GENERALIZATIONS OF SEVERAL EXPLICIT FORMULAS FOR EULER POLYNOMIALS

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Abstract

In this paper, we employ tools of classical linear operator theory to derive new explicit formulas for Euler polynomials. Additionally, we present an explicit formula for generalized Euler polynomials. These results generalize recent identities by Wei, Qi, and Guo. We offer a simplified approach to understand these classical polynomials.

1. Introduction and Preliminaries

Throughout this paper, we let $\mathrm{End}(\mathbb{C}[x])$ denote the algebra of endomorphisms of the $\mathbb{C}$-vector space $\mathbb{C}[x]$ and let $(\mathbb{C}[x])^{\mathbb{N}}$ denote the set of polynomial sequences. An operator on $\mathbb{C}[x]$ is any element of $\mathrm{End}(\mathbb{C}[x])$. Operators appeared early in mathematics to simplify the writing of formulas for polynomial sequences and remarkable numbers, and they were also studied as mathematical objects to discover their properties. With operator functions at our disposal, we can solve many partial differential equations. Operators are also used in quantum mechanics and in a wide range of mathematical domains. Some of the most well-known operators include the *identity operator* I, defined by $I(x^n) = x^n$ for all $n \in \mathbb{N}_0$, the *translation operator* T_r, defined for any complex number r by $T_r(x^n) = (x + r)^n$ for all $n \in \mathbb{N}_0$, the *differential operator* D, defined by $D(x^0) = 0$ and $D(x^n) = nx^{n-1}$ for all $n \in \mathbb{N}$,

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the *multiplication operator* M_x, defined by $M_x(x^n) = x^{n+1}$ for all $n \in \mathbb{N}_0$, the *finite difference operator* Δ_r , defined by $\Delta_r = T_r - I$, and the *backward difference operator* ∇_r, defined by $\nabla_r = I - T_{-r}$. Note that T_1, Δ_1, and ∇_1 are simply denoted by T, Δ, and ∇, respectively. For convenience, we denote the identity operator I as 1. It is clear that $\Delta = T - 1$ and $\nabla = 1 - T^{-1}$. Moreover, according to Taylor's formula, for any $P \in \mathbb{C}[x]$, we have

$$TP(x) = P(x+1) = \sum_{j=0}^{\infty} \frac{D^j P(x)}{j!} = e^D P(x).$$

From this, it follows that $T = e^D$. More generally, we have $T_r = e^{rD} = \sum_{j=0}^{\infty} \frac{r^j D^j}{j!}$.

A *composition operator* is an operator that commutes with the differential operator. It is well-known that the set $\mathfrak{C}$ of composition operators is a commutative subalgebra of the algebra $\mathrm{End}(\mathbb{C}[x])$.

We now introduce a key mapping between polynomial sequences and operators. Consider the map defined as

$$L : (\mathbb{C}[x])^{\mathbb{N}_0} \to \mathrm{End}(\mathbb{C}[x]), \quad \{A_n(x)\}_{n\geq 0} \mapsto \Omega_A,$$

where Ω_A is the operator defined by $\Omega_A(x^n) = A_n(x)$ for all $n \in \mathbb{N}_0$. Clearly, L is bijective, and its inverse is given by

$$L^{-1} : \mathrm{End}(\mathbb{C}[x]) \to (\mathbb{C}[x])^{\mathbb{N}_0}, \quad \Omega \mapsto \{\Omega(x^n)\}_{n\in\mathbb{N}_0}.$$

For any $A \in (\mathbb{C}[x])^{\mathbb{N}}$, the image Ω_A under L is called the *operator associated* with A. Conversely, for any operator Ω, the polynomial sequence $\{\Omega(x^n)\}_{n\geq 0}$ is said to be *associated* with Ω . Two important properties should be noted:

1. Let $A = \{A_n(x)\}_{n\geq 0} \in (\mathbb{C}[x])^{\mathbb{N}_0}$. Then, $D(A_0(x)) = 0$ and $D(A_n(x)) = nA_{n-1}(x)$ for $n \geq 1$, if and only if $D \circ \Omega_A = \Omega_A \circ D$.

