#A67 INTEGERS 25 (2025)

ON k-PELL-LUCAS NUMBERS CLOSE TO A POWER OF 2

Mohamadou Bachabi
Department of Mathematics, Université d'Abomey-Calavi (UAC), Institut de Mathématiques et de Sciences Physiques (IMSP), Dangbo, Benin
mohamadoubachabi96@gmail.com

Bakary Kourouma
Department of Mathematics, Gamal Abdel Nasser University, Faculty of Science, Conakry, Guinea
kouroumabakr22@gmail.com

Alain Togbé
Department of Mathematics and Statistics, Purdue University Northwest, Hammond, Indiana
atogbe@pnw.edu

Received: 10/17/24, Revised: 3/8/25, Accepted: 7/8/25, Published: 8/15/25

Abstract

For $k \geq 2$, let $\left(Q_n^{(k)}\right)_{n \geq 2-k}$ be the k-generalized Pell-Lucas sequence which starts with $0, \cdots, 0, 2, 2$ (k terms) and each term afterwards is given by the linear recurrence

$$Q_n^{(k)} = 2Q_{n-1}^{(k)} + Q_{n-2}^{(k)} + \cdots + Q_{n-k}^{(k)}, \quad \text{for} \quad n \geq 2.$$

An integer n is said to be close to a positive integer m if n satisfies $|n - m| < \sqrt{m}$. In this paper, we solve the Diophantine inequality $\left|Q_n^{(k)} - 2^m\right| < 2^{m/2}$, in positive unknowns k, m, and n.

1. Introduction

Let k, r be integers with $k \geq 2$ and $r \neq 0$. Let the linear recurrence sequence $\left(G_n^{(k)}\right)_{n \geq 2-k}$ of order k be defined by

$$G_n^{(k)} = rG_{n-1}^{(k)} + G_{n-2}^{(k)} + \ldots + G_{n-k}^{(k)},$$

for $n \geq 2$ with the initial conditions

$$G_{-(k-2)}^{(k)} = G_{-(k-3)}^{(k)} = \cdots = G_{-1}^{(k)} = 0, \quad G_0^{(k)} = a, \text{ and } G_1^{(k)} = b.$$

DOI: 10.5281/zenodo.16881744

For $(a,b,r) = (0,1,1)$, the sequence $\left(G_n^{(k)}\right)_{n\geq 2-k}$ is called the *k-generalized Fibonacci sequence* $\left(F_n^{(k)}\right)_{n\geq 2-k}$ [8]. For $(a,b,r) = (0,1,2)$ and $(a,b,r) = (2,2,2)$, the sequence $\left(G_n^{(k)}\right)_{n\geq 2-k}$ is called the *k-generalized Pell sequence* $\left(P_n^{(k)}\right)_{n\geq 2-k}$ and the *k-generalized Pell-Lucas sequence* $\left(Q_n^{(k)}\right)_{n\geq 2-k}$, respectively [14]. The terms of these sequences are called *k-generalized Fibonacci numbers*, *k-generalized Pell numbers,* and *k-generalized Pell-Lucas numbers*, respectively. When $k = 2$, we have the usual Fibonacci, Pell, and Pell-Lucas sequences, $(F_n)_{n\geq 0}$, $(P_n)_{n\geq 0}$, and $(Q_n)_{n\geq 0}$, respectively.

We need the following definition of closeness.

Definition 1. An integer n is said to be *close* to a positive integer m if n satisfies

$$|n-m| < \sqrt{m}.$$

After the introduction of the previous definition by Chern and Cui in 2014 [10], they determined the Fibonacci numbers that are close to a power of 2. Their work was extended by Bravo, Gomez, and Herrera [4], who characterized all terms $F_n^{(k)}$ that are close to a power of 2. In parallel, Açikel, Irmak, and Szalay [1] studied k-generalized Lucas numbers that are close to powers of 2. More recently, Bachabi and Togbé [2] determined the k-generalized Pell numbers in the same context.

As a continuation of the work done in [2], in this paper we study the k-Pell-Lucas numbers that are close to a power of 2. More precisely, we will prove the following theorem.

Theorem 1. *All the solutions $(Q_n^{(k)}, k, n, m)$ of the inequality*

$$\left|Q_n^{(k)} - 2^m\right| < 2^{m/2}, \tag{1}$$

in positive integers $k,\ n,\ m$ with $k \geq 2$, are given by

$$(2,k,1,1),\ k\geq 2, \quad (6,k,2,3),\ k\geq 2, \quad (16,k,3,4),\ k\geq 3,$$

$$(34,2,4,5), \quad (260,3,6,8), \quad \textit{and} \quad (32774,4,11,15).$$

We deduce the following consequence.

