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ON $[j, k]$-OVERPARTITIONS WITH EVEN PARTS DISTINCT

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Abstract

Let $\overline{ped}_{j,k}(n)$ denote the number of $[j, k]$-overpartitions of a positive integer n with even parts distinct and where the first occurrence of each distinct part congruent to j modulo k may be overlined. In this paper, we establish many infinite families of congruences modulo powers of 2 for $\overline{ped}_{9,9}(n)$ and congruences modulo powers of 2 and 3 for $\overline{ped}_{9,18}(n)$. For example, for any $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+3} \cdot 5^{2\beta+1}(5n+k) + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+1} + 1}{4}\right) \equiv 0 \pmod{32},$$

where $k = 0, 1, 3,$ or 4.

1. Introduction

An *overpartition* of a positive integer n is a partition in which the first occurrence of each distinct part may be overlined. For example, the overpartitions of 3 are

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

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Let $\overline{p}(n)$ denote the number of overpartitions of n with $\overline{p}(0) = 1$. Corteel and Lovejoy [7] obtained the following generating function for $\overline{p}(n)$:

$$\sum_{n=0}^{\infty} \overline{p}\,(n)\,q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.$$

For more information about $\overline{p}(n)$, one can see [1, 6, 10, 12, 18, 19, 21, 22]. Throughout the paper, we use the standard q-series notation, and f_k is defined as

$$f_k := \left(q^k;q^k\right)_\infty = \prod_{n=0}^{\infty}(1-q^{nk}).$$

Ramanujan's general theta function $f(a,b)$ is defined for $|ab| < 1$ by

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}.$$

By using Jacobi's triple product identity [2, Entry 19, p. 35], the function $f(a,b)$ can be written as

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The most important special cases of $f(a,b)$ are as follows:

$$\phi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1},$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1.$$

If we replace q by $-q$ in (1), we get

$$\phi(-q) := f(-q,-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = \frac{f_1^2}{f_2}.$$

For positive integers j and k such that $k > j \geq 1$, an $[j,k]$-*overpartition* of n is a partition in which the first occurrence of each distinct part congruent to j modulo k may be overlined. Let $\overline{p}_{j,k}(n)$ denote the number of such overpartitions of n with $\overline{p}_{j,k}(0) = 1$. The generating function for $\overline{p}_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}_{j,k}\,(n)\,q^n = \frac{(-q^j;q^k)_\infty}{(q;q)_\infty}.$$

For example, the $[5,10]$-overpartitions of 5 are

$$5,\ \bar{5},\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

Mahadeva Naika et al. [14] obtained many infinite families of congruences modulo powers of 2 for $\overline{pr}_{j,k}(n)$, the number of $[j,k]$-regular overpartitions of n in which none of the parts are congruent to j modulo k. For example, for all $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{pr}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta+1}(24(5n+i)+23)\right) \equiv 0 \pmod{64},$$

where $i = 0, 1, 2,$ or 4. For more information about $[j,k]$-overpartitions, one can see [13, 16, 17].

Let $\overline{ped}_{j,k}(n)$ denote the number of $[j,k]$-overpartitions of a positive integer n with even parts distinct and where the first occurrence of each distinct part congruent to j modulo k may be overlined. The generating function for $\overline{ped}_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{ped}_{j,k}(n)\, q^n = \frac{(q^4;q^4)_\infty(-q^j;q^k)_\infty}{(q;q)_\infty}. \tag{2}$$

In [15], the authors proved many infinite families of congruences modulo powers of 2 for $\overline{ped}_{3,3}(n)$ and $\overline{ped}_{3,6}(n)$. For example, for any $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{ped}_{3,6}\left(8 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+2} n + c_1 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{64},$$

where $c_1 \in \{23, 47, 71, 119\}$.

In this paper, we establish many infinite families of congruences modulo powers of 2 for $\overline{ped}_{9,9}(n)$ and congruences modulo powers of 2 and 3 for $\overline{ped}_{9,18}(n)$. For example, for any $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+3} \cdot 5^{2\beta+1}(5n+k) + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+1}+1}{4}\right) \equiv 0 \pmod{32},$$

where $k = 0, 1, 3,$ or 4.

2. Preliminary Results

Define

$$\zeta = \frac{f_1 f_2}{q f_9 f_{18}}$$

and

$$T = \frac{f_3^4 f_6^4}{q^3 f_9^4 f_{18}^4}.$$

Let H be the "huffing" operator defined by

$$H\left(\sum a_n q^n\right) = \sum a_{3n} q^{3n}.$$

From Chan [5, (11)-(19)], for each $i \geq 1$, we have

$$H\left(\frac{1}{\zeta^i}\right) = \sum_{j=1}^{i} \frac{m_{i,j}}{T^j},$$

where $m_{i,j}$'s are defined in the following matrix. The $m_{i,j}$'s form a matrix, the first six rows of which are