2. Let $A = \{A_n(x)\}_{n\geq 0} \in (\mathbb{C}[x])^{\mathbb{N}_0}$ satisfying $D(A_0(x)) = 0$ and $D(A_n(x)) = nA_{n-1}(x)$ for all $n \in \mathbb{N}$. Then we have

$$A_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} A_k(y). \tag{1}$$

Recall that a sequence of polynomials $A = \{A_n(x)\}_{n\geq 0}$ is said to be an *Appell polynomial sequence* if $D(A_n(x)) = nA_{n-1}(x)$ for all $n \in \mathbb{N}$ and $A_0(x)$ is a nonzero constant polynomial. This is equivalent to the conditions $D(A_0(x)) = 0$, $A_0(x) \neq 0$, and $D(A_n(x)) = nA_{n-1}(x)$ for all $n \in \mathbb{N}$. By considering the following power series in $\mathbb{C}[[z]]$,

$$S_E(z) = \frac{2}{e^z + 1} = \left(1 + \frac{1}{2}\sum_{k=1}^{\infty} \frac{z^k}{k!}\right)^{-1},$$

we define the composition operator Ω_E by $\Omega_E = S_E(D) = \frac{2}{e^D+1}$. The operator Ω_E is called the *Euler operator*. For $\alpha \in \mathbb{C}$, we define the *generalized Euler operator of order* α by $\Omega_{E^{(\alpha)}} = \Omega_E^\alpha$. Euler polynomials $\{E_n(x)\}_{n\geq 0}$ can be defined via operators as $E_n(x) = \Omega_E(x^n)$. In turn, generalized Euler polynomials $\{E_n^{(\alpha)}(x)\}_{n\geq 0}$ can be defined as in [7] $E_n^{(\alpha)}(x) = \Omega_{E^{(\alpha)}}(x^n)$. The classical Euler polynomials are then obtained by taking $\alpha = 1$.

In 2015, Wei and Qi [12] introduced four new explicit expressions for the Euler numbers. The first expression provides a representation for the Euler number E_{2n} in terms of a specific determinant. It states that

$$E_{2n} = (-1)^n \left| \binom{i}{j-1} \cos\left((i-j+1)\frac{\pi}{2}\right) \right|_{(2n)\times(2n)},$$

where $|c_{ij}|_{n\times n}$ denotes the determinant of the matrix $(c_{ij})_{1\leq i,j\leq n}$. The other three theorems provide explicit expressions for E_{2n} and E_n as follows:

$$E_{2n} = (2n+1)\sum_{k=1}^{2n}(-1)^k \frac{1}{2^k(k+1)}\binom{2n}{k}\sum_{j=0}^{k}\binom{k}{j}(2j-k)^{2n} \quad \text{([12, Theorem 1.2])}, \tag{1}$$

$$E_n = 1 + \sum_{k=1}^{n}\frac{(k+1)!}{2^k}S(n,k)\sum_{\ell=1}^{k}(-1)^\ell \frac{2^\ell}{\ell+1}\binom{\ell+1}{k-\ell} \quad \text{([12, Theorem 1.3])}, \tag{2}$$

$$E_n = 1 + \sum_{\ell=1}^{n}(-1)^\ell \frac{1}{\ell+1}\sum_{k=0}^{n-\ell}\frac{(k+\ell+1)!}{2^k}\binom{\ell+1}{k}S(n,k+\ell) \quad \text{([12, Theorem 1.3])}, \tag{3}$$

$$E_{2n} = \sum_{k=1}^{2n}(-1)^k\frac{1}{2^k}\sum_{\ell=0}^{2k}(-1)^\ell\binom{2k}{\ell}(k-\ell)^{2n} \quad \text{([12, Theorem 1.4])}, \tag{4}$$

where $S(n,k)$ denotes the Stirling number of the second kind. These results are obtained by exploiting remarkable properties of Bell partial polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ and the Faà di Bruno formula. The Bell partial polynomials are also known as Bell polynomials of the second kind [2, p. 134, Theorem A] and can be defined as

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{1\leq q\leq n,\ell_q\in\mathbb{N}\\ \sum_{q=1}^n q\ell_q=n\\ \sum_{q=1}^n \ell_q=k}} \frac{n!}{\prod_{q=1}^{n-k+1}\ell_q}\prod_{q=1}^{n-k+1}\left(\frac{x_q}{q!}\right)^{\ell_q} \quad (n\geq k\geq 0),$$

and the Faà di Bruno formula [2, p. 134, Theorem C] is given by

$$\frac{\mathrm{d}^n (f\circ g)(x)}{\mathrm{d}\,x^n} = \sum_{k=1}^{n} f^{(k)}(g(x))B_{n,k}\left(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)\right) \quad (n\geq 1).$$

It is worth noting that Formulas (1) and (4) can be seen as variants of Formulas (36) and (37) presented in [4].