Corollary 1. *Let $n,\ m,\ k$ be defined as in Theorem 1, the only solutions to the Diophantine equation $Q_n^{(k)} = 2^m$ are*

$$Q_1^{(k)} = 2^1 = 2,\ k\geq 2 \quad \textit{and} \quad Q_3^{(k)} = 2^4 = 16,\ k\geq 3.$$

Note that this theorem gives the solutions to the Diophantine equation

$$Q_n^{(k)} = 2^m + e \quad \text{with} \quad |e| < 2^{m/2}. \tag{2}$$

For the proof of Theorem 1 we use the properties of the k-Pell-Lucas sequence, Baker's method based on linear forms in logarithms of algebraic numbers, and the Baker-Davenport reduction method [3]. Here, for the reduction method, we will use a modified version of the result due to Bravo, Gómez, and Luca (Lemma 1 of [5]).

2. Preliminary Results

This section is devoted to collecting a few definitions, notations, properties, and results, which will be used in the remainder of this paper.

2.1. Properties of the k-Generalized Pell-Lucas Sequence

The characteristic polynomial of the k-generalized Pell-Lucas sequence is

$$\Phi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1.$$

The above polynomial is irreducible over $\mathbb{Q}[x]$ and it has one positive real root $\alpha := \alpha(k)$ which is located between $\phi^2(1-\phi^{-k})$ and ϕ^2 with $\phi = \frac{1+\sqrt{5}}{2}$, and which lies outside the unit circle (see [17]). The other roots are strictly contained in the unit circle. To simplify the notation, we will omit the dependence of α on k whenever no confusion may arise.

The Binet-type formula for $Q_n^{(k)}$, found in [17], is

$$Q_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 2) g_k(\alpha_i) \alpha_i^n = \sum_{i=1}^{k} \frac{2(\alpha_i - 1)^2}{(k+1)\alpha_i^2 - 3k\alpha_i + k - 1} \alpha_i^n, \tag{3}$$

where the α_i are the roots of the characteristic polynomial $\Phi_k(x)$ and the function g_k is given by

$$g_k(z) := \frac{z-1}{(k+1)z^2 - 3kz + k - 1}, \tag{4}$$

for $k \geq 2$.

Additionally, it was also shown in [17] that the roots located inside the unit circle have a very minimal influence in formula (3), as can be seen by the inequality

$$\left| Q_n^{(k)} - (2\alpha - 2) g_k(\alpha) \alpha^n \right| < 2, \tag{5}$$

for $n \geq 2 - k$. Furthermore, it was shown by Şiar and Keskin in [17, Lemma 10] that the inequalities

$$\alpha^{n-1} < Q_n^{(k)} < 2\alpha^n \tag{6}$$

hold, for $n \geq 1$ and $k \geq 2$.

Lemma 1 ([6], Lemma 1 and [7], Lemma 2.3)**.** *Let $k \geq 2$ be an integer. Then, we have*

(a) $0.276 < g_k(\alpha) < 0.5$ *and* $|g_k(\alpha_i)| < 1, \quad for \quad 2 \leq i \leq k.$

(b) $\phi^2\left(1-\phi^{-k}\right) < \alpha < \phi^2.$

Definition 2. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$, and let $\alpha = \alpha^{(1)}, \ldots, \alpha^{(d)}$ be its conjugates. The *logarithmic height* of α is defined by

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\left(\max\{|\alpha^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. For any algebraic numbers α and β, we have the following properties [19, Property 3.3]:

$$h(\alpha\beta) \leq h(\alpha) + h(\beta),$$
$$h(\alpha \pm \beta) \leq \log 2 + h(\alpha) + h(\beta).$$

Moreover, for any integer n,

$$h(\alpha^n) \leq |n| h(\alpha).$$

With the above notation, Şiar et al. [18] showed that the logarithmic height of $g_k(\alpha)$ satisfies

$$h(g_k(\alpha)) < 5\log k, \quad \text{for} \quad k \geq 2. \tag{7}$$

Lemma 2. *Let α be the dominant root of the characteristic polynomial $\Phi_k(x)$ and consider the function $g_k(x)$ defined in (4). If $k \geq 50$ and $n > 1$ are integers satisfying $n < \phi^{k/2}$, then the following inequalities hold:*
(i) ([17], Equation 30)

$$\left|(2\alpha-2)\alpha^n - 2\phi^{2n+1}\right| < \frac{4\phi^{2n}}{\phi^{k/2}},$$