3	0	0	0	0	0	0	$\cdots$
2	27	0	0	0	0	0	$\cdots$
1	27	243	0	0	0	0	$\cdots$
0	18	324	2187	0	0	0	$\cdots$
0	5	270	3645	19683	0	0	$\cdots$
0	1	126	3645	39366	177147	0	$\cdots$

with $m_{i,1} = 0$ for $i \geq 4$ and, for $j \geq 2$, we have

$$m_{i,j} = 9m_{i-1,j-1} + 3m_{i-2,j-1} + m_{i-3,j-1}.$$

In fact $m_{4i-3,j} = 0$ for $j \leq i-1$, so we can write

$$H\left(\frac{1}{\zeta^{4i-3}}\right) = \sum_{j=1}^{4i-3} \frac{m_{4i-3,j}}{T^j} = \sum_{j=1}^{3i-2} \frac{m_{4i-3,i+j-1}}{T^{i+j-1}} = \sum_{j=1}^{3i-2} \frac{a_{i,j}}{T^{i+j-1}}, \tag{3}$$

where

$$a_{i,j} = m_{4i-3,i+j-1}.$$

Similarly, $m_{4i-1,j} = 0$ for $j \leq i-1$, so we can write

$$H\left(\frac{1}{\zeta^{4i-1}}\right) = \sum_{j=1}^{4i-1} \frac{m_{4i-1,j}}{T^j} = \sum_{j=1}^{3i} \frac{m_{4i-1,i+j-1}}{T^{i+j-1}} = \sum_{j=1}^{3i} \frac{b_{i,j}}{T^{i+j-1}}, \tag{4}$$

where

$$b_{i,j} = m_{4i-1,i+j-1}.$$

Also, $m_{4i,j} = 0$ for $j \leq i$, so we can write

$$H\left(\frac{1}{\zeta^{4i}}\right) = \sum_{j=1+i}^{4i} \frac{m_{4i,j}}{T^j} = \sum_{j=1}^{3i} \frac{m_{4i,i+j}}{T^{i+j}} = \sum_{j=1}^{3i} \frac{c_{i,j}}{T^{i+j}}, \tag{5}$$

where

$$c_{i,j} = m_{4i,i+j}.$$

Equation (3) can be written as

$$H\left(\left(q\frac{f_9 f_{18}}{f_1 f_2}\right)^{4i-3}\right) = \sum_{j=1}^{3i-2} a_{i,j} \left(q^3 \frac{f_9^4 f_{18}^4}{f_3^4 f_6^4}\right)^{i+j-1},$$

which implies that

$$H\left(q^{i-3}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-3}\right)=\sum_{j=1}^{3i-2}a_{i,j}q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j-1}.$$

Equation (4) can be written as

$$H\left(\left(q\frac{f_9f_{18}}{f_1f_2}\right)^{4i-1}\right)=\sum_{j=1}^{3i}b_{i,j}\left(q^3\frac{f_9^4f_{18}^4}{f_3^4f_6^4}\right)^{i+j-1},$$

which implies that

$$H\left(q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-1}\right)=\sum_{j=1}^{3i}b_{i,j}q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j-3}.$$

Similarly, Equation (5) can be written as

$$H\left(\left(q\frac{f_9f_{18}}{f_1f_2}\right)^{4i}\right)=\sum_{j=1}^{3i}c_{i,j}\left(q^3\frac{f_9^4f_{18}^4}{f_3^4f_6^4}\right)^{i+j},$$

which implies that

$$H\left(q^{i}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i}\right)=\sum_{j=1}^{3i}c_{i,j}q^{3j}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j}.$$

Lemma 1 ([9])**.** *The following* 2*-dissections hold:*

$$f_1^2=\frac{f_2f_8^5}{f_4^2f_{16}^2}-2q\frac{f_2f_{16}^2}{f_8}, \tag{6}$$

$$f_1^4=\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}. \tag{7}$$

Lemma 2 ([9])**.** *The following* 2*-dissections hold:*

$$f_1f_3=\frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2}-q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}, \tag{8}$$

$$\frac{f_3^3}{f_1}=\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}. \tag{9}$$

Lemma 3. *The following* 3*-dissections hold:*

$$\frac{f_2^2}{f_1}=\frac{f_6f_9^2}{f_3f_{18}}+q\frac{f_{18}^2}{f_9}, \tag{10}$$

$$\frac{f_1^2}{f_2}=\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}, \tag{11}$$

$$\frac{f_2}{f_1^2}=\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}. \tag{12}$$

Equations (10) and (11) can be found in [9]. Equation (12) is from [3].

Lemma 4. *The following* 3*-dissections hold:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}, \tag{13}$$

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}, \tag{14}$$

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q \frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3}. \tag{15}$$

Equations (13), (14), and (15) can be found in [9], [11], and [4], respectively.

