PROPERTIES OF EHRHART POLYNOMIALS WHOSE ROOTS LIE ON THE CANONICAL LINE

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Abstract

We study a class of polynomials that have all of their roots on the canonical line and share certain properties with Ehrhart polynomials. Braun showed in 2008 that the roots of Ehrhart polynomials are bounded quadratically and Higashitani in 2012 provided examples for polytopes whose Ehrhart polynomial roots come close to this bound. In the case of polynomials of the aforementioned class, we present an improved bound. As a side effect this confirms a special case of a conjecture posed by Braun and M. Develin. Furthermore, we show that in low dimensions, the polytopes with the widest spread of Ehrhart polynomial roots on the canonical line are the standard reflexive simplices.

1. Introduction

Let $P \subset \mathbb{R}^d$ be a convex polytope. We call P a *lattice polytope* if all of its vertices lie in $\mathbb{Z}^d$. Going forward, for the sake of simplicity, we will always assume that P has full dimension. We define the *Ehrhart polynomial* of P as the function

$$E_P(k) = |kP \cap \mathbb{Z}^d|,$$

where kP denotes the dilation of P by the non-negative integer factor k and $E_P(0)$ is set to be 1 for every P. In other words, E_P counts the lattice points in the integer dilations of P. The namesake of the Ehrhart polynomial, Eugène Ehrhart, proved in [7] that these functions are indeed polynomials.

A related notion is the *Ehrhart series* of a polytope. It is defined as the generating function of the Ehrhart polynomial, i.e., as the formal power series

$$\mathrm{ehr}_P(t) = \sum_{k \geq 0} E_P(k)\, t^k$$

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and can be rewritten as a rational function

$$\mathrm{ehr}_P(t) = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where h_P^* denotes a polynomial of degree at most d with non-negative integer coefficients [22]. We call this polynomial the *h^*-polynomial* of P. Its vector of coefficients is also often called the *δ-vector* of P. The constant coefficient of h_P^* is always equal to 1.

The Ehrhart polynomial can be fully retrieved from the h^*-polynomial by performing a change of basis:

$$E_P(z) = \sum_{k=0}^{d} h_k^* \binom{z+d-k}{d}. \tag{1}$$

Here h_k^* refers to the k-th coefficient of the h^*-polynomial.

The Ehrhart polynomial and the h^*-polynomial encode information about the underlying lattice polytope P. For example, the degree of E_P is equal to the dimension of P, the leading coefficient of E_P is the volume of P, and the second highest degree is half the boundary volume of P, where these volumes are suitably normalized (i.e., the volumes of the polytope and each of its facets are defined with respect to the volume of the empty hypercubes in the sublattices they lie in). Other important properties are captured by the h^*-polynomial. For instance, $h_1^* = |P \cap \mathbb{Z}^d| - d - 1$, $h_d^* = |P^\circ \cap \mathbb{Z}^d|$, where P° denotes the interior of P, and $h_P^*(1)$ is equal to the normalized volume (or lattice volume) of P. See [3] for a comprehensive introduction to Ehrhart theory.

One particularly remarkable property reflected in the h^*-polynomial is that of *reflexivity*. A lattice polytope P is called reflexive if its polar dual is also a lattice polytope. Reflexive polytopes rose to popularity after Batyrev [1] noticed their connection to string theory. Around the same time, Hibi showed in [11] that P is reflexive if and only if $h_P^*(t) = \sum_{k=0}^{d} h_k^* t^k$ is *palindromic*, i.e., its coefficients satisfy $h_k^* = h_{d-k}^*$ for every k.

1.1. Ehrhart Polynomial Roots

One popular aspect in the study of Ehrhart theory is the root distribution of Ehrhart polynomials. For every lattice polytope P, the Ehrhart polynomial E_P has rational coefficients. Hence, we may notice right away that by the complex conjugate root theorem, $z \in \mathbb{C}$ being a root of E_P implies that its complex conjugate $\bar{z}$ is also a root. However, there are more advanced patterns as well.

If P is reflexive, its Ehrhart polynomial roots exhibit a peculiar behavior: they are distributed symmetrically across the *canonical line*

$$\mathrm{CL} = \left\{ z \in \mathbb{C} \colon \mathrm{Re}(z) = -\frac{1}{2} \right\},$$

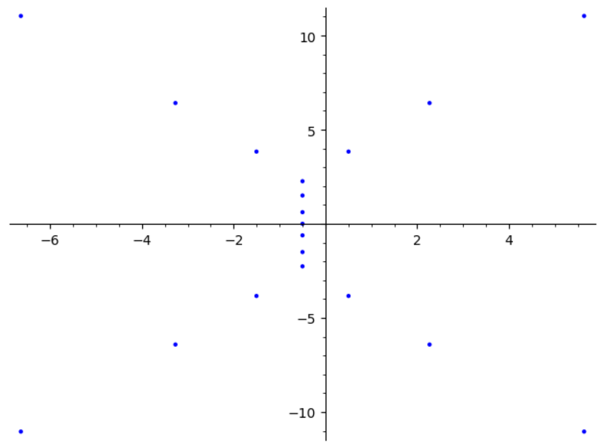

Figure 1: The root distribution of the Ehrhart polynomial of a 20-dimensional reflexive non-CL polytope studied in [18].

which is to say that if $z \in \mathbb{C}$ is a root of E_P, then so is $-z-1$ (see the example in Figure 1). This behavior is a consequence of the following statement.

Lemma 1 (Corollary 2.2 in [11])**.** *Let f be a polynomial of degree d, and define $h^*(t) = (1-t)^{d+1} \sum_{k \geq 0} f(k)\, t^k$. Then f satisfies the functional equation*

$$f(z-1) = (-1)^d f(-z) \tag{2}$$

if and only if h^* has palindromic coefficients.