(ii) ([17], Lemma 13)

$$|g_k(\alpha) - g_k(\phi^2)| < \frac{4k}{\phi^k}.$$

Lemma 3 ([16], Lemma 2.3)**.** *Let $k \geq 50$ and suppose that $n < \phi^{k/2}$. Then*

$$(2\alpha-2)g_k(\alpha)\alpha^n = \frac{2\phi^{2n+1}}{\phi+2}(1+\xi), \quad \textit{where} \quad |\xi| < \frac{1.25}{\phi^{k/2}}.$$

2.2. Linear Form in Logarithms

Matveev (Corollary 2.3 of [15]) or (Theorem 9.4 of [9]) proved the following result.

Theorem 2. *Let $\eta_1, \ldots, \eta_s$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree $d_{\mathbb{K}}$. Let $d_1, \ldots, d_s$ be non-zero integers such that*

$$\Gamma := \eta_1^{d_1} \cdots \eta_s^{d_s} - 1 \neq 0.$$

Then

$$-\log|\Gamma| \leq 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log D) \cdot B_1 \cdots B_s,$$

where

$$D \geq \max\{|d_1|, \ldots, |d_s|\},$$

and

$$B_j \geq \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\}, \text{ for all } j = 1, \ldots, s.$$

2.3. The Reduction Method

Here, we present the following result due to Bravo, Gómez, and Luca (Lemma 1 of [5]), which is a generalization of the results of Baker and Davenport (Lemma of [3]) and Dujella and Pethö (Lemma 5(a) of [11]).

Lemma 4. *Let M be a positive integer, let p/q be a convergent of the continued fraction of an irrational number τ such that $q > 6M$, and let A, B, and μ be real numbers with $A > 0$ and $B > 1$. Further, let $\varepsilon = ||\mu q|| - M \cdot ||\tau q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < |m\tau - n + \mu| < AB^{-k},$$

in positive integers m, n, and k with

$$m \leq M \text{ and } k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Note that Lemma 4 cannot be applied for $\mu = 0$ (since then $\varepsilon < 0$). For this case, we use the following technical result from Diophantine approximation, known as Legendre's criterion. This comes from the theory of continued fractions (see [13], pages 30 and 37).

Lemma 5. *Let τ be an irrational number.*

(i) If x, y are positive integers such that

$$\left|\tau - \frac{y}{x}\right| < \frac{1}{2x^2},$$

then $y/x = p_k/q_k$ is a convergent of τ.

(ii) *Let M be a positive real number and $p_0/q_0,\ p_1/q_1,\ \ldots$ be all the convergents of the continued fraction $[a_0, a_1, a_2, \ldots]$ of τ. Let N be the smallest positive integer such that $q_N > M$. Put $a(M) = \max\{a_k : k = 0, 1, \ldots, N\}$. Then, the inequality*

$$\left|\tau - \frac{y}{x}\right| > \frac{1}{(a(M)+2)x^2}$$

holds for all pairs (x, y) of integers with $0 < x < M$.

2.4. Other Useful Results

We conclude this section by recalling the following results that we will need.

Lemma 6 (Lemma 2.2 of [20]). *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1-e^{-a}} \cdot |e^x - 1|.$$

Lemma 7 (Lemma 7 of [12]). *If $\ell \geq 1$, $T > (4\ell^2)^\ell$, and $T > x/(\log x)^\ell$, then*

$$x < 2^\ell T (\log T)^\ell.$$

Theorem 3 (Theorem 1 of [4]). *The Diophantine inequality*

$$|F_n^{(k)} - 2^m| < 2^{m/2}$$

has two parametric families of solutions $(n,\ k,\ m)$ with $n,\ k \geq 2$, and $m \geq 0$, namely

1. $(n,\ k,\ m) = (t,\ k,\ t-2)$ *for* $2 \leq t \leq k+1$, *and*
2. $(n,\ k,\ m) = (k+2+t,\ k,\ k+t)$ *for* $0 \leq t \leq \max\{x \in \mathbb{Z} : 2+x < 2^{1+(k-x/2)}\}$.
3. *In addition, we have the sporadic solution* $(n,\ k,\ m) = (12,\ 3,\ 9)$.