Lemma 5 ([9])**.** *The following* 5*-dissection holds:*

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{16}$$

where

$$R(q) = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

Lemma 6 ([9])**.** *The following* 7*-dissection holds:*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)}\right), \tag{17}$$

where $A(q) = f(-q^3,-q^4)$*,* $B(q) = f(-q^2,-q^5)$ *and* $C(q) = f(-q,-q^6)$*.*

3. Congruences for $\overline{ped}_{9,9}(n)$

Theorem 1. *For all* $n \geq 0$ *and* $\alpha \geq 0$*, we have, modulo 32,*

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+2} n + \frac{3^{4\alpha+2}-1}{2}\right) q^n \equiv 4\frac{f_1^3 f_3 f_4 f_6}{f_2^2} + 16q f_6^6, \tag{18}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+6} n + \frac{19\cdot 3^{4\alpha+5}-1}{2}\right) \equiv \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4} n + \frac{19\cdot 3^{4\alpha+3}-1}{2}\right), \tag{19}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 5^{2\beta} n + \frac{19\cdot 3^{4\alpha+3}\cdot 5^{2\beta}-1}{2}\right) q^n \equiv 16 f_1 f_6^3, \tag{20}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 5^{2\beta+1}(5n+i) + \frac{23\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}-1}{2}\right) \equiv 0, \tag{21}$$

where $i = 0, 1, 2,$ *or* 4*.*

Proof. Setting $j = k = 9$ in (2), we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}(n)\, q^n = \frac{f_4 f_{18}}{f_1 f_9}. \tag{22}$$

Employing (15) in (22) and then collecting the coefficients of q^{3n+1} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}(3n+1)\, q^n = \frac{f_2^2 f_3^2 f_{12}}{f_1^4 f_6}. \tag{23}$$

Substituting (12) in (23), we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}(9n+4)\, q^n = 4\frac{f_2^6 f_3^9 f_4}{f_1^{13} f_6^3} + 16q\frac{f_2^3 f_4 f_6^6}{f_1^{10}}. \tag{24}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m} \equiv f_{2k}^{m} \pmod{2}, \tag{25}$$
$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{4}, \tag{26}$$
$$f_k^{8m} \equiv f_{2k}^{4m} \pmod{8}, \tag{27}$$
$$f_k^{16m} \equiv f_{2k}^{8m} \pmod{16}. \tag{28}$$

Invoking (25) and (27) in (24), we find that, modulo 32,

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}(9n+4)\, q^n \equiv 4\frac{f_1^3 f_3 f_4 f_6}{f_2^2} + 16q f_6^6,$$

which is the $\alpha = 0$ case of (18). Now assume that Congruence (18) is true for $\alpha \geq 0$. Employing (12) and (13) in (18), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+3}n + \frac{3^{4\alpha+2}-1}{2}\right) q^n \equiv 4\frac{f_2^2 f_3^6 f_6^3}{f_{12}^3} + 8q\frac{f_1 f_3^3 f_4 f_6^3}{f_2^2} + 16q\frac{f_1 f_{12}^3}{f_3^3}, \tag{29}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+3}n + \frac{3^{4\alpha+3}-1}{2}\right) q^n \equiv 16 f_2^6 + 20\frac{f_1 f_2 f_3^3 f_6^6}{f_{12}^3} + 16q f_{12}^3, \tag{30}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+3}n + \frac{5\cdot 3^{4\alpha+2}-1}{2}\right) q^n \equiv 8\frac{f_3^6 f_4}{f_2} + 16q\frac{f_3^3 f_{12}^3}{f_1}. \tag{31}$$

Employing (13) and (14) in (30), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+4}n + \frac{3^{4\alpha+3}-1}{2}\right) q^n \equiv 16f_4 + 20\frac{f_1^2 f_3^4 f_4}{f_2 f_6^2}, \tag{32}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+4}n + \frac{3^{4\alpha+4}-1}{2}\right) q^n \equiv 12\frac{f_1^3 f_3 f_4 f_6}{f_2^2} + 16f_4^3 + 16qf_6^6, \tag{33}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+4}n + \frac{5\cdot 3^{4\alpha+3}-1}{2}\right) q^n \equiv 24\frac{f_2^3 f_3^6}{f_4}. \tag{34}$$

Utilizing (12) and (13) in (33), we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+5}n + \frac{3^{4\alpha+4}-1}{2}\right) q^n \equiv 12\frac{f_2^2 f_3^6 f_6^3}{f_{12}^3} + 16f_4 + 16qf_1 f_3^9 + 24q\frac{f_3^3 f_4 f_6^3}{f_1^3}, \tag{35}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+5}n + \frac{3^{4\alpha+5}-1}{2}\right) q^n \equiv 16f_2^6 + 28\frac{f_1 f_2 f_3^3 f_6^6}{f_{12}^3}, \tag{36}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+5}n + \frac{5\cdot 3^{4\alpha+4}-1}{2}\right) q^n \equiv 24\frac{f_3^6 f_4}{f_2} + 16q\frac{f_3^3 f_{12}^3}{f_1}. \tag{37}$$

Substituting (13) and (14) in (36), we get

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+6}n + \frac{3^{4\alpha+5}-1}{2}\right) q^n \equiv 16f_4 + 28\frac{f_1^2 f_3^4 f_4}{f_2 f_6^2}, \tag{38}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+6}n + \frac{3^{4\alpha+6}-1}{2}\right) q^n \equiv 4\frac{f_1^3 f_3 f_4 f_6}{f_2^2} + 16qf_6^6, \tag{39}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+6}n + \frac{5\cdot 3^{4\alpha+5}-1}{2}\right) q^n \equiv 8\frac{f_2^3 f_6^6}{f_4}. \tag{40}$$