A reflexive polytope whose Ehrhart polynomial roots do not just lie symmetrically across CL, but *on* CL is called a *CL-polytope*. The term was coined in [9], where the geometric properties of these polytopes were studied in low dimension. The origins of the study of CL-polytopes, however, dates back further. Within the framework of the *local Riemann hypothesis*, Bump, Choi, Kurlberg, and Vaaler proved the CL-ness of *cross-polytopes*[6, Theorems 4 and 6], i.e., the polytopes of the form

$$\operatorname{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d\} \subset \mathbb{R}^d,$$

where the e_k refer to the unit vectors of $\mathbb{R}^d$ and conv denotes the convex hull operator. Shortly thereafter, Rodriguez-Villegas gave a criterion for the CL-ness of a polytope in terms of its h^*-polynomial [20]. This led to Golyshev conjecturing that smooth Fano polytopes of dimension up to 5 have all their Ehrhart polynomial roots on CL, which has since been confirmed in [10]. In the same paper, the authors show that the conjecture does not hold in higher dimensions, which makes it clear that the study of CL-polytopes cannot be reduced to that of smooth Fano polytopes.

1.2. Bounds for Ehrhart Polynomial Roots

The study of the bounds of Ehrhart polynomial roots goes back to [2] and starts with the following theorem.

Theorem 1 (Theorem 1.2 in [2]). *(a) The roots of Ehrhart polynomials of lattice polytopes of dimension d are bounded in norm by $1 + (d + 1)!$.*

(b) All real roots of Ehrhart polynomials of d-dimensional lattice polytopes lie in the half-open interval $\left[-d, \left\lfloor \frac{d}{2} \right\rfloor\right)$.

The authors noticed that this bound was far from being optimal and conjectured, based on experimental data, the following.

Conjecture 1 (Conjecture 1.4 in [2]). All roots α of Ehrhart polynomials of lattice polytopes of dimension d satisfy $-d \leq \mathrm{Re}(\alpha) \leq d - 1$.

This conjecture holds true for the real roots of Ehrhart polynomials of degree 5 or less, but has been disproven in general by counterexamples in [13] and [18]. Meanwhile, Braun gave an improvement of the bound in Theorem 1.

Theorem 2 (Theorem 1 in [4]). *If P is a lattice polytope of dimension d, then all the roots of E_P lie inside the disc with center $-\frac{1}{2}$ of radius $d\left(d - \frac{1}{2}\right)$.*

Braun obtained this result by studying a larger class of polynomials, called *Stanley non-negative polynomials* (*SNN-polynomials*). They are defined as the class of non-zero polynomials f such that $h^*(t) = (1-t)^{\deg f + 1} \sum_{k \geq 0} f(k)\, t^k$ has only non-negative coefficients. Notice that for every (not necessarily reflexive) lattice polytope P, its Ehrhart polynomial E_P lies in $\mathfrak{S}$. SNN-polynomials were also used in [5] to give a bound for the imaginary part of Ehrhart polynomial roots.

Theorem 3 (Theorem 2.3 in [5]). *For the polynomial $M_d(t) = \binom{t+d}{d} + \binom{t}{d}$, which is not an Ehrhart polynomial, if β_d is the root of $M_d(t)$ of maximal norm, then*

$$\left| \beta_d + \frac{1}{2} \right| = \frac{d^2}{\pi} + O(1)$$

as d goes to infinity.

The authors also conjecture the following.

Conjecture 2 (Conjecture 2.4 in [5]). The root of the polynomial $M_d(t)$ with largest norm has the maximal imaginary part among all roots of degree d polynomials in $\mathfrak{S}$.

In this study, we will use a similar idea to study the roots of CL-polytopes and define the class $\mathfrak{C} \subset \mathbb{R}[z]$ of *CL-polynomials*. Its elements are the polynomials of the form

$$f(z) = b(z)(z^2 + z + c_0)(z^2 + z + c_1) \cdots (z^2 + z + c_m), \tag{3}$$

where the c_k are real numbers $\geq \frac{1}{4}$ and

$$b(z) = \begin{cases} a & \text{if } \deg f \text{ is even,} \\ a(2z+1) & \text{otherwise} \end{cases}$$

for a non-zero real number a. Notice that if P is a CL-polytope, E_P does indeed fall into this class: if $-\frac{1}{2} + \alpha i$ is a root of E_P with $\alpha > 0$, then so is $-\frac{1}{2} - \alpha i$ and E_P is divisible by $z^2 + z + \frac{1}{4} + \alpha^2$. If E_P has odd degree, then $-\frac{1}{2}$ is necessarily a root, thus E_P is divisible by $2z+1$. Furthermore, notice that every $f \in \mathfrak{C}$ satisfies Equation (2) and thus

$$h^*(t) = (1-t)^{\deg f + 1} \sum_{k \geq 0} f(k)\, t^k$$

is a palindromic polynomial.

However, not every CL-polynomial is an SNN-polynomial. For example, for $f(z) = \frac{2}{15}\left(z^2 + z + \frac{13}{4}\right)\left(z^2 + z + \frac{1}{4}\right)$, we get $h^*(t) = 1 + \frac{2}{3}t - \frac{2}{15}t^2 + \frac{2}{3}t^3 + t^4$. Hence, we will focus on the class $\mathfrak{C} \cap \mathfrak{S}$.