Theorem 4 (Theorem 1.1 of [10]). *There are only 8 Fibonacci numbers which are close to a power of 2. Namely, the solutions $(F_n, 2^m)$ of the inequality*

$$|F_n - 2^m| < 2^{m/2}$$

are $(1, 2), (2, 2), (3, 2), (3, 4), (5, 4), (8, 8), (13, 16)$, *and* $(34, 32)$.

3. Proof of Theorem 1

In this section, we give all details about the proof of our main theorem. We establish some preliminary results.

The following result gives us the bounds of m in terms of n.

Lemma 8. *If (m, n, k) is a solution of Diophantine Inequality (1) with $n \geq 1$, $m \geq 2$, and $n \geq k+1$, then we have the inequalities*

$$0.69n - 1.69 < m < 1.39n + 3.78.$$

Proof. Combining Inequality (6) with Equation (1), we have

$$2^{m-1} \leq 2^m - 2^{m/2} < Q_n^{(k)} < 2\alpha^n \leq \alpha^{n+2}$$

and

$$\alpha^{n-1} < Q_n^{(k)} < 2^m + 2^{m/2} < 2^{m+1}.$$

Since $2^{m-1} < \alpha^{n+2}$ and $\alpha^{n-1} < 2^{m+1}$, it follows that $(m-1)\log 2 < (n+2)\log\alpha$ and $(n-1)\log\alpha < (m+1)\log 2$. So, we get

$$(n-1)\frac{\log\alpha}{\log 2} - 1 < m < (n+2)\frac{\log\alpha}{\log 2} + 1.$$

Because $\phi^2(1-\phi^{-2}) < \alpha < \phi^2$, by Lemma 1(b), for $k \geq 2$, we deduce that

$$0.69n - 1.69 < m < 1.39n + 3.78.$$

This finishes the proof of the lemma. □

The following result gives an upper bound of m and n in terms of k.

Lemma 9. *If the integers n, k, and m satisfy Diophantine Equation (2) with $n \geq k+1$, then we have the following estimates:*

$$n < 1.7 \cdot 10^{15} \cdot k^4 \cdot \log^3 k \quad \text{and} \quad m < 2.37 \cdot 10^{15} \cdot k^4 \cdot \log^3 k.$$

Proof. Let us express Equation (2) as follows:

$$2^m - (2\alpha - 2)g_k(\alpha)\alpha^n = Q_n^{(k)} - (2\alpha-2)g_k(\alpha)\alpha^n - e.$$

Taking the absolute value of both sides, we obtain

$$|2^m - (2\alpha-2)g_k(\alpha)\alpha^n| \leq \left|Q_n^{(k)} - (2\alpha-2)g_k(\alpha)\alpha^n\right| + |e| < 2 + 2^{m/2}.$$

Dividing through by $(2\alpha-2)g_k(\alpha)\alpha^n$, we get

$$\left|\frac{1}{(2\alpha-2)g_k(\alpha)}\alpha^{-n}2^m - 1\right| < \frac{1}{(\alpha-1)g_k(\alpha)\alpha^n}\left(1 + 2^{(m-2)/2}\right).$$

Since the inequalities $0.276 < g_k(\alpha) < 0.5$ hold for $k \geq 2$ (see Lemma 1) and $2^{m-1} < \alpha^{n+2}$, we deduce that

$$\begin{aligned}\left|\frac{1}{(2\alpha-2)g_k(\alpha)}\alpha^{-n}2^m - 1\right| &< 5.87 \cdot \left(\frac{1}{\alpha^n} + \frac{2^{-1/2}\left(\alpha^{n+2}\right)^{1/2}}{\alpha^n}\right) \\ &< 5.87 \cdot \left(\frac{1}{\alpha^{n/2}} + \frac{2^{-1/2}\cdot\alpha}{\alpha^{n/2}}\right) \\ &< 5.87 \cdot \left(\frac{1+2^{-1/2}\cdot\phi^2}{\alpha^{n/2}}\right) \\ &< \frac{16.74}{\alpha^{n/2}}. \qquad (8)\end{aligned}$$

