B_{j,\chi_f}$$

with $c_{r,j}$ as in Equation (8). *If $r \geq 2$ is even and $\chi \in \mathcal{X}^+$ has the conductor f, then*

$$y(\chi|i^r \cot^r(\pi/n)) = y(\chi|1) + \sum_{\substack{1 \leq j \leq r \\ j \equiv 0 \bmod 2}} c_{r,j}\, \frac{(2n)^j}{jf^j} \prod_{p\,|\,n} \left(1 - \frac{\overline{\chi}_f(p)}{p^j}\right) B_{j,\chi_f}$$

with $y(\chi|1)$ as in Equation (11) *and $c_{r,j}$ as in Equation* (8).

Theorem 1 is given in [8, Cor. 13] (for a preliminary version see [7, Cor. 4.4]). We think, however, that our simple access to this theorem, i.e., via cotangent derivatives and the old result (5), deserves to be noted.

4. Another Form of the Coefficients $c_{r,j}$

In the paper [3], formulas for $y(\chi|i^r\cot^r(\pi/n))$, $\chi \in \mathcal{X}$ primitive, are given that involve coefficients $d_{r,j}$ seemingly different from the above $c_{r,j}$. In this section we use properties of character coordinates in order to show that $c_{r,j}$ and $d_{r,j}$ are basically the same.

For an integer $r \geq 1$, we put

$$\mathcal{X}^r = \begin{cases} \mathcal{X}^-, & \text{if } r \text{ is odd;} \\ \mathcal{X}^+, & \text{if } r \text{ is even.} \end{cases}$$

Let $\chi \in \mathcal{X}^r$ be a primitive character mod n. Formulas (2.29) and (2.30) of [3] say, in our terminology,

$$y(\chi|i^r\cot^r(\pi/n)) = -2^r \sum_{\substack{1\leq j\leq r \\ j\equiv r \bmod 2}} d_{r,j}B_{j,\chi}/j!, \tag{12}$$

with

$$d_{r,j} = \sum_{\substack{j_1,\ldots,j_r\geq 0 \\ j+2j_1+\ldots+2j_r=r}} \prod_{t=1}^{r} B_{2j_t}/(2j_t)!. \tag{13}$$

Here the numbers B_{2j_t} are ordinary Bernoulli numbers.

On the other hand, Equation (5) says, since χ is primitive,

$$y(\chi|i^j\cot_{j-1}(\pi/n)) = \frac{(-1)^r 2^j}{j}B_{j,\chi}$$

for the numbers $j \equiv r \bmod 2$, $1 \leq j \leq n$. Hence we may express the numbers $B_{j,\chi}$ of Equation (12) in terms of $y(\chi|i^j\cot_{j-1}(\pi/n))$. Thereby, we obtain

$$y(\chi|i^r\cot^r(\pi/n)) = \sum_{\substack{1\leq j\leq r \\ j\equiv r \bmod 2}} \frac{(-1)^{r+1}2^{r-j}}{(j-1)!}\, d_{r,j}\, y(\chi|i^j\cot_{j-1}(\pi/n))$$

The $\mathbb{Q}$-linearity of the χ-coordinate yields

$$y(\chi|i^r\cot^r(\pi/n)) = y(\chi| \sum_{\substack{1\leq j\leq r \\ j\equiv r \bmod 2}} \frac{(-1)^{r+1}2^{r-j}}{(j-1)!}\, d_{r,j}\, i^j\cot_{j-1}(\pi/n)). \tag{14}$$

Now suppose that $n = p$ is a prime. First we assume that r is odd. Then all characters $\chi \in \mathcal{X}^r$ are primitive. Thus, Equation (14) holds for all $\chi \in \mathcal{X}^r$. However, for $\chi \in \mathcal{X}^{r+1}$ both sides of Equation (14) vanish. So this equation holds for all $\chi \in \mathcal{X}$. In Section 2 we have seen that this means

$$i^r\cot^r(\pi/p) = \sum_{\substack{1\leq j\leq r \\ j\equiv r \bmod 2}} \frac{(-1)^{r+1}2^{r-j}}{(j-1)!} d_{r,j}\, i^j\cot_{j-1}(\pi/p).$$

On the other hand, Equation (9) implies

$$i^r \cot^r(\pi/p) = \sum_{\substack{1 \le j \le r \\ j \equiv r \bmod 2}} c_{r,j} i^j \cot_{j-1}(\pi/p).$$