1.3. Interlacing Polynomials

In the course of this study, we will make use of the theory of interlacing polynomials, which has garnered attention in combinatorics after it was used in [16] and [17] to prove the Kadison–Singer problem as well as the existence of bipartite Ramanujan graphs. In Ehrhart theory, interlacing has been used to show that the symmetric edge polytopes from multipartite graphs of the forms $K_{2,n}$, $K_{3,n}$, $K_{1,1,n}$, $K_{1,2,n}$, and $K_{1,1,1,n}$ are CL [14, 15], among other examples.

The general idea behind interlacing polynomials is that the roots of two polynomials, f and g, whose degrees differ by 1, all lie on a real curve embedded in the complex numbers, where they alternate.

Definition 1. Let f and g be polynomials of degrees d and $d+1$, respectively. Further, let L be a totally ordered subset of $\mathbb{C}$. We say that f *L-interlaces* g or *f and g interlace on L* if all the roots $a_1, \dots, a_d$ of f and $b_1, \dots, b_{d+1}$ of g lie on L and satisfy

$$b_1 \leq a_1 \leq b_2 \leq a_2 \leq \cdots \leq a_d \leq b_{d+1}$$

with respect to the ordering on L.

We quickly recall some results we need in the course of this paper, two of which come from the extensive work of Fisk [8]. In the following, we will consider CL-interlacing by choosing

$$-\frac{1}{2} + ai \prec -\frac{1}{2} + bi \quad \text{if and only if} \quad a < b$$

as our total order.

Proposition 1 (Theorem 2.1.10, [19]). *Let f and g be CL-polynomials with degrees d and $d+1$, respectively. Let h_f^* and h_g^* be the polynomials $(1+t)^{d+1}\sum_{k\geq 0} f(k)\, t^k$ and $(1+t)^{d+2}\sum_{k\geq 0} g(k)\, t^k$, respectively. Assume h_f^* and h_g^* also have degrees d and $d+1$ and their roots interlace on the unit circle. Then f CL-interlaces g.*

Proposition 2 (Lemma 1.26, [8], "Leibnitz Rule"). *Suppose that f, f_1, g, g_1 are polynomials with positive leading coefficients, and with all real roots. Assume that f and g have no common roots. If f_1 $\mathbb{R}$-interlaces f and g_1 $\mathbb{R}$-interlaces g, then f_1g_1 $\mathbb{R}$-interlaces $fg_1 + f_1g$ which in turn $\mathbb{R}$-interlaces fg, fg_1, and f_1g. In particular, $fg_1 + f_1g$ has all real roots.*

Proposition 3 (Corollary 1.41, [8]). *Suppose that $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ are sequences of polynomials with all real roots that converge to polynomials f and g, respectively. If f_n and g_n $\mathbb{R}$-interlace for all positive integers n, then f and g $\mathbb{R}$-interlace.*

All the statements from [8] refer to interlacing on the real line, but can be transported to any line of the form $c_1\,\mathbb{R} + c_2$ for complex numbers c_1, c_2 by performing an appropriate affine transformation. Further, since roots are invariant under scaling, positive leading coefficients can always be obtained.

1.4. Outline of this Paper

In Section 2, we prove Conjecture 2 in the case of CL-polynomials.

Theorem (Theorem 4). The root of the polynomial $M_d(t)$ with largest norm has the maximal imaginary part among all roots of degree d polynomials in $\mathfrak{C} \cap \mathfrak{S}$.

Furthermore, in Theorem 4, we show that in dimension $d \leq 9$, the standard reflexive simplices, a family of CL-polytopes, are the lattice polytopes whose roots on CL come closest to the bound.

In Section 3 we present a sufficient condition for a given $f \in \mathfrak{C}$ to lie in $\mathfrak{S}$.

Theorem (Proposition 5). Let f be a CL-polynomial of degree d. Assume that the c_k are ordered by size. Then $f \in \mathfrak{S}$ if the c_k satisfy

$$\frac{1}{4} \leq c_k \leq \begin{cases} 2k+2, & d \text{ is odd,} \\ 2k+1, & d \text{ is even.} \end{cases}$$

While this condition is only sufficient, we find a number of examples of CL-polytopes whose Ehrhart polynomials satisfy it.

2. Possible Roots of Polynomials in $\mathfrak{C} \cap \mathfrak{S}$

Let Ω_d denote the set of points $z \in \mathrm{CL}$ such that there exists a polynomial $f \in \mathfrak{C} \cap \mathfrak{S}$ of degree d with $f(z) = 0$. In the course of this section, we will characterize the sets Ω_d for every non-negative integer d, using techniques from [4]. We start with some helpful definitions.

Let a bracketed term with a lower integer index refer to the *Pochhammer symbol* $(x)_j = x(x-1)(x-2)\cdots(x-j+1)$, where $(x)_0 := 1$. For positive integers d and j, we define the functions

$$p_j^d(z) = \begin{cases} (z+d-j)_d + (z+j)_d & \text{if } 2j \neq d, \\ (z+j)_d & \text{if } 2j = d. \end{cases}$$

If a degree d polynomial f is in $\mathfrak{C}$, with the help of Equation (1), it can be expressed in terms of the p_j^d,

$$d!\ f(z) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} h_k^* p_k^d(z),$$

where h_k^* refers to the k-th coefficient of the polynomial

$$h^*(t) = (1-t)^{d+1} \sum_{k=0}^{d} h_k^*\, t^k.$$

Notice however, that for $j > 0$, the p_j^d themselves are not in $\mathfrak{C}$.

Lastly, let f be a polynomial with root set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$. We define the *CL-span* of f as the set $\operatorname{clspan} f := \operatorname{conv}\{A \cap \mathrm{CL}\}$. If $\operatorname{clspan} f$ is non-empty, it is an interval of CL.