Let

$$\Gamma_1 := \frac{1}{(2\alpha-2)g_k(\alpha)}\alpha^{-n}2^m - 1.$$

Observe that $\Gamma_1 \neq 0$. If $\Gamma_1 = 0$, then $(2\alpha-2)g_k(\alpha)\alpha^n = 2^m$. Using the $\mathbb{Q}$-automorphisms $\sigma_i : \alpha \mapsto \alpha_i$, $i \geq 2$, of the Galois extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ and Lemma 1, we find that

$$16 \leq 2^m = |g_k(\alpha_i)|\,|\alpha_i^n|\,|2\alpha_i - 2| < 4,$$

which is a contradiction. Thus $\Gamma_1 \neq 0$. Now, we can apply Theorem 2 to Γ_1. Let us consider

$$\eta_1 = \frac{1}{(2\alpha-2)g_k(\alpha)},\ \eta_2 = \alpha,\ \eta_3 = 2, \quad d_1 = 1,\ d_2 = -n,\ d_3 = m.$$

The numbers $\eta_1,\ \eta_2,\ \eta_3$ are elements of the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ with $d_{\mathbb{K}} = k$. We have

$$h(\eta_2) = \frac{\log\alpha}{k} < \frac{2\log\phi}{k} \quad \text{and} \quad h(\eta_3) = \log 2.$$

Moreover, we get

$$\max\{kh(\eta_2), |\log\eta_2|, 0.16\} < 0.97 = A_2,$$

and

$$\max\{kh(\eta_3), |\log\eta_3|, 0.16\} \leq k\log 2 = A_3.$$

Using the properties of the logarithmic height and (7), we obtain

$$\begin{aligned}h(\eta_1) &\leq \log 2 + h(\alpha-1) + h(g_k(\alpha)) \\ &< 2\log 2 + \frac{\log\alpha}{k} + 5\log k \\ &< 2\log 2 + \log\phi + 5\log k \\ &< 7.8\log k,\end{aligned}$$

for $k \geq 2$. So, we can take

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 7.8k \log k = A_1.$$

Finally, from Lemma 8, we can choose $D = 6n > 1.39n + 3.78 \geq \max\{1, m, n\}$, for $n \geq 1$. Thus, Theorem 2 tells us that

$$\log |\Gamma_1| \geq -7.51 \cdot 10^{11} \cdot k^4 \cdot \log k \cdot (1 + \log k)(1 + \log(6n)).$$

By the facts $1 + \log(6n) < 2.3 \log n$ and $1 + \log k < 2.5 \log k$, which hold for $n \geq 9$ and $k \geq 2$, we obtain

$$\log |\Gamma_1| > -4.32 \cdot 10^{12} \cdot k^4 \cdot \log^2 k \cdot \log n. \tag{9}$$

Combining Inequalities (8) and (9), we get

$$n < 1.8 \cdot 10^{13} \cdot k^4 \cdot \log^2 k \cdot \log n.$$

Thus, we obtain

$$\frac{n}{\log n} < 1.8 \cdot 10^{13} \cdot k^4 \cdot \log^2 k.$$

Applying Lemma 7 with $T = 1.8 \cdot 10^{13} \cdot k^4 \cdot \log^2 k$, $x = n$, and $\ell = 1$, we have

$$\begin{aligned} n &< 2 \cdot (1.8 \cdot 10^{13} \cdot k^4 \cdot \log^2 k) \cdot \log(1.8 \cdot 10^{13} \cdot k^4 \cdot \log^2 k) \\ &< (3.6 \cdot 10^{13} \cdot k^4 \cdot \log^2 k) \cdot (30.53 + 4 \log k + 2 \log \log k) \\ &< 1.7 \cdot 10^{15} \cdot k^4 \cdot \log^3 k. \end{aligned} \tag{10}$$

In the above inequalities, we have used the fact that

$$30.53 + 4 \log k + 2 \log(\log k) < 47 \log k$$

holds for $k \geq 2$. Finally, using Inequality (10) and Lemma 8, we obtain

$$m < 2.37 \cdot 10^{15} \cdot k^4 \cdot \log^3 k.$$

This completes the proof of Lemma 9. □

The last preliminary result established is the following lemma.

Lemma 10. *There is no solution for Inequality* (1) *with* $k > 350$ *and* $n \geq k + 1$.