Equation (39) is the $\alpha+1$ case of (18). By induction, Congruence (18) holds for all integer $\alpha \geq 0$. Employing (6) and (7) in (32), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+4}n + \frac{3^{4\alpha+3}-1}{2}\right) q^n \equiv 16f_2 + 20\frac{f_4^5 f_6^2}{f_2 f_3^4 f_8^2} \tag{41}$$

and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+4}n + \frac{7\cdot 3^{4\alpha+3}-1}{2}\right) q^n \equiv 24f_2 f_4^3 + 16qf_2 f_{12}^3. \tag{42}$$

Equation (42) implies

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}n + \frac{7\cdot 3^{4\alpha+3}-1}{2}\right) q^n \equiv 24f_1 f_2^3 \tag{43}$$

and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}n + \frac{19\cdot 3^{4\alpha+3}-1}{2}\right) q^n \equiv 16 f_1 f_6^3. \tag{44}$$

Employing (6) and (7) in (38), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+6}n + \frac{3^{4\alpha+5}-1}{2}\right) q^n \equiv 16 f_2 + 28\frac{f_4^5 f_6^2}{f_2 f_3^4 f_8^2} \tag{45}$$

and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+6}n + \frac{7\cdot 3^{4\alpha+5}-1}{2}\right) q^n \equiv 8 f_2 f_4^3 + 16 q f_2 f_{12}^3. \tag{46}$$

Equation (46) implies

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+6}n + \frac{7\cdot 3^{4\alpha+5}-1}{2}\right) q^n \equiv 8 f_1 f_2^3 \tag{47}$$

and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+6}n + \frac{19\cdot 3^{4\alpha+5}-1}{2}\right) q^n \equiv 16 f_1 f_6^3. \tag{48}$$

From Equations (44) and (48), we obtain (19). Equation (44) is the $\beta = 0$ case of (20). Suppose that Congruence (20) is true for $\beta \geq 0$. Using (16) in (20), we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 5^{2\beta+1}n + \frac{23\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}-1}{2}\right) q^n \equiv 16 q^3 f_5 f_{30}^3,$$

which implies (21) and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 5^{2\beta+2}n + \frac{19\cdot 3^{4\alpha+3}\cdot 5^{2\beta+2}-1}{2}\right) q^n \equiv 16 f_1 f_6^3,$$

which proves that Congruence (20) is true for $\beta+1$. By induction, Congruence (20) is true for all $\alpha, \beta \geq 0$. □

Theorem 2. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have, modulo 16,*

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+6}n + \frac{13\cdot 3^{4\alpha+5}-1}{2}\right) \equiv \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}n + \frac{13\cdot 3^{4\alpha+3}-1}{2}\right) \equiv 0, \tag{49}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+6}n + \frac{7\cdot 3^{4\alpha+5}-1}{2}\right) \equiv \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}n + \frac{7\cdot 3^{4\alpha+3}-1}{2}\right), \tag{50}$$

$$\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+5}n + \frac{11\cdot 3^{4\alpha+4}-1}{2}\right) \equiv \overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+3}n + \frac{11\cdot 3^{4\alpha+2}-1}{2}\right) \equiv 0, \tag{51}$$

$$\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+6}n+\frac{11\cdot 3^{4\alpha+5}-1}{2}\right)\equiv \overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+4}n+\frac{11\cdot 3^{4\alpha+3}-1}{2}\right)\equiv 0, \tag{52}$$

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 5^{2\beta}n+\frac{7\cdot 3^{4\alpha+3}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 8f_1^7, \tag{53}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 5^{2\beta+1}(5n+i)+\frac{11\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}-1}{2}\right)\equiv 0, \tag{54}$$

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}\cdot 7^{2\gamma}n+\frac{5\cdot 3^{4\alpha+2}\cdot 7^{2\gamma}-1}{2}\right)q^n\equiv 8f_1^5, \tag{55}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}\cdot 7^{2\gamma+1}(7n+j)+\frac{11\cdot 3^{4\alpha+2}\cdot 7^{2\gamma+1}-1}{2}\right)\equiv 0, \tag{56}$$

where $i=0,2,3,$ *or* 4 *and* $j=0,2,3,4,5,$ *or* 6.

Proof. From Equations (41) and (45), we obtain (49). In view of Equations (43) and (47), we obtain (50). Equation (43) becomes, modulo 16,

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}n+\frac{7\cdot 3^{4\alpha+3}-1}{2}\right)q^n\equiv 8f_1^7,$$

which is the $\beta=0$ case of (53). Now assume that Congruence (53) is true for $\beta\geq 0$. Utilizing (16) in (53), we find that

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 5^{2\beta+1}n+\frac{11\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}-1}{2}\right)q^n\equiv 8qf_5^7,$$

which implies (54) and

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 5^{2\beta+2}n+\frac{7\cdot 3^{4\alpha+3}\cdot 5^{2\beta+2}-1}{2}\right)q^n\equiv 8f_1^7,$$

which shows that Congruence (53) is true for $\beta+1$. By induction, Congruence (53) holds for all $\alpha,\beta\geq 0$. Equation (31) implies, modulo 16,