2.1. An Upper Bound for the Roots of CL-Polynomials

The main result in this subsection is a proof of Conjecture 2 in the case of CL-polytopes. Notice that the polynomials M_d mentioned in this conjecture are equal to the polynomials p_0^d above.

We start with a useful lemma telling us that CL-polynomials take on either exclusively real values or exclusively imaginary values on CL, depending on their degree.

Lemma 2. *Let $f \in \mathfrak{C}$ be a degree d polynomial. Then for every $z_0 \in \mathrm{CL}$, we find that $f(z_0) \in \mathbb{R}\, i^d$.*

Proof. We can use once again the functional equation in Lemma 1,

$$f(z-1) = (-1)^d f(-z).$$

Further, we can use that for any $z_0 \in \mathrm{CL}$, the equality $-z_0 - 1 = \overline{z_0}$ holds. Since f has real coefficients, we obtain the equality

$$\overline{f(z_0)} = f(\overline{z_0}) = f(-z_0 - 1) = (-1)^d f(z_0),$$

which implies the statement. □

Remark 1. Notice that this result holds more generally for polynomials with palindromic h^*-polynomials, which includes polynomials not in $\mathfrak{C}$.

Lemma 2 enables us to find roots of $p_j^d\big|_{\mathrm{CL}}$ using a variant of the Intermediate Value Theorem. We use this to study the limit behavior and the extremal roots these functions. In the following, we will use the convention that t is a real number. Its purpose will be to parametrize CL via $it - \frac{1}{2}$.

Lemma 3. *Let d and j be non-negative integers with $2j \leq d$. Then*

(a) $\lim_{t\to\infty} p_j^d\left(it - \frac{1}{2}\right) i^{-d} = \infty$,

(b) For $2j \neq d$, the expression $p_j^d\left(it - \frac{1}{2}\right) = 0$ holds if and only if

$$\left(it - \frac{1}{2} + d - j\right)_{d-2j} \in \mathbb{R}\, i^{d-2j+1}.$$

(c) $\operatorname{clspan} p_j^d \subset \operatorname{clspan} p_{j-1}^d$ *for every j with $0 < 2j \leq d$.*

Proof. We begin with (a). We have

$$p_j^d\left(it - \frac{1}{2}\right) = \begin{cases} \left(it - \frac{1}{2} + d - j\right)_d + \left(it - \frac{1}{2} + j\right)_d & \text{if } 2j \neq d, \\ \left(it - \frac{1}{2} + j\right)_d & \text{if } 2j = d. \end{cases}$$

Observe that this results in a degree d polynomial with leading coefficient $2i^d t$ if $2j < d$ and $i^d t$ if $2j = d$. Multiplying by i^{-d} makes the leading coefficient positive, which proves the statement.

For (b), we start by noticing the identity

$$(z + m - n)_m = (-1)^m \overline{(z+n)_m}, \tag{4}$$

where m and n are non-negative integers. Next, we rewrite p_j^d as follows:

$$p_j^d(z) = (z+j)_{2j}\left((z-j)_{d-2j} + (z+d-j)_{d-2j}\right) = (z+j)_{2j}\; p_{d-j}^{d-2j}(z). \tag{5}$$

Since $\left(it - \frac{1}{2} + j\right)_{2j} \neq 0$ for all t, the polynomials p_j^d and p_{d-j}^{d-2j} have the same CL-span. It follows that $p_j^d\left(it - \frac{1}{2}\right) = 0$ if and only if

$$\left(it - \frac{1}{2} + d - j\right)_{d-2j} = -\left(it - \frac{1}{2} + -j\right)_{d-2j} = (-1)^{d-2j+1}\overline{\left(it - \frac{1}{2} + d - j\right)_{d-2j}}.$$

The second equality follows from Equation (4). From these equalities, we can see that $\left(it - \frac{1}{2} + d - j\right)_{d-2j}$ is an element of $\mathbb{R}\, i^{d-2j+1}$.

For (c), we first notice that if $d = 2j$, the polynomial p_j^d has an empty CL-span. Without loss of generality, we can assume that d is odd. Thanks to (b), we have

(i) $p^d_{j-1}\left(it-\frac{1}{2}\right)=0$ if and only if $\left(it-\frac{1}{2}-j+1\right)_{d-2j+2}\in\mathbb{R}$,

(ii) $p^d_j\left(it-\frac{1}{2}\right)=0$ if and only if $\left(it-\frac{1}{2}-j\right)_{d-2j}\in\mathbb{R}$.

We make another observation: Statement (b) is equivalent to the following statement.

(b') For $2j\neq d$, the expression $p^d_j\left(it-\frac{1}{2}\right)=0$ holds if and only if

$$\sum_{k=0}^{d-2j-1}\arg\left(it-\frac{1}{2}+d-j-k\right)\in\{0,\pi\},$$

where $\arg(z)$ denotes the complex argument of z.