We aim to generalize the identities (1), (2), (3), and (4) to Euler polynomials. To this end, the paper is structured as follows: Following the presentation of operators, some of their properties, and the results from [12] in Section 1, Section 2 introduces lemmas establishing connections between operators, polynomial sequences, and generating functions. These lemmas will be instrumental in proving our main theorems, which are presented and proved in Section 4, after establishing several auxiliary results in Section 3.

2. Lemmas

In this section, we present some lemmas that establish a connection between operators, polynomial sequences, and generating functions.

Lemma 2.1. *Let $\{A_n(x)\}_{n\geq 0}$ be a polynomial sequence associated with a composition operator $S(D)$. Then, for any nonzero complex number λ, we have*

$$A_{n+1}(x) - xA_n(x) = (S'(D))(x^n) \quad (n \geq 0), \tag{5}$$

$$A_n(\lambda x) = \lambda^n S\left(\frac{D}{\lambda}\right)(x^n) \quad (n \geq 0), \tag{6}$$

$$(-1)^n A_n(\lambda - x) = e^{-\lambda D} S(-D)(x^n) \quad (n \geq 0). \tag{7}$$

Proof. Differentiating the relation

$$S(z) = e^{-xz} \sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!} \text{ with respect to } z,$$

we get

$$S'(z) = -xe^{-xz} \sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!} + e^{-xz} \sum_{n=0}^{\infty} A_{n+1}(x) \frac{z^n}{n!}.$$

Thus,

$$S'(z) = e^{-xz} \sum_{n=0}^{\infty} (A_{n+1}(x) - xA_n(x)) \frac{z^n}{n!}.$$

Hence, Formula (5) follows.

We have the generating function

$$\sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!} = S(z)e^{xz}. \tag{8}$$

Substituting x by λx and z by $\frac{z}{\lambda}$ in (8), we obtain

$$\sum_{n=0}^{\infty} \frac{A_n(\lambda x)}{\lambda^n} \frac{z^n}{n!} = S\left(\frac{z}{\lambda}\right) e^{xz}.$$

Thus, Formula (6) follows.

Exploiting Formula (5), we get

$$\begin{aligned}(-1)^n A_n(\lambda - x) &= e^{-\lambda D}(-1)^n A_n(-x)\\ &= e^{-\lambda D} S(-D)(x^n).\end{aligned}$$

Thus, Formula (7) follows. □

First, recall that according to Lemma 2.1, for any nonzero complex number q and any Appell polynomial sequence $\{A_n(x)\}_{n\geq 0}$, the polynomial sequence $\left\{\frac{1}{q^n} A_n(qx)\right\}_{n\geq 0}$ is also an Appell polynomial sequence. Moreover, if $\Omega_A = S(D)$ is the Appell operator associated to the sequence $\{A_n(x)\}_{n\geq 0}$, then $S\left(\frac{D}{q}\right)$ is the Appell operator associated to the Appell polynomial sequence $\left\{\frac{1}{q^n} A_n(qx)\right\}_{n\geq 0}$. Applying Lemma 2.1 to the sequence of classical Euler polynomials $\{E_n(x)\}_{n\geq 0}$, whose Appell associated operator is $S_E(D) = \frac{2}{e^D+1}$, we can show that for any nonzero complex number q, the polynomial sequence $\left\{\frac{1}{q^n} E_n(qx)\right\}_{n\geq 0}$ is an Appell polynomial sequence whose associated Appell operator is $S_E\left(\frac{D}{q}\right) = \frac{2}{e^{D/q}+1}$. In other words, we have

$$\frac{1}{q^n} E_n(qx) = \left(\frac{2}{e^{D/q}+1}\right)(x^n).$$

In particular, for $q = \frac{1}{2}$, we get

$$2^n E_n\left(\frac{x}{2}\right) = \left(\frac{2}{e^{2D}+1}\right)(x^n).$$

The next two lemmas will facilitate the proof of new generalizations of the theorems of Qi and Guo [8] and Wei and Qi [12].