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+3}n+\frac{5\cdot 3^{4\alpha+2}-1}{2}\right)q^n\equiv 8\frac{f_2f_3^3}{f_1}. \tag{57}$$

Using (9) in (57), we find that

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}n+\frac{5\cdot 3^{4\alpha+2}-1}{2}\right)q^n\equiv 8f_1^5,$$

which is the $\gamma = 0$ case of Congruence (55). Now assume that Congruence (55) is true for $\gamma \geq 0$. Using (17) in (55) and then collecting the coefficients of q^{7n+3} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}\cdot 7^{2\gamma+1}n+\frac{11\cdot 3^{4\alpha+2}\cdot 7^{2\gamma+1}-1}{2}\right)q^n \equiv 8qf_7^5,$$

which implies (56) and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}\cdot 7^{2\gamma+2}n+\frac{5\cdot 3^{4\alpha+2}\cdot 7^{2\gamma+2}-1}{2}\right)q^n \equiv 8f_1^5,$$

which implies that Congruence (55) is true for $\gamma+1$. By induction, Congruence (55) holds for all $\alpha, \gamma \geq 0$. From Equations (31) and (37), we arrive at (51). In view of Congruences (34) and (40), we obtain (52). □

Theorem 3. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have, modulo 16,*

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}n+\frac{13\cdot 3^{4\alpha+2}-1}{2}\right) \equiv 0, \tag{58}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}n+\frac{19\cdot 3^{4\alpha+2}-1}{2}\right) \equiv 0, \tag{59}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+\frac{7\cdot 3^{4\alpha+2}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 8f_1^7, \tag{60}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+i)+\frac{11\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}-1}{2}\right) \equiv 0, \tag{61}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+5}n+\frac{13\cdot 3^{4\alpha+4}-1}{2}\right) \equiv 0, \tag{62}$$

$$\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+5}n+\frac{7\cdot 3^{4\alpha+4}-1}{2}\right) \equiv \overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+2}-1}{2}\right), \tag{63}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 7^{2\gamma}n+\frac{5\cdot 3^{4\alpha+3}\cdot 7^{2\gamma}-1}{2}\right)q^n \equiv 8f_1^5, \tag{64}$$

$$\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}\cdot 7^{2\gamma+1}(7n+j)+\frac{11\cdot 3^{4\alpha+3}\cdot 7^{2\gamma+1}-1}{2}\right) \equiv 0, \tag{65}$$

where $i = 0, 2, 3,$ or 4 and $j = 0, 2, 3, 4, 5,$ or 6.

Proof. Equation (29) becomes, modulo 16,

$$\sum_{n=0}^{\infty} \overline{ped}_{9,9}\left(3^{4\alpha+3}n+\frac{3^{4\alpha+2}-1}{2}\right)q^n \equiv 4\frac{f_2^2f_3^2f_6}{f_{12}}+8q\frac{f_2f_3^3f_6^3}{f_1}. \tag{66}$$

Employing (6) and (9) in (66), we get

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+3}n+\frac{3^{4\alpha+2}-1}{2}\right)q^n\equiv 4\frac{f_1^2f_3^2f_{12}}{f_6^3}+8q\frac{f_3^3f_6^3}{f_1} \tag{67}$$

and

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+2}-1}{2}\right)q^n\equiv 8\frac{f_2^3f_3^3}{f_1}+8qf_2f_{12}^3. \tag{68}$$

Substituting (8) and (9) in (67) and then comparing the coefficients of q^{2n+1} on both sides of the resultant equation. we arrive at (58). Using (9) in (68), we obtain (59) and

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+2}-1}{2}\right)q^n\equiv 8f_1^7,$$

which is the $\beta=0$ case of (60). The rest of the proofs of Identities (60) and (61) are similar to the proofs of Identities (53) and (54), so we omit the details. Equation (35) becomes, modulo 16,

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(3^{4\alpha+5}n+\frac{3^{4\alpha+4}-1}{2}\right)q^n\equiv 12\frac{f_2^2f_3^2f_6}{f_{12}}+8q\frac{f_2f_3^3f_6^3}{f_1}. \tag{69}$$

Utilizing (6) and (9) in (69), we have

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+5}n+\frac{3^{4\alpha+4}-1}{2}\right)q^n\equiv 12\frac{f_1^2f_3^2f_{12}}{f_6^3}+8q\frac{f_3^3f_6^3}{f_1} \tag{70}$$

and

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+5}n+\frac{7\cdot 3^{4\alpha+4}-1}{2}\right)q^n\equiv 8\frac{f_2^3f_3^3}{f_1}+8qf_2f_{12}^3. \tag{71}$$

Employing (8) and (9) in (70) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we get (62). From Equations (68) and (71), we obtain (63). Equation (34) implies, modulo 16,

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(2\cdot 3^{4\alpha+4}n+\frac{5\cdot 3^{4\alpha+3}-1}{2}\right)q^n\equiv 8\frac{f_2f_3^3}{f_1}. \tag{72}$$