We reverse the order of the sum.

$$\sum_{k=0}^{d-2j-1}\arg\left(it-\frac{1}{2}+d-j-k\right)=\sum_{k=0}^{d-2j-1}\arg\left(it+j+\frac{1}{2}+k\right).$$

Hence, we see that for positive t, we get $0<\arg\left(it+j+\frac{1}{2}+k\right)<\frac{\pi}{2}$. Also for each k, the function $\arg\left(it+j+\frac{1}{2}+k\right)$ is monotonic and tends to $\frac{\pi}{2}$ as t tends to ∞. Thus we can rewrite the arguments with error terms $\varepsilon_k(t)$

$$\arg\left(it+j+\frac{1}{2}+k\right)=\frac{\pi}{2}-\varepsilon_k(t).$$

Summarizing all this, we can restate (i) and (ii) for positive t:

(i') $p^d_{j-1}\left(it-\frac{1}{2}\right)=0$ if and only if

$$\frac{(d-2j)\pi}{2}-\sum_{k=0}^{d-2j-1}\varepsilon_k(t)\in\left\{0,\frac{\pi}{2}\right\},$$

(ii') $p^d_j\left(it-\frac{1}{2}\right)=0$ if and only if

$$\frac{(d-2j-2)\pi}{2}-\sum_{k=1}^{d-2j-2}\varepsilon_k(t)\in\left\{0,\frac{\pi}{2}\right\}.$$

Since d is odd, the equality

$$\left\{\frac{(d-2j)\pi}{2},\frac{(d-2j-2)\pi}{2}\right\}=\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$$

holds As a consequence, $t>0$ is a root of p^d_{j-1} if and only if

$$\sum_{k=0}^{d-2j-1}\varepsilon_k(t)\in\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}.$$

Since the $\varepsilon_k(t)$ are monotonic functions, there exists an $a > 0$ such that

$$\begin{cases} \displaystyle\sum_{k=0}^{d-2j-1} \varepsilon_k(t) = \frac{\pi}{2} & \text{if } t = a, \\ \displaystyle\sum_{k=0}^{d-2j-1} \varepsilon_k(t) < \frac{\pi}{2} & \text{if } t < a. \end{cases}$$

We can conclude that the CL-span of p^d_{j-1} is bounded by the values $\pm ia - \frac{1}{2}$. Finally, we see that

$$\begin{cases} \displaystyle\sum_{k=1}^{d-2j-2} \varepsilon_k(t) < \frac{\pi}{2} & \text{if } t = a, \\ \displaystyle\sum_{k=1}^{d-2j-2} \varepsilon_k(t) < \frac{\pi}{2} & \text{if } t < a, \end{cases}$$

which implies that the values $\pm ia - \frac{1}{2}$ lie outside the CL-span of p^d_j. □

Finally, we may discuss the bound of the roots.

Theorem 4. *For every degree d polynomial $f \in \mathfrak{C} \cap \mathfrak{S}$, the inclusion* $\operatorname{clspan} f \subseteq \operatorname{clspan} p^d_0$ *holds.*

Proof. Let $b > 0$ be a real number such that $ib - \frac{1}{2} \notin \operatorname{clspan} p^d_0$. By Lemma 3(c), we also get $ib - \frac{1}{2} \notin \operatorname{clspan} p^d_j$ for every integer j with $0 < 2j \leq d$. Write

$$i^{-d}\, d!\, f\left(ib - \frac{1}{2}\right) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} h^*_k\, i^{-d}\, p^d_k \left(ib - \frac{1}{2}\right),$$

where the h^*_k are non-negative real numbers. Lemma 3(a) indicates that the value of $i^{-d}\, d!\, f\left(ib - \frac{1}{2}\right)$ is greater than 0 and thus not a root. □

2.2. The Standard Reflexive Simplex

The following is a classical result by Hibi.

Theorem 5 (Hibi's Lower Bound Theorem [12])**.** *Let P be a lattice polytope of dimension d with h^*-polynomial $h^*(t) = \sum_{k=0}^{d} h^*_k t^k$. Further, suppose that $h^*_d \neq 0$. Then the inequalities $h^*_1 \leq h^*_k$ hold for every $1 \leq k < d$.*

The theorem shows that the polynomials p^d_0 are not themselves Ehrhart polynomials of any polytope. Hence it is natural to ask which CL-polytopes have Ehrhart polynomials with large extremal roots. In dimension at most 9, this question can be answered by the *standard reflexive simplex*, which is given by

$$\Delta^d_{sr} = \operatorname{conv}\left\{ e_1, e_2, \ldots, e_d, -\sum_{k=1}^{d} e_k \right\},$$

where the e_k are a lattice basis of $\mathbb{Z}^d$.

We can write Δ^d_{sr} as a union of simplices

$$\Delta_e = \operatorname{conv}\left(\left\{0, e_1, e_2, \ldots, e_d, -\sum_{k=1}^{d} e_k\right\} \setminus \{e\}\right),$$

where e is an element of $\left\{e_1, e_2, \ldots, e_d, -\sum_{k=1}^{d} e_k\right\}$. This is a unimodular triangulation into $d+1$ elements and implies that Δ^d_{sr} has lattice volume $d+1$. Thus $h^*_{\Delta^d_{sr}}(1) = d+1$ (see Introduction) and using Hibi's Lower Bound Theorem, we can see that $h^*_k = 1$ for every coefficient of Δ^d_{sr}.

Proposition 4. *For every reflexive polytope P of dimension $d \leq 9$, the inclusion* $\operatorname{clspan} E_P \subseteq \operatorname{clspan} E_{\Delta^d_{sr}}$ *holds.*

Proof. There are two cases: $d \leq 5$ and $5 < d \leq 9$. In the case $d \leq 5$, we verify with a computer that

$$\operatorname{clspan} p^d_1 \subset \operatorname{clspan} E_{\Delta^d_{sr}} \subset \operatorname{clspan} p^d_0.$$

Let $ia - \frac{1}{2}$ be the boundary point of $\operatorname{clspan} E_{\Delta^d_{sr}}$ in the upper half plane. Lemma 3(a) implies that for $j > 0$ and $b \geq a$,

$$i^{-d} p^d_j\left(ib - \frac{1}{2}\right) > 0.$$

Assume the Ehrhart polynomial of P is given by

$$E_P(z) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} h^*_k \, p^d_k(z).$$

Since $h^*_0 = 1$,

$$i^{-d} E_P\left(ib - \frac{1}{2}\right) > i^{-d} E_{\Delta^d_{sr}}\left(ib - \frac{1}{2}\right) \geq 0.$$