Proof. Referring to Lemma 9, we have

$$n < 1.7 \cdot 10^{15} \cdot k^4 \cdot \log^3 k < \phi^{k/2}, \quad \text{for } k > 350.$$

Thus, from Lemma 3, Equation (2), and Inequality (5), we have

$$\left|2^m - \frac{2\phi^{2n+1}}{\phi+2}\right| = \left|\frac{2\phi^{2n+1}}{\phi+2}\xi + Q_n^{(k)} - (2\alpha-2)g_k(\alpha)\alpha^n - e\right|$$
$$< \frac{2.5\phi^{2n+1}}{(\phi+2)\cdot\phi^{k/2}} + 2 + 2^{m/2}.$$

Multiplying through by $(\phi+2)/2\phi^{2n+1}$, and using the facts that $2 < \phi^2$ and $m < 1.39n + 3.78 < 1.5n$ for $n > 34$ (see Lemma 8), we obtain

$$\begin{aligned}
|\Gamma_2| &< \frac{1.25}{\phi^{k/2}} + \frac{\phi+2}{\phi^{2n+1}} + 2^{m/2}\cdot\frac{\phi+2}{2\phi^{2n+1}} \\
&< \frac{1.25}{\phi^{k/2}} + \frac{3.62}{\phi^{2n+1}} + \frac{1.81\phi^m}{\phi^{2n+1}} \\
&< \frac{4.87}{\phi^{k/2}} + \frac{1.81}{\phi^{n/2}} \\
&< \frac{6.68}{\phi^{k/2}},
\end{aligned} \tag{11}$$

where

$$\Gamma_2 := (\phi+2)\phi^{-2n-1}2^{m-1} - 1.$$

Observe that $\Gamma_2 \neq 0$. Indeed, we have $2^{m-1} = \frac{\phi^{2n+1}}{\phi+2}$, which is impossible because the left-hand side is an integer whereas it can be seen that the right-hand side is irrational as $n > 351$ and $m > 240$. So, we can apply Theorem 2 to Γ_2 with

$$\eta_1 = \phi+2, \quad \eta_2 = \phi, \quad \eta_3 = 2, \quad d_1 = 1, \quad d_2 = -2n-1, \text{ and } d_3 = m-1.$$

Since $\mathbb{K} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\phi)$, it follows that $d_{\mathbb{K}} = 2$. Also, we have

$$h(\eta_2) = \frac{\log\phi}{2} \quad \text{and} \quad h(\eta_3) = \log 2.$$

Moreover, one has

$$h(\eta_1) \le h(\phi) + h(2) + \log 2 \le \frac{\log\phi}{2} + 2\log 2 < 1.63.$$

Thus, we can take

$$A_1 := 3.26, \quad A_2 := \log\phi, \quad \text{and} \quad A_3 := 2\log 2.$$

Here we can take $D = 2n+1$. Using the fact that $1 + \log(2n+1) < 1.8\log n$, which holds for $n \ge 9$, from Theorem 2 we get

$$\log|\Gamma_2| > -3.8\cdot 10^{12}\cdot\log n. \tag{12}$$

Next, we put (11) and (12) together to obtain

$$k < 1.58 \cdot 10^{13} \cdot \log n. \tag{13}$$

By Lemma 9, we have

$$\log n < \log(1.7 \cdot 10^{15} \cdot k^4 \cdot \log^3 k) < 10.9 \log k, \quad \text{for } k \geq 350. \tag{14}$$

Using Inequalities (13) and (14), we obtain

$$k < 1.73 \cdot 10^{14} \log k.$$

It follows that

$$k < 6.3 \cdot 10^{15}.$$

We deduce that

$$n < 1.29 \cdot 10^{83}.$$

In order to reduce the above bounds of n, we put

$$\Lambda_2 = -\log\left(\frac{1}{\phi+2}\right) - (2n+1)\log\phi + (m-1)\log 2 = \log(\Gamma_2 + 1).$$

Hence, $\Lambda_2 \neq 0$ because $\Gamma_2 \neq 0$. So, we get

$$0 < |\Lambda_2| < \frac{13.36}{\phi^{k/2}}.$$

Dividing through by $\log 2$, we get

$$|(2n+1)\tau - (m-1) + \mu| < \frac{20}{\phi^{k/2}}, \tag{15}$$

with

$$\tau = \frac{\log\phi}{\log 2} \quad \text{and} \quad \mu = \frac{\log\left(\frac{1}{\phi+2}\right)}{\log 2}.$$

Now, we apply Lemma 4 with $A = 20,\ B = \phi$, and $M = 2.58 \cdot 10^{83}$. Using Maple, we find that q_{165} satisfies the hypotheses of Lemma 4, and we get

$$\frac{k}{2} < 415.14.$$

Thus, for $k \leq 830$, using Lemma 9, we obtain

$$n < 2.45 \cdot 10^{29}.$$

We apply again Lemma 4 to Inequality (15) with $A = 20,\ B = \phi,\ M = 4.9 \cdot 10^{29}$ to obtain

$$\frac{k}{2} < 157.73.$$

We obtain $k \leq 315$, which is a contradiction to the fact that $k > 350$. Therefore, we deduce that Inequality (1) does not admit any solution for $k > 350$. This completes the proof of Lemma 10. □

Proof of Theorem 1. For this, two cases will be considered according to the values of n.