Substituting (9) in (72), we have

$$\sum_{n=0}^{\infty}\overline{ped}_{9,9}\left(4\cdot 3^{4\alpha+4}n+\frac{5\cdot 3^{4\alpha+3}-1}{2}\right)q^n\equiv 8f_1^5,$$

which is the $\gamma=0$ case of (64). The rest of the proofs of Identities (64) and (65) are similar to the proofs of Identities (55) and (56), so we omit the details. □

4. Congruences for $\overline{ped}_{9,18}(n)$

Theorem 4. *For all $n \geq 0$ and $\alpha \geq 0$, we have, modulo 64,*

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+2}n + \frac{3^{4\alpha+3}+1}{4}\right) q^n \equiv 12\frac{f_1^4 f_3^6}{f_2^2}, \tag{73}$$

$$\overline{ped}_{9,18}\left(3^{4\alpha+5}n + \frac{7 \cdot 3^{4\alpha+4}+1}{4}\right) \equiv \overline{ped}_{9,18}\left(3^{4\alpha+3}n + \frac{7 \cdot 3^{4\alpha+2}+1}{4}\right), \tag{74}$$

$$\overline{ped}_{9,18}\left(3^{4\alpha+5}n + \frac{11 \cdot 3^{4\alpha+4}+1}{4}\right) \equiv \overline{ped}_{9,18}\left(3^{4\alpha+3}n + \frac{11 \cdot 3^{4\alpha+2}+1}{4}\right), \tag{75}$$

$$\overline{ped}_{9,18}\left(3^{4\alpha+6}n + \frac{5 \cdot 3^{4\alpha+5}+1}{4}\right) \equiv \overline{ped}_{9,18}\left(3^{4\alpha+4}n + \frac{5 \cdot 3^{4\alpha+3}+1}{4}\right). \tag{76}$$

Proof. Setting $j = 9$ and $k = 18$ in (2), we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(n)\, q^n = \frac{f_4 f_{18}^2}{f_1 f_9 f_{36}}. \tag{77}$$

Using (15) in (77) and then extracting the terms involving q^{3n+1} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(3n+1)\, q^n = \frac{f_2^2 f_3^2}{f_1^4}. \tag{78}$$

Subtituting (12) in (78), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(9n+1)\, q^n = \frac{f_2^8 f_3^{12}}{f_1^{14} f_6^6} + 16q\frac{f_2^5 f_3^3 f_6^3}{f_1^{11}}, \tag{79}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(9n+4)\, q^n = 4\frac{f_2^7 f_3^9}{f_1^{13} f_6^3} + 16q\frac{f_2^4 f_6^6}{f_1^{10}}, \tag{80}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(9n+7)\, q^n = 12\frac{f_2^6 f_3^6}{f_1^{12}}. \tag{81}$$

Invoking (28) in (81), we find that, modulo 64,

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(9n+7)\, q^n \equiv 12\frac{f_1^4 f_3^6}{f_2^2},$$

which is the $\alpha = 0$ case of (73). Now assume that Congruence (73) is true for $\alpha \geq 0$. Employing (11) in (73), we get

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+3}n + \frac{3^{4\alpha+3}+1}{4}\right) q^n \equiv 12\frac{f_1^6 f_3^4}{f_6^2}, \tag{82}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+3}n + \frac{7\cdot 3^{4\alpha+2}+1}{4}\right) q^n \equiv 16 f_1^3 f_2 f_3 f_6, \tag{83}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+3}n + \frac{11\cdot 3^{4\alpha+2}+1}{4}\right) q^n \equiv 48 f_2^2 f_3^6. \tag{84}$$

Utilizing (13) in (82), we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+4}n + \frac{3^{4\alpha+3}+1}{4}\right) q^n \equiv 12 \frac{f_1^2 f_6^2}{f_3^4} + 32q \frac{f_3^3 f_6^3}{f_1}, \tag{85}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+4}n + \frac{5\cdot 3^{4\alpha+3}+1}{4}\right) q^n \equiv 56 \frac{f_1^3 f_3 f_6}{f_2} + 32q \frac{f_6^6}{f_2}, \tag{86}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+4}n + \frac{3^{4\alpha+5}+1}{4}\right) q^n \equiv 44 \frac{f_1^4 f_3^6}{f_2^2}. \tag{87}$$

Using (11) in (87), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+5}n + \frac{3^{4\alpha+5}+1}{4}\right) q^n \equiv 44 \frac{f_1^6 f_3^4}{f_6^2}, \tag{88}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+5}n + \frac{7\cdot 3^{4\alpha+4}+1}{4}\right) q^n \equiv 16 f_1^3 f_2 f_3 f_6, \tag{89}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+5}n + \frac{11\cdot 3^{4\alpha+4}+1}{4}\right) q^n \equiv 48 f_2^2 f_3^6. \tag{90}$$

In view of Equations (83) and (89), we obtain (74). From Equations (84) and (90), we obtain (75). Substituting (13) in (88), we arrive at