Lemma 2.2. *We have*

$$\frac{2+2z}{2+2z+z^2} = \sum_{k=0}^{\infty} \alpha_k z^k,$$

where

$$\alpha_k = \frac{k+1}{2^k} \sum_{\ell=0}^{k} (-1)^\ell \frac{2^\ell}{\ell+1} \binom{\ell+1}{k-\ell}. \tag{9}$$

Proof. Easily, we have

$$\frac{2+2z}{2+2z+z^2} = \frac{1}{z+1+i} + \frac{1}{z+1-i} = \sum_{k=0}^{\infty} \alpha_k z^k,$$

where

$$\alpha_k = \left(\frac{1+i}{2}\right)^{k+1} + \left(\frac{1-i}{2}\right)^{k+1}. \tag{10}$$

Now, recall the well-known identity for any positive integer n:

$$x^n + y^n = \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-\ell} \binom{n-\ell}{\ell} (-xy)^\ell (x+y)^{n-2\ell}. \tag{11}$$

Applying Identity (11) to Equation (10), with $x = \frac{1+i}{2}$ and $y = \frac{1-i}{2}$, we deduce

$$\alpha_k = \sum_{\ell=0}^{\lfloor \frac{k+1}{2} \rfloor} \frac{k+1}{k+1-\ell} \binom{k+1-\ell}{\ell} \left(-\frac{1}{2}\right)^\ell.$$

Changing ℓ to $k-\ell$, we obtain

$$\alpha_k = \sum_{\ell=m}^{k} \frac{k+1}{\ell+1} \binom{\ell+1}{k-\ell} \left(-\frac{1}{2}\right)^{k-\ell},$$

where $m = k - \left\lfloor \frac{k+1}{2} \right\rfloor$. As $\binom{\ell+1}{k-\ell} = 0$ for $0 \leq k < m$, we deduce Formula (9). $\square$

Lemma 2.3. *The sequence of polynomials $\{2^n E_n\left(\frac{x+1}{2}\right)\}_{n\geq 0}$ is an Appell sequence with associated operator $e^D S_E(2D)$. Equivalently, we have*

$$2^n E_n\left(\frac{x+1}{2}\right) = (e^D S_E(2D))(x^n) = \left(\frac{2e^D}{2+e^{2D}}\right)(x^n).$$

Furthermore, we also have

$$\frac{2e^D}{2+e^{2D}} = \frac{2+2\Delta}{2+2\Delta+\Delta^2} = \sum_{k=0}^{\infty} \alpha_k \Delta^k,$$

where

$$\alpha_k = \frac{k+1}{2^k} \sum_{\ell=0}^{k} (-1)^\ell \frac{2^\ell}{\ell+1} \binom{\ell+1}{k-\ell}.$$

Proof. From the definitions of T and $S_E(D)$, we have

$$2^n E_n\left(\frac{x+1}{2}\right) = T\left(2^n E_n\left(\frac{x}{2}\right)\right) = (e^D S_E(2D))(x^n) = \left(\frac{2e^D}{2+e^{2D}}\right)(x^n).$$

Next, using the identity $e^D = 1 + \Delta$, we obtain

$$\frac{2e^D}{2+e^{2D}} = \frac{2(1+\Delta)}{2+(1+\Delta)^2} = \frac{2+2\Delta}{2+2\Delta+\Delta^2} = \sum_{k=0}^{\infty} \alpha_k \Delta^k,$$

where

$$\alpha_k = \frac{k+1}{2^k} \sum_{\ell=0}^{k} (-1)^\ell \frac{2^\ell}{\ell+1} \binom{\ell+1}{k-\ell}.$$

This completes the proof of the lemma. □

The following lemma is well-known and can be found in [9, p. 205].