`Case 1:` $1 \le n \le k$. First, observe that for $1 \le n \le k$, we have $Q_n^{(k)} = 2F_{2n}$ (see Lemma 10 of [17]), where F_n is the nth Fibonacci number. Hence, Inequality (1) becomes

$$\left|F_{2n} - 2^{m-1}\right| < 2^{(m-2)/2}. \tag{16}$$

By Theorems 3 and 4, we deduce that the solutions $(F_{2n}, n, m-1)$ of Inequality (16) are $(1,1,0)$, $(3,2,2)$, and $(8,3,3)$. Therefore, the solutions $(Q_n^{(k)}, k, n, m)$ of Inequality (1) are

$$(2,k,1,1),\ k \ge 2, \quad (6,k,2,3),\ k \ge 2, \text{ and } (16,k,3,4),\ k \ge 3.$$

`Case 2:` $n \ge k+1$. We assume that $n \ge k+1$. Furthermore, considering the solutions of Inequality (16) and checking for the small values for n, we may assume that $n \ge 9$. Then applying Lemma 8, we have $m > 4$. Next by Lemma 9 we get

$$n < 1.7 \cdot 10^{15} \cdot k^4 \cdot \log^3 k \quad \text{and} \quad m < 2.37 \cdot 10^{15} \cdot k^4 \cdot \log^3 k.$$

Subsequently, we will discuss different cases depending on the size of k.

`Case 2.1:` $2 \le k \le 350$. To reduce the above bound on n (see Lemma 9), we assume that $n \ge 12$ and we put

$$\Lambda_1 := m\log 2 - n\log\alpha + \log\left(\frac{1}{(2\alpha-2)g_k(\alpha)}\right) = \log(\Gamma_1 + 1).$$

By Inequality (8), we obtain

$$0 < |\Gamma_1| < \frac{16.74}{\alpha^{n/2}} < 0.94, \quad \text{for } n \ge 12.$$

Applying Lemma 6 with $a := 0.94$, $x := \Gamma_1$ we have

$$0 < |\Lambda_1| < \frac{51}{\alpha^{n/2}}.$$

Dividing through by $\log\alpha$, we get

$$|m\tau - n + \mu| < \frac{106}{\alpha^{n/2}}, \tag{17}$$

where

$$\tau = \frac{\log 2}{\log\alpha} \quad \text{and} \quad \mu = \frac{\log\left(\frac{1}{(2\alpha-2)g_k(\alpha)}\right)}{\log\alpha}.$$

Now, we apply Lemma 4 to (17), for $3 \le k \le 350$, by putting

$$M = M_k := \lfloor 2.37 \cdot 10^{15} \cdot k^4 \cdot \log^3 k \rfloor,\ A = 106, \text{ and } B = \alpha.$$

And a quick computation with Mathematica reveals that

$$\frac{n}{2} < 76.42, \quad \text{for} \quad k \in [3, 350].$$

Note that Lemma 4 cannot be applied to (17) for $k = 2$ because $\mu = 0$. So, for $k = 2$, we rewrite Inequality (17) as follows:

$$\left|\tau - \frac{n}{m}\right| < \frac{106}{m\alpha^{n/2}}. \tag{18}$$

If $\dfrac{106}{m\alpha^{n/2}} \geq \dfrac{1}{2m^2}$, then

$$\frac{n}{2} \leq \frac{\log(212m)}{\log\alpha} \leq \frac{\log(212 \cdot 2.37 \cdot 10^{15} \cdot 2^4 \cdot \log^3 2)}{\log\alpha} < 48.15.$$

If $\dfrac{106}{m\alpha^{n/2}} < \dfrac{1}{2m^2}$, then we apply Lemma 5 to (18) using

$$\tau = \frac{\log 2}{\log\alpha}, \quad M = 2.37 \cdot 10^{15} \cdot 2^4 \cdot \log^3 2, \quad x = m, \quad \text{and} \quad y = n.$$

After a computation using Maple, we obtain that 32 is the smallest positive integer such that $q_{32} > M$ and $a(M) = 100$. So, we have

$$\frac{1}{102m^2} < \left|\tau - \frac{n}{m}\right|. \tag{19}$$

Combining Inequality (19) with Inequality (18), we get

$$\frac{n}{2} < \frac{\log(106 \cdot 102m)}{\log\alpha} \leq \frac{\log(10812 \cdot 2.37 \cdot 10^{15} \cdot 2^4 \cdot \log^3 2)}{\log\alpha} < 52.61.$$

In all cases, we see that $n/2 < 76.42$.