In the case $5 < d \leq 9$, we can verify with a computer that

$$\operatorname{clspan} p^d_2 \subset \operatorname{clspan} E_{\Delta^d_{sr}} \subset \operatorname{clspan} p^d_1.$$

Let $ia - \frac{1}{2}$ be the boundary point of $\operatorname{clspan} E_{\Delta^d_{sr}}$ in the upper half plane. Lemma 3(a) implies that for $j > 1$ and $b \geq a$,

$$i^{-d} p^d_j\left(ib - \frac{1}{2}\right) > 0.$$

Assume the Ehrhart polynomial of P is given by

$$E_P(z) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} h^*_k \, p^d_k(z).$$

Since $h_0^* = 1$ and, by Hibi's Lower Bound Theorem, $h_k^* \geq h_1^*$ for $k > 1$, we get

$$i^{-d} E_P\left(ib - \frac{1}{2}\right) \geq h_1^* \sum_{k=1}^{d} i^{-d} p_k^d \left(ib - \frac{1}{2}\right) + i^{-d} p_0^d \left(ib - \frac{1}{2}\right)$$
$$> i^{-d} E_{\Delta_{sr}^d}\left(ib - \frac{1}{2}\right) \geq 0,$$

which concludes the proof. □

For higher degrees, it is no longer true that $\operatorname{clspan} p_2^d \subset \operatorname{clspan} E_{\Delta_{sr}^d} \subset \operatorname{clspan} p_1^d$ and thus Hibi's Lower Bound Theorem can no longer guarantee that the h_k^* for $k \geq 3$ are large enough to balance out h_3^*. In particular, in degree 10, for $2 \leq m \leq 14$ the polynomial

$$f(z) = p_0^5(z) + p_1^5(z) + m p_2^5(z) + p_3^5(z) + p_4^5(z) + p_5^5(z)$$

is a CL-polynomial whose extremal roots have a larger absolute imaginary part than those of the Ehrhart polynomial of Δ_{sr}. We still conjecture the following.

Conjecture 3. Let P be a reflexive polytope of dimension d whose h^*-polynomial is unimodal, i.e., satisfies the inequalities

$$h_0^* \leq h_1^* \leq \cdots \leq h^*_{\lfloor \frac{d}{2} \rfloor} \geq \cdots \geq h_{d-1}^* h_d^*,$$

where h_k^* is the k-th coefficient of h_P^*. Then $\operatorname{clspan} E_P \subseteq \operatorname{clspan} E_{\Delta_{sr}^d}$.

Remark 2. Table 1 gives an overview of the behavior of the maximal root of p_0^d, the maximal root of the standard reflexive simplex, and the bounds from Theorems 2 and 3 for selected values of d. The values were obtained using SAGEMATH [21].

2.3. Connectedness of the Set of Possible Roots

We return to the characterization of the sets Ω_d we defined in the very beginning of this section. After establishing a sharp bound, it is natural to ask, which values on CL within that bound can be assumed by the roots of an appropriate degree d polynomial in $\mathfrak{C}$.

Lemma 4. *For any positive integer d, the polynomial p_0^d CL-interlaces p_0^{d+1}.*

Proof. Equation (1) tells us that

$$h^*_{p_0^d}(t) = (1+t)^{d+1} \sum_{k \geq 0} p_0^d(k)\, t^k = d! \left(1 + t^d\right).$$

An analogous results holds for p_0^{d+1}. The roots of $h^*_{p_0^d}$ and $h^*_{p_0^{d+1}}$ are

$$\exp\left(\frac{1}{d}(1+2n)\pi i\right) \quad \text{and} \quad \exp\left(\frac{1}{d+1}(1+2n)\pi i\right)$$

d	α_d	β_d	$\frac{d^2}{\pi}$	$d(d-\frac{1}{2})$
2	0.866	0.645	1.273	3
3	2.398	1.658	2.865	7.5
4	4.603	3.040	5.093	14
5	7.457	4.761	7.958	22.5
6	10.952	6.811	11.459	33
7	15.085	9.186	15.597	45.5
8	19.857	11.882	20.372	60
9	25.267	14.899	25.783	76.5
10	31.313	18.236	31.831	95
20	126.802	69.147	127.324	390
30	285.956	151.904	286.479	885
100	3182.575	1622.493	3183.099	9950
150	7161.449	3627.845	7161.972	22425

Table 1: Comparison of the maximal roots $i\beta_d - \frac{1}{2}$ of $E_{\Delta^d_{sr}}$ to the maximal roots $i\alpha_d - \frac{1}{2}$ of p^d_0 to the bounds from Theorems 3 and 2.

respectively, where n ranges from 0 to $d-1$ (resp. d). These roots interlace on the unit circle and hence, by Proposition 1, they interlace on the canonical line. □

Lemma 5. *For any positive integer d and every positive real number w, the polynomial p^d_0 CL-interlaces by $p^{d+1}_0 + w(2z+1)p^d_0$.*

Proof. Since w is positive, we can without loss of generality assume that $w = 1$. We start with the case when d is odd. From Lemma 4 we know that p^d_0 CL-interlaces p^{d+1}_0. Further, $2z+1$ trivially CL-interlaces $(2z+1)^2$. Since

$$p^{d+1}_0\left(-\frac{1}{2}\right) = \left(-\frac{1}{2} + d + 1\right)_{d+1} + \left(-\frac{1}{2}\right)_{d+1}$$

is not a root, p^{d+1}_0 does not share a root with $(2z+1)^2$. Hence, by Proposition 2, $(2z+1)p^d_0$ interlaces $(2z+1)(p^{d+1}_0 + (2z+1)p^d_0)$. Dividing $2z+1$ from both expressions yields the statement.