We shall see below that, for a sufficiently large prime p, the numbers $i^j \cot_{j-1}(\pi/p)$, $1 \le j \le r$, $j \equiv r \bmod 2$, are $\mathbb{Q}$-linearly independent. Under this assumption, we may compare the coefficients on the right-hand sides of the last two identities and get

$$c_{r,j} = \frac{(-1)^{r+1} 2^{r-j}}{(j-1)!}\, d_{r,j}, \quad 1 \le j \le r, j \equiv r \bmod 2. \tag{15}$$

If r is even, there are minor differences. Again, let $n = p$ be a prime. Then all characters $\chi \in \mathcal{X}^r$ are primitive except the principal character χ_0. Hence Equation (14) holds for all $\chi \in \mathcal{X} \smallsetminus \{\chi_0\}$. Suppose that

$$\begin{aligned} &y(\chi_0 | \cot^r(\pi/p)) = C_1 \in \mathbb{Q} \text{ and} \\ &y(\chi_0 | \sum_{\substack{1 \le j \le r \\ j \equiv r \bmod 2}} \frac{(-1)^{r+1} 2^{r-j}}{(j-1)!}\, d_{r,j}\, i^j \cot_{j-1}(\pi/n)) = C_2 \in \mathbb{Q}. \end{aligned}$$

Now we use Formula (11) and $\varphi(p) = p - 1$. Therefore, the χ-coordinate of $i^r \cot^r(\pi/p) - C_1/(p-1)$ agrees withe the χ-coordinate of

$$\sum_{\substack{1 \le j \le r \\ j \equiv r \bmod 2}} \frac{(-1)^{r+1} 2^{r-j}}{(j-1)!}\, d_{r,j}\, i^j \cot_{j-1}(\pi/n) - C_2/(p-1)$$

for each $\chi \in \mathcal{X}$, and so these numbers are equal. Thus,

$$i^r \cot^r(\pi/p) = \sum_{\substack{1 \le j \le r \\ j \equiv r \bmod 2}} \frac{(-1)^{r+1} 2^{r-j}}{(j-1)!}\, d_{r,j}\, i^j \cot_{j-1}(\pi/n) + (C_1 - C_2)/(p-1).$$

In a similar way as above, we use the fact that the family $(1; i^j \cot_{j-1}(\pi/p) : 1 \le j \le r, j \equiv r \bmod 2)$ is $\mathbb{Q}$-linearly independent for a sufficiently large prime p. Equation (10) implies

$$i^r \cot^r(\pi/n) = \sum_{\substack{1 \le j \le r \\ j \equiv r \bmod 2}} c_{r,j}\, i^j \cot_{j-1}(\pi/n) + 1.$$

On comparing the coefficients on the right-hand side of the last two identities, we see that Equation (15) holds also in this case. Altogether, we have the following theorem.

Theorem 2. *The coefficients $c_{r,j}$ and $d_{r,j}$ are connected by Equation* (15)*.*

We still have to show the aforesaid linear independence. By Equation (4), the conjugates of $i\cot(\pi/p)$ are just the numbers

$$i\cot(\pi k/p),\ \ 1 \le k \le p-1.$$

Since the cotangent function is strictly monotonous in $(0,\pi)$, these numbers are pairwise different. Accordingly, the minimal polynomial of $i\cot(\pi/p)$ (over $\mathbb{Q}$) has the degree $p-1$. This means that the numbers

$$i^j\cot(\pi/p)^j,\ \ j = 0,\dots,p-2,$$

are $\mathbb{Q}$-linearly independent.

Now we choose p such that $p-2 \ge r$. If r is odd, the families $(i^j\cot^j : 1 \le j \le r, j \equiv r \bmod 2)$ and $(i^j\cot_{j-1} : 1 \le j \le r, j \equiv r \bmod 2)$ span the same $\mathbb{Q}$-vector space. This is also true for the families $(i^j\cot^j(\pi/p) : 1 \le j \le r, j \equiv r \bmod 2)$ and $(i^j\cot_{j-1}(\pi/p) : 1 \le j \le r, j \equiv r \bmod 2)$. Accordingly, one of the latter families is $\mathbb{Q}$-linearly independent if, and only if, the other is $\mathbb{Q}$-linearly independent. But we have shown the $\mathbb{Q}$-linear independence of the first of the latter families. So the second family is also $\mathbb{Q}$-linearly independent.

If r is even, we work with the families $(1; i^j\cot^j(\pi/p) : 1 \le j \le r, j \equiv r \bmod 2)$ and $(1; i^j\cot_{j-1}(\pi/p) : 1 \le j \le r, j \equiv r \bmod 2)$ instead.

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