Finally, using a Maple program to search all the solutions of Equation (2) with $2 \leq k \leq 350$, $0 \leq n \leq 152$, $0 \leq m \leq 215$ (as $m < 1.39n + 3.78$), and $n \geq k + 1$, we obtain the remaining solutions of this equation mentioned in our main theorem.

`Case 2.2:` $k > 350$. By Lemma 10, we deduce that Inequality (1) does not admit any solution in this case. This complete the proof of Theorem 1. □

Acknowledgements. The authors would like to express their gratitude to the reviewer for carefully reading this paper and the remarks which have qualitatively improved the work. This paper was completed when the third author visited Gamal Abdel Nasser University of Conakry (Guinea). He thanks the institution for the great working environment, the hospitality and the support.

References

[1] A. Açikel, N. Irmak, and L. Szalay, The k-generalized Lucas numbers close to a power of 2, *Math. Slovaca* **73** (4) (2023), 871–882.

[2] M. Bachabi and A. Togbé, On k-Pell numbers close to power of 2, *Rev. Colombiana Mat.* **58** (2024) 1, 67–80.

[3] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Q. J. Math.* **20** (2) (1969), 129–137.

[4] J. J. Bravo, C. A. Gómez, and J. L. Herrera, k-Fibonacci numbers close to a power of 2, *Quaest. Math.* **44** (12) (2021), 1681–1690.

[5] J. J. Bravo, C. A. Gómez, and F. Luca, Power of two as sums of two k-Fibonacci numbers, *Miskolc Math. Notes* **17** (2016), 85–100.

[6] J. J. Bravo and J. L. Herrera, Repdigits in generalized Pell sequences, *Arch. Math. (Brno)* **56** (4) (2020), 249–262.

[7] J. J. Bravo, J. L. Herrera, and F. Luca, On a generalization of the Pell sequence, *Math. Bohem.* **146** (2) (2021), 199–213.

[8] J. J. Bravo and F. Luca, Powers of two in generalized Fibonacci sequences, *Rev. Colombiana Mat.* **46** (1) (2012), 67–79.

[9] Y. Bugeaud, M. Maurice, and S. Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. Math.* **163** (2006), 969–1018.

[10] S. Chern and A. Cui, Fibonacci numbers close to a power of 2, *Fibonacci Quart.* **52** (4) (2014), 344–348.

[11] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Q. J. Math.* **49** (2) (1998), 291–306.

[12] S. Guzmán and F. Luca, Linear combinations of factorials and S-units in a binary recurrence sequence, *Ann. Math. Qué.* **38** (2014), 169–188.

[13] A. Ya. Khinchin, *Continued Fractions*, P. Noordhoff Ltd., Groningen, 1963.

[14] E. Kılıç and D. Taşcı, The generalized Binet formula, representation and sums of the generalized order-k, Pell Numbers, *Taiwanese J. Math.*, **10** (6) (2006), 1661–1670.

[15] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125-180. English translation in *Izv. Math.* **64** (2000), 1217–1269.

[16] B. K. Patel and B. P. Tripathy, k-Pell-Lucas numbers which are concatenations of two repdigits, preprint, `arXiv:2303.05293`

[17] Z. Şiar and R. Keskin, On perfect powers in k-generalized Pell-Lucas sequence, *Math. Notes* **114** (2023), 936–948.

[18] Z. Şiar, R. Keskin, and E. S. Öztaş, On perfect powers in k-generalized Pell sequence, *Math. Bohem.*, (2022), 1–12.

[19] M. Waldshmidt, *Diophantine Approximation on Linear Algebraic Groups: Transcendence Properties of the Exponential Function in Several Variables*, Springer-Verlag, Berlin, 2000.

[20] B. M. M. de Weger, *Algorithms for Diophantine Equations*, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.