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+6}n + \frac{3^{4\alpha+5}+1}{4}\right) q^n \equiv 44 \frac{f_1^2 f_6^2}{f_3^4} + 32q \frac{f_3^3 f_6^3}{f_1}, \tag{91}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+6}n + \frac{5\cdot 3^{4\alpha+5}+1}{4}\right) q^n \equiv 56 \frac{f_1^3 f_3 f_6}{f_2} + 32q \frac{f_6^6}{f_2}, \tag{92}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+6}n + \frac{3^{4\alpha+7}+1}{4}\right) q^n \equiv 12 \frac{f_1^4 f_3^6}{f_2^2}. \tag{93}$$

Equation (93) is the $\alpha+1$ case of (73). So, by induction, Congruence (73) holds for all $\alpha \geq 0$. In view of Congruences (86) and (92), we arrive at (76). □

Theorem 5. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have, modulo 32,*

$$\overline{ped}_{9,18}\left(3^{4\alpha+6}n + \frac{3^{4\alpha+5}+1}{4}\right) \equiv \overline{ped}_{9,18}\left(3^{4\alpha+4}n + \frac{3^{4\alpha+3}+1}{4}\right), \tag{94}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+\frac{7\cdot 3^{4\alpha+2}\cdot 5^{2\beta}+1}{4}\right)q^n \equiv 16f_1^7, \tag{95}$$

$$\overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+i)+\frac{11\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}+1}{4}\right) \equiv 0, \tag{96}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+\frac{19\cdot 3^{4\alpha+2}\cdot 5^{2\beta}+1}{4}\right)q^n \equiv 16f_1f_6^3, \tag{97}$$

$$\overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+j)+\frac{23\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}+1}{4}\right) \equiv 0, \tag{98}$$

$$\overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}n+\frac{23\cdot 3^{4\alpha+2}+1}{4}\right) \equiv 0, \tag{99}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+\frac{11\cdot 3^{4\alpha+2}\cdot 5^{2\beta}+1}{4}\right)q^n \equiv 16f_2f_3^3, \tag{100}$$

$$\overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+k)+\frac{7\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}+1}{4}\right) \equiv 0, \tag{101}$$

where $i = 0, 2, 3,$ *or* 4, $j = 0, 1, 2,$ *or* 4, *and* $k = 0, 1, 3,$ *or* 4.

Proof. From Equations (85) and (91), we obtain (94). Equation (83) becomes, modulo 32,

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+2}+1}{4}\right)q^n \equiv 16\frac{f_2^3f_3^3}{f_1}. \tag{102}$$

Employing (9) in (102), we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+2}+1}{4}\right)q^n \equiv 16f_1^7 \tag{103}$$

and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}n+\frac{19\cdot 3^{4\alpha+2}+1}{4}\right)q^n \equiv 16f_1f_6^3. \tag{104}$$

Equation (103) is the $\beta = 0$ case of (95). The rest of the proofs of Identities (95) and (96) are similar to the proofs of Identities (53) and (54), so we omit the details. Equation (104) is the $\beta = 0$ case of (97). The rest of the proofs of Identities (97) and (98) are similar to the proofs of Identities (20) and (21), so we omit the details. Equation (84) becomes, modulo 32,

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+3}n+\frac{11\cdot 3^{4\alpha+2}+1}{4}\right)q^n \equiv 16f_4f_6^3,$$

which implies (99) and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+3}n+\frac{11\cdot 3^{4\alpha+2}+1}{4}\right)q^n \equiv 16f_2f_3^3,$$

which is the $\beta = 0$ case of (100). Suppose that Congruence (100) is true for $\beta \geq 0$. Using (16) in (100), we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+3} \cdot 5^{2\beta+1} n + \frac{7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+1} + 1}{4}\right) q^n \equiv 16 q^2 f_{10} f_{15}^3,$$

which implies (101) and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+3} \cdot 5^{2\beta+2} n + \frac{11 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+2} + 1}{4}\right) q^n \equiv 16 f_2 f_3^3,$$

which implies that Congruence (100) is true for $\beta + 1$. By induction, Congruence (100) holds for all $\alpha, \beta \geq 0$. □

Theorem 6. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+4} \cdot 5^{2\beta} n + \frac{13 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} + 1}{4}\right) q^n \equiv 8 f_1^{13} \pmod{16}, \tag{105}$$

$$\overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+4} \cdot 5^{2\beta+1}(5n+i) + \frac{17 \cdot 3^{4\alpha+3} \cdot 5^{2\beta+1} + 1}{4}\right) \equiv 0 \pmod{16}, \tag{106}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+4} \cdot 7^{2\gamma} n + \frac{5 \cdot 3^{4\alpha+3} \cdot 7^{2\gamma} + 1}{4}\right) q^n \equiv 8 f_1^{5} \pmod{16}, \tag{107}$$

$$\overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+4} \cdot 7^{2\gamma+1}(7n+j) + \frac{11 \cdot 3^{4\alpha+3} \cdot 7^{2\gamma+1} + 1}{4}\right) \equiv 0 \pmod{16}, \tag{108}$$

$$\overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+4} n + \frac{3^{4\alpha+3} + 1}{4}\right) \equiv \begin{cases} 4 \pmod{8} & \text{if } n \text{ is a pentagonal number,} \\ 0 \pmod{8} & \text{otherwise,} \end{cases} \tag{109}$$

where $i = 0, 1, 3,$ or 4, and $j = 0, 2, 3, 4, 5,$ or 6.