In the case where d is even, p^{d+1}_0 has a root at $-\frac{1}{2}$ due to symmetry. The root has multiplicity 1, because if it had a higher multiplicity, p^d_0 would need to have two roots at $-\frac{1}{2}$ as well due to interlacing, but we already saw that this is not the case. Hence we define polynomials

$$g_k(z) = z^2 + z + \frac{1}{4} + \varepsilon_k^2,$$

where $\varepsilon_1 > \varepsilon_2 > \cdots$ is a sequence of positive reals that goes to 0. The roots of g_k are $-\frac{1}{2} \pm \epsilon_k i$. Hence, they are CL-interlaced by $2z+1$ and, with appropriately chosen ϵ_k, none of them have a common root with p_0^{d+1}. Hence, by Proposition 2, $(2z+1)p_0^d$ interlaces $(2z+1)p_0^{d+1}(z)+p_0^d(z)\,g_k(z)$. Using Proposition 3, we get that $(2z+1)(p_0^{d+1}(z)+(2z+1)p_0^d(z))$ interlaces $(2z+1)p_0^{d+1}(z)$ and dividing by $2z+1$ again yields the statement. □

Lemma 6. *Let f be a degree d SNN-polynomial. Then $(2z+1)f(z)$ is also an SNN-polynomial.*

Proof. Since by Equation (1) f is a non-negative linear combination of polynomials $\binom{z+d-k}{d}$, we may restrict ourselves to these. Using $z=(z+d-k+1)-(d-k+1)$ and then

$$\binom{z+d-k}{d}=\binom{z+d-k+1}{d+1}-\binom{z+d-k}{d+1},$$

we get

$$z\binom{z+d-k}{d}=k\binom{z+d-k+1}{d+1}+(d-k+1)\binom{z+d-k}{d+1}$$

and thus

$$(2z+1)\binom{z+d-k}{d}=(2k+1)\binom{z+d-k+1}{d+1}+(2(d-k)+1)\binom{z+d-k}{d+1}.$$

This is a positive linear combination of polynomials $\binom{z+(d+1)+k}{d+1}$. Hence, $(2z+1)f(z)$ is an SNN-polynomial. □

With these three lemmas, we can prove the main statement of this subsection. For simplicity, we will use the convention

$$h_f^*(t)=(1+t)^d\sum_{k\geq 0}f(k)\,t^k$$

for any degree d polynomial f.

Theorem 6. *For every positive integer d, the set Ω_d is connected.*

Proof. In the case $d=1$, the set $\Omega_1=\{-\frac{1}{2}\}$ is a singleton and hence connected.

Consider the case $d=2$. Let c be a positive real number. Then $h_{f_c}^*(t)=1+ct+t^2$ corresponds to an SNN-polynomial f_c whose roots are

$$-\frac{1}{2}\pm\frac{\sqrt{c^2-4c-12}}{2c+4}.$$

For $0\leq c\leq 6$, the roots of f_c lie on CL and f_0 is exactly p_0^2, which marks the boundary of Ω_2. The roots of f_6 are both $-\frac{1}{2}$. Since the roots depend continuously on c, the set Ω_2 is connected.

The proof for higher degrees $d > 2$ can be built inductively. First, take an element $z_0 \in \Omega_d$ and a degree d polynomial $f \in \mathfrak{C} \cap \mathfrak{S}$ with $f(z_0) = 0$. The polynomial $g(z) = (2z+1)\,f(z)$ is a degree $d+1$ polynomial with $g(z_0) = 0$ and it is in $\mathfrak{C}$. By Lemma 6, g lies also in $\mathfrak{S}$ and thus, $z_0 \in \Omega_{d+1}$.

Now, pick $z_0 = ci - \frac{1}{2} \in \operatorname{clspan} p_0^{d+1} \setminus \Omega_d$ in the upper half plane. Denote the roots of p_0^d by $b_k i - \frac{1}{2}$, where $b_m < b_n$ if $m < n$. Analogously, we denote the roots of p_0^{d+1} by $a_k i - \frac{1}{2}$. From Lemma 4, it follows that $a_d < b_d < c < a_{d+1}$. Define the function $g(z) = (2z+1)p_0^d(z)$. Lemmas 2 and 3(a) imply

$$i^{-d-1} p_0^{d+1}(z_0) < 0 \quad \text{and} \quad i^{-d-1}(2z_0+1)p_0^d(z_0) > 0.$$

Thus, for an appropriate number $w > 0$, the linear combination

$$\lambda(z) = p_0^{d+1}(z) + w(2z+1)p_0^d(z)$$

satisfies $\lambda(z_0) = 0$. In particular, $\lambda \in \mathfrak{S}$. Since by Lemma 4 λ is interlaced by p_0^d, it follows that $\lambda \in \mathfrak{C}$. Thus $z_0 \in \Omega_{d+1}$. □

3. Inequalities for $\mathfrak{C} \cap \mathfrak{S}$

In Equation (3), we define CL-polynomials in terms of parameters $c_k \geq \frac{1}{4}$. Every c_k corresponds to a pair of roots $-\frac{1}{2} \pm \sqrt{c_k - \frac{1}{4}}i$, which is a fact we used several times throughout the previous section. Thus, Theorem 4 can be interpreted as an inequality that gives a necessary condition for the c_k to correspond to an SNN-polynomial.

Theorem 7 (Restatement of Theorem 4)**.** *Let f be a CL-polynomial of degree d with parameters c_k. If $f \in \mathfrak{S}$, the inequality*

$$c_k \leq m_0^d$$

is satisfied for every k, where m_0^d is the maximal parameter of p_0^d.