Lemma 2.4. *For any formal series $S(z) \in \mathbb{C}[[x]]$, we have*

$$S(D)M_x - M_x S(D) = S'(D). \tag{12}$$

3. Auxiliary Results

In contemporary mathematical literature, numerous explicit formulas have been established for classical Bernoulli and Euler numbers and polynomials. Notably, the articles [3, 5, 6, 11] and the references therein offer intriguing explicit formulas for Bernoulli and Euler polynomials, employing composition operators.

Recall that

$$\Delta = T - 1$$

and

$$\Delta^k = (e^D - 1)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{jD}.$$

As seen before, Ω_B and Ω_E denote the composition operators associated with the Bernoulli and Euler polynomial sequences. Then we can write

$$\Omega_B = \frac{\ln(1+\Delta)}{1+\Delta} = \sum_{j=0}^{\infty} (-1)^k \frac{\Delta^k}{k+1}$$

and

$$\Omega_E = \frac{2}{2+\Delta} = \frac{1}{\frac{1}{2}\Delta+1} = \sum_{j=0}^{\infty} (-1)^k \frac{\Delta^k}{2^k}.$$

Hence the following formulas hold:

$$B_n(x) = \sum_{k=0}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k+1} \binom{k}{j} (x+j)^n \tag{13}$$

and

$$E_n(x) = \sum_{k=0}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{2^k} \binom{k}{j} (x+j)^n. \tag{14}$$

The following proposition introduces a new explicit formula obtained by applying composition operators.

Proposition 3.1. *For any positive integers n and m with $m \geq n$, we have*

$$2^n E_n\left(\frac{x+1}{2}\right) = \sum_{k=0}^{m} \frac{k!}{2^{(k-1)/2}} \cos\left(\frac{(3k-1)\pi}{4}\right) S(n,k,x), \tag{15}$$

where $S(n,k,x)$ denotes the generalized Stirling polynomials defined by

$$S(n,k,x) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n.$$

Note that $S(n,k,0) = S(n,k)$.

Proof. Consider the composition operator $\Omega_{E^*} = \frac{2e^D}{e^{2D}+1}$. We can express this operator as a series in Δ as follows:

$$\begin{aligned} \Omega_{E^*} &= \frac{2e^D}{e^{2D}+1} = \frac{2+2\Delta}{1+(1+\Delta)^2} = -\sum_{k=1}^{\infty} \frac{1}{2^k} \operatorname{Re}\left((i-1)^{k+1}\right) \Delta^k \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{(k-1)/2}} \cos\left(\frac{(3k-1)\,\pi}{4}\right) \Delta^k. \end{aligned} \tag{16}$$

Applying this last operator to x^n, and noting that $\Delta^m(x^n) = 0$ for $m > n$, yields Formula (15). □

In 2018, Bounebirat et al. [1] established the following explicit formula for the Euler numbers:

$$E_n = -\sum_{k=1}^{n} \frac{k!}{2^k} \operatorname{Re}\left((i-1)^{k+1}\right) S(n,k) \quad (n \geq 0). \tag{17}$$

Note that Formula (15) extends the explicit formula (17) to Euler polynomials. The following two classical recurrence formulas are well-known [10, pp. 96 and 103]:

$$B_n^{(\alpha+1)}(x) = \left(1 - \frac{n}{\alpha}\right) B_n^{(\alpha)}(x) - n\left(\frac{x}{\alpha} - 1\right) B_{n-1}^{(\alpha)}(x) \tag{18}$$

and

$$E_{n+1}^{(\alpha)}(x) = xE_n^{(\alpha)}(x) - \frac{\alpha}{2} E_n^{(\alpha+1)}(x+1). \tag{19}$$

In the following proposition, we establish an analogous recurrence formula [10] to

$$B_n^{(\alpha)}(x) = \left(x - \frac{\alpha}{2}\right) B_{n-1}^{(\alpha)}(x) - \frac{\alpha}{n} \sum_{k=0}^{n-2} (-1)^{n-k} \binom{n}{k} B_{n-k}(0) B_k^{(\alpha)}(x). \tag{20}$$

Our formula is verified by the generalized Euler polynomials.

Proposition 3.2. *For any $\alpha \in \mathbb{C}$ and any integer $n \geq 1$, we have*

$$E_n^{(\alpha)}(x) = \left(x - \frac{\alpha}{2}\right) E_{n-1}^{(\alpha)}(x) + \frac{\alpha}{2} \sum_{k=0}^{n-2} (-1)^{n-k} \binom{n-1}{k} E_{n-k}(0) E_k^{(\alpha)}(x). \tag{21}$$

Proof. Consider the formal series $S_{E^{(\alpha)}}(z) = \left(\frac{2}{e^z+1}\right)^{\alpha}$. Differentiating, we obtain

$$S'_{E^{(\alpha)}}(z) = -\frac{\alpha}{2} z e^z \left(\frac{2}{e^z+1}\right)^{\alpha+1}.$$

According to Relation (12), for any integer $n \geq 1$, we have

$$S'_{E^{(\alpha)}}(D)(x^n) = (S_{E^{(\alpha)}}(D) M_x - M_x S_{E^{(\alpha)}}(D))(x^n).$$

This identity leads to the relation

$$\frac{1}{2} \alpha E_n^{(\alpha+1)}(x+1) = E_{n+1}^{(\alpha)}(x) - x E_n^{(\alpha)}(x). \tag{22}$$

Now, note that

$$E_n^{(\alpha+1)}(x+1) = \Omega_E^{\alpha} \Omega_E^{\alpha} (x+1)^n = \Omega_E^{\alpha} E_n(x+1) = (-1)^n \Omega_E^{\alpha} (E_n(-x)).$$

Expanding $E_n(-x)$, we get

$$\Omega_E^{\alpha} \left\{ (-1)^n \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(0)(-x)^k \right\} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E_{n-k}(0) E_k^{(\alpha)}(x). \tag{23}$$

Finally, by substituting this into (22) we immediately derive Formula (21). □

4. Main Results

The following result is a generalization of Theorem 1.2 in [12].

Theorem 4.1. *For any positive integers m and n such that $m \geq \left\lfloor \frac{n}{2} \right\rfloor$, we have*

$$E_n(x) = (m+1) \sum_{k=0}^{m} \frac{(-1)^k}{2^k (k+1)} \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} \left(x - \frac{1}{2} + \frac{2j-k}{2}\right)^n. \tag{24}$$

Proof. We prove Formula (24) using appropriate composition operators. To begin we observe that Formula (24) is equivalent to

$$2^n E_n\left(\frac{x+1}{2}\right) = \sum_{k=0}^{m} \frac{(-1)^k}{2^k}\binom{m+1}{k+1}\sum_{j=0}^{k}\binom{k}{j}(x+2j-k)^n \quad (2m+1 \geq n).$$

We then have

$$2^n E_n\left(\frac{x+1}{2}\right) = T\left(2^n E_n\left(\frac{x}{2}\right)\right) = \left(\frac{2e^D}{e^{2D}+1}\right)(x^n) = \Phi^{-1}(x^n),$$

where Φ is the Appell operator defined as

$$\Phi = \frac{e^D + e^{-D}}{2} = 1 + \sum_{k=1}^{\infty}\frac{D^{2k}}{(2k)!}.$$

Consider now the operator

$$\Psi = \Phi - 1 = \sum_{k=1}^{\infty}\frac{D^{2k}}{(2k)!}. \tag{25}$$

Since Ψ has order 2 (see [9, p. 201]), it follows that Ψ^{m+1} has order $2m+2$. Thus, for $2m+2 > n$,

$$\Psi^{m+1}(x^n) = 0$$

and

$$(1-\Psi^{m+1})(x^n) = x^n.$$

That is, for $m \geq \left\lfloor \frac{n}{2} \right\rfloor$, we have

$$2^n E_n\left(\frac{x+1}{2}\right) = \Phi^{-1}(x^n) = \Phi^{-1}(1-\Psi^{m+1})(x^n).$$

Moreover, the composition operator $\Phi^{-1}(1-\Psi^{m+1})$ can be expressed as a polynomial in Φ. Indeed, according to (25), we have

$$\Phi^{-1}(1-\Psi^{m+1}) = \Phi^{-1}(1-(1-\Phi)^{m+1}) = \sum_{k=0}^{m}\binom{m+1}{k+1}(-1)^k\Phi^k.$$