However, the condition in Theorem 7 is very far from being sufficient to characterize the class $\mathfrak{C} \cap \mathfrak{S}$. For example, the polynomial $f(z) = \frac{1}{400}(z^2 + z + 20)^2$ has its roots at around $-\frac{1}{2} \pm 4.44i$, which by Table 1 lies within Ω_4, but

$$h_f^*(t) = 1 - \frac{379}{100}t + \frac{564}{100}t^2 - \frac{379}{100}t^3 + t^4.$$

In the following, we give a sufficient condition. We base it on a computational lemma.

Lemma 7. *Let d be a positive integer and $j \leq d$ be a non-negative integer. Further, let $c \geq \frac{1}{4}$ be a real number. Then*

$$(z^2+z+c)\binom{z+d-j}{d} = \alpha\binom{z+d-j+2}{d+2} + \beta\binom{z+d-j+1}{d+2} + \gamma\binom{z+d-j}{d+2},$$

where

$$\begin{aligned}\alpha &= j^2+j+c,\\ \beta &= 2(dj-j^2+d+1-c),\\ \gamma &= d^2-2dj-j+j^2+d+c.\end{aligned}$$

Proof. By adapting the technique used in Lemma 6,

$$(z+c)\binom{z+d-j}{d} = (j+c)\binom{z+d-j+1}{d+1} + (d-j+1-c)\binom{z+d-j}{d+1}.$$

Equation (1) implies that

$$\sum_{k\geq 0}\binom{k+d-j}{d}t^k = \frac{t^j}{(1-t)^{d+1}},$$

which means that we can write

$$\sum_{k\geq 0}(k+c)\binom{k+d-j}{d}t^k = \frac{(j+c)t^j + (d-j+1-c)t^{j+1}}{(1-t)^{d+2}}.$$

We can use the same identity to compute $\sum_{k\geq 0} k^2\binom{k+d-j}{d}t^k$ by applying it twice with $c=0$ both times. Thus, $\sum_{k\geq 0} k^2\binom{k+d-j}{d}t^k$ is given by

$$\frac{j^2t^j + \Big(j(d+2-j)+(d+1-j)(j+1)\Big)t^{j+1} + (d+1-j)^2t^{j+2}}{(1-t)^{d+3}}.$$

Summing up gives the values for α, β, and γ as stated. □

Proposition 5. *Let f be a CL-polynomial of degree d. Assume that the c_k are ordered by size. Then $f \in \mathfrak{S}$ if the c_k satisfy*

$$\frac{1}{4} \leq c_k \leq \begin{cases} 2k+2, & d \text{ is odd} \\ 2k+1, & d \text{ is even.}\end{cases} \tag{6}$$

Proof. The proof proceeds inductively. The idea is to take a degree d element f of $\mathfrak{C} \cap \mathfrak{S}$ and multiply it with $z^2 + z + c$, where c is chosen so that it preserves non-negativity of the coefficients of the h^*_f. That is in particular the case when the three factors from Lemma 7, $\alpha = j^2 + j + c$, $\beta = 2(dj - j^2 + d + 1 - c)$, and $\gamma = d^2 - 2dj - j + j^2 + d + c$, are non-negative. Since c is positive, α and γ are always non-negative. For β, the largest possible choice for c is $d+1$ since j ranges from 0 up to d.

To complete the induction, we only have to look at the cases of $d = 1$ and $d = 2$. We start with the former. If f has degree 1, it is of the form $z + \frac{1}{2}$ and $h^*_f(t) = 1 + t$. Thus $c_0 \leq 2$, $c_1 \leq 4$, $c_2 \leq 6$, etc. If f has degree 2, it is of the form $z^2 + z + c_0$ and has $h^*_f(t) = c_0 + 2(1 - c_0)t + c_0 t^2$. Thus, $c_0 \leq 1$, $c_1 \leq 3$, $c_2 \leq 5$, etc. □

The class of CL-polynomials that satisfy this proposition trivially includes the Ehrhart polynomials $E_{[-1,1]^d}$ of reflexive hypercubes since they satisfy

$$c_1 = c_2 = \cdots = c_d = \frac{1}{4}.$$

It is possible to construct further examples.

Example 1. Let P be a CL-polytope of dimension d. Then there exists a non-negative integer n, such that the Ehrhart polynomial of $P \times [-1,1]^n$ satisfies Inequalities (6).

If E_P is defined by the parameters $c_1, c_2, \ldots, c_d$, then $E_{P\times[-1,1]^n}$ is defined by the parameters $\frac{1}{4}, \frac{1}{4}, \ldots, \frac{1}{4}, c_1, c_2, \ldots, c_d$, where $\frac{1}{4}$ appears n times. For the c_k, this changes the equations to

$$\frac{1}{4} \leq c_k \leq \begin{cases} 2(k+n)+2, & d \text{ is odd} \\ 2(k+n)+1, & d \text{ is even.} \end{cases}$$

which is always satisfied for a sufficiently large n.

Using the same idea, we also get another example.

Example 2. Let P be a CL-polytope of dimension d and let Q be a CL-polytope of dimension $2m+1$. Then there exists a non-negative integer n, such that the Ehrhart polynomial of $P \times Q^n$ satisfies Inequalities (6).

However, there exist counter-examples as well. The Ehrhart polynomial of standard reflexive 4-simplex Δ^4_{sr} does not satisfy Inequalities 6:

$$E_{\Delta^4_{sr}}(k) = \frac{5}{24}(x^2 + x + 0.505558989151154)(x^2 + x + 9.49444101084885).$$

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