Consequently, for $m \geq \left\lfloor \frac{n}{2} \right\rfloor$,

$$2^n E_n\left(\frac{x+1}{2}\right) = \sum_{k=0}^{m}\binom{m+1}{k+1}(-1)^k\Phi^k(x^n). \tag{26}$$

On the other hand, we have

$$\Phi^k = \left(\frac{e^D + e^{-D}}{2}\right)^k = \frac{e^{-kD}}{2^k}(e^{2D}+1)^k = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}e^{(2j-k)D},$$

thus,

$$\Phi^k(x^n) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x+2j-k)^n. \tag{27}$$

Finally, from (26) and (27), we deduce that for $m \geq \lfloor \frac{n}{2} \rfloor$

$$2^n E_n\left(\frac{x+1}{2}\right) = \sum_{k=0}^{m}\binom{m+1}{k+1}\frac{(-1)^k}{2^k}\sum_{j=0}^{k}\binom{k}{j}(x+2j-k)^n.$$

Therefore, the proof is complete. □

The following result is a generalization of Formula (1.5) in [12, Theorem 1.3].

Theorem 4.2. *For any positive integer n, we have*

$$E_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\frac{(k+1)!}{2^k}S(n,k,2x-1)\sum_{\ell=0}^{k}(-1)^\ell\frac{2^\ell}{\ell+1}\binom{\ell+1}{k-\ell}. \tag{28}$$

Proof. Let us prove the following equivalent formula:

$$2^n E_n\left(\frac{x+1}{2}\right) = \sum_{k=0}^{n}\frac{(k+1)!}{2^k}S(n,k,x)\sum_{\ell=0}^{k}(-1)^\ell\frac{2^\ell}{\ell+1}\binom{\ell+1}{k-\ell}. \tag{29}$$

We know from Lemma 2.3 that

$$2^n E_n\left(\frac{x+1}{2}\right) = \left(\frac{2+2\Delta}{1+(1+\Delta)^2}\right)(x^n) = \left(\sum_{k=0}^{\infty}\alpha_k\Delta^k\right)(x^n),$$

where

$$\alpha_k = \frac{k+1}{2^k}\sum_{\ell=0}^{k}(-1)^\ell\frac{2^\ell}{\ell+1}\binom{\ell+1}{k-\ell}.$$

Hence, we deduce that

$$\begin{aligned}2^n E_n\left(\frac{x+1}{2}\right) &= \sum_{k=0}^{n}\alpha_k\Delta^k(x^n)\\ &= \sum_{k=0}^{n}\alpha_k k!S(n,k,x)\\ &= \sum_{k=0}^{n}\frac{(k+1)!}{2^k}\sum_{\ell=0}^{k}(-1)^\ell\frac{2^\ell}{\ell+1}\binom{\ell+1}{k-\ell}S(n,k,x),\end{aligned}$$

as required, completing this proof. □

The following result is a generalization of Formula (1.6) in [12, Theorem 1.3].

Theorem 4.3. *For any integer $n \geq 0$, we have*

$$E_n(x) = \frac{1}{2^n} \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell+1} \sum_{k=0}^{n-\ell} \frac{(k+\ell+1)!}{2^k} \binom{\ell+1}{k} S(n, k+\ell, 2x-1). \tag{30}$$

Proof. Let us prove the following equivalent formula:

$$2^n E_n\left(\frac{x+1}{2}\right) = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell+1} \sum_{k=0}^{n-\ell} \frac{(k+\ell+1)!}{2^k} \binom{\ell+1}{k} S(n, k+\ell, x). \tag{31}$$

We know from Lemma 2.3 that

$$2^n E_n\left(\frac{x+1}{2}\right) = \left(\frac{2+2\Delta}{2+2\Delta+\Delta^2}\right)(x^n) = \sum_{k=0}^{\infty} \alpha_k \Delta^k(x^n),$$

where

$$\alpha_k = \frac{k+1}{2^k} \sum_{\ell=0}^{k} (-1)^\ell \frac{2^\ell}{\ell+1} \binom{\ell+1}{k-\ell}.$$

Similarly, by taking into account the fact that $S(n, k+\ell, x) = \frac{1}{(k+\ell)!} \Delta^{k+\ell}(x^n)$, the right-hand side of (31) becomes

$$\begin{aligned} \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell+1} \sum_{k=0}^{n-\ell} \frac{(k+\ell+1)!}{2^k} \binom{\ell+1}{k} S(n, k+\ell, x) &= \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell+1} \sum_{j=0}^{n-\ell} \frac{j+\ell+1}{2^j} \binom{\ell+1}{j} \Delta^{j+\ell}(x^n) \\ &= \sum_{\ell=0}^{n} \beta_k \Delta^k(x^n), \end{aligned}$$

with

$$\begin{aligned} \beta_k &= \sum_{j+\ell=k} \frac{(-1)^\ell}{\ell+1} \frac{j+\ell+1}{2^j} \binom{\ell+1}{j} \\ &= \sum_{\ell=0}^{k} \frac{(-1)^\ell}{\ell+1} \frac{k+1}{2^{k-\ell}} \binom{\ell+1}{k-\ell} \\ &= \frac{k+1}{2^k} \sum_{\ell=0}^{k} (-1)^\ell \frac{2^\ell}{\ell+1} \binom{\ell+1}{k-\ell} \\ &= \alpha_k. \end{aligned}$$

Thus, Formula (31) follows. □

Formula (4) was reaffirmed in 2017 by Qi and Guo [8] . In the next theorem, we extend it to generalized Euler polynomials.

Theorem 4.4. *For any $m \geq \left[\frac{n}{2}\right]$, we have*

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{m} \frac{(-1)^k}{2^{n+k}} \binom{\alpha+k-1}{k} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (2x-\alpha-k+j)^n. \tag{32}$$

Proof. Let us prove the following equivalent formula:

$$2^n E_n^{(\alpha)}\left(\frac{x+\alpha}{2}\right) = \sum_{k=0}^{m} \frac{(-1)^k}{2^k} \binom{\alpha+k-1}{k} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (x-k+j)^n. \tag{33}$$

We have

$$\begin{aligned} 2^n E_n^{(\alpha)}\left(\frac{x+\alpha}{2}\right) &= T_\alpha\left(2^n E_n^{(\alpha)}\left(\frac{x}{2}\right)\right) \\ &= \left(T_\alpha S_{E_n^{(\alpha)}}(2D)\right)(x^n) \\ &= e^{\alpha D}\left(\frac{2}{e^{2D}+1}\right)^{\alpha}(x^n) \\ &= \left(\frac{2e^D}{e^{2D}+1}\right)^{\alpha}(x^n). \end{aligned} \tag{34}$$

Let us set

$$\Psi_\alpha = (1+\delta)^{-\alpha}$$

and

$$\delta = \frac{1}{2} e^{-D} \Delta^2.$$

Then from

$$\frac{2z}{z^2+1} = \frac{1}{1+\frac{(z^2-1)}{2z}},$$

we derive that

$$\Psi_\alpha = \sum_{k=0}^{\infty} \binom{-\alpha}{k} \delta^k = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha+k-1}{k} \delta^k.$$

Moreover, since the order of δ is equal to 2, we have for any integer $m \geq \left\lfloor \frac{n}{2} \right\rfloor$,

$$\Psi_\alpha(x^n) = \sum_{k=0}^{m} (-1)^k \binom{\alpha+k-1}{k} \delta^k(x^n). \tag{35}$$

The calculation of $\delta^k(x^n)$ gives

$$\delta^k = \frac{1}{2^k} e^{-kD} \Delta^{2k} = \frac{1}{2^k} T_{-k}(T_1-1)^{2k} = \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} T_{j-k}.$$

This yields

$$\delta^k(x^n) = \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (x-k+j)^n. \tag{36}$$

Finally, by inserting (36) in (35), we get

$$\Psi_\alpha(x^n) = \sum_{k=0}^{m} (-1)^k \binom{\alpha+k-1}{k} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (x-k+j)^n.$$

Formula (33) follows immediately by inserting this last formula in (34). Thus, the proof is complete. □

5. Conclusion

It is well-known that the use of formal power series is a powerful tool for studying certain sequences of numbers and polynomials. Throughout this study, we have observed that the use of composition operators represents both a natural approach and an effective, well-suited tool for the investigation of Appell polynomial sequences. The results we have obtained suggest broad and promising avenues for further research in this area.

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