Proof. Equation (85) becomes

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+4} n + \frac{3^{4\alpha+3} + 1}{4}\right) q^n \equiv 12 f_1^2 \pmod{16}. \tag{110}$$

Substituting (6) in (110), we have

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+4} n + \frac{3^{4\alpha+3} + 1}{4}\right) q^n \equiv 12 \frac{f_1 f_4}{f_2^2} \pmod{16} \tag{111}$$

and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2 \cdot 3^{4\alpha+4} n + \frac{13 \cdot 3^{4\alpha+3} + 1}{4}\right) q^n \equiv 8 f_1^{13} \pmod{16},$$

which is the $\beta = 0$ case of (105). The rest of the proofs of Identities (105) and (106) are similar to the proofs of Identities (53) and (54), so we omit the details. Equation (86) becomes

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{4\alpha+4}n + \frac{5\cdot 3^{4\alpha+3}+1}{4}\right) q^n \equiv 8\frac{f_2 f_3^3}{f_1} \pmod{16}. \tag{112}$$

Employing (9) in (112), we obtain

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(2\cdot 3^{4\alpha+4}n + \frac{5\cdot 3^{4\alpha+3}+1}{4}\right) q^n \equiv 8 f_1^5 \pmod{16},$$

which is the $\gamma = 0$ case of (107). The rest of the proofs of Identities (107) and (108) are similar to the proofs of Identities (55) and (56), so we omit the details. From Equation (111), we arrive at (109). □

Theorem 7. *For all $n \geq 0$, we have*

$$\overline{ped}_{9,18}(27n+16) \equiv 0 \pmod{9}, \tag{113}$$
$$\overline{ped}_{9,18}(27n+25) \equiv 0 \pmod{9}, \tag{114}$$
$$\overline{ped}_{9,18}(243n+142) \equiv 0 \pmod{27}, \tag{115}$$
$$\overline{ped}_{9,18}(243n+223) \equiv 0 \pmod{27}, \tag{116}$$
$$\overline{ped}_{9,18}(729n+547) \equiv 0 \pmod{27}. \tag{117}$$

Proof. From the binomial theorem, it is easy to see that for positive integers ℓ and k,

$$f_\ell^{9k} \equiv f_{3\ell}^{3k} \pmod{9}. \tag{118}$$

Invoking (118) in (81), we see that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(9n+7)\, q^n \equiv 12\frac{f_2^6 f_3^3}{f_1^3} \pmod{27}. \tag{119}$$

Using (10) in (119), we get

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(27n+7)\, q^n \equiv 12\frac{f_2^3 f_3^6}{f_6^3} + 12q\frac{f_1^3 f_6^6}{f_3^3} \pmod{27}, \tag{120}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(27n+16)\, q^n \equiv 9 f_1 f_2^2 f_3^3 \pmod{27}, \tag{121}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}(27n+25)\, q^n \equiv 9 f_1^2 f_2 f_6^3 \pmod{27}. \tag{122}$$

From Equations (121) and (122), we obtain (113) and (114) respectively. Employing (13) in (120) and then collecting the coefficients of q^{3n+2} from both sides of the resultant equation, we find that

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\,(81n+61)\,q^n \equiv 9f_3^2f_6^2 \pmod{27},$$

which implies (115), (116), and

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\,(243n+61)\,q^n \equiv 9f_1^2f_2^2 \pmod{27}. \tag{123}$$

Utilizing (14) in (123) and then extracting the coefficients of q^{3n+2} from both sides of the resultant equation, we obtain (117). $\square$

Theorem 8 (Theorem 5, [8])**.** *For each $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{2\alpha+5}n+\frac{3^{2\alpha+5}+1}{4}\right)q^n \equiv 9\sum_{i=1}^{\infty} x_{2\alpha,i}q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-3} \pmod{27}, \tag{124}$$

$$\sum_{n=0}^{\infty} \overline{ped}_{9,18}\left(3^{2\alpha+6}n+\frac{3^{2\alpha+7}+1}{4}\right)q^n \equiv 9\sum_{i=1}^{\infty} x_{2\alpha+1,i}q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-1} \pmod{27}, \tag{125}$$

where the coefficient vectors $X_\alpha = (x_{\alpha,1}, x_{\alpha,2}, \dots)$ are given by

$$X_0 = (x_{0,1}, x_{0,2}, x_{0,3}, \dots) = (1, 0, 0, \dots)$$

and

$$X_{\alpha+1} = X_\alpha A \quad if \quad \alpha \quad is \quad even,$$
$$X_{\alpha+1} = X_\alpha B \quad if \quad \alpha \quad is \quad odd,$$

where $A = (a_{i,j})_{i,j\geq 1}$ and $B = (b_{i,j})_{i,j\geq 1}$.

Proof. Equation (123) can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{ped}_{9,18}\,(243n+61)\,q^n &\equiv 9\frac{f_3f_6}{f_1f_2}\\ &\equiv 9\sum_{n=0}^{\infty} a_3\,(n)\,q^n \pmod{27},\end{aligned} \tag{126}$$

where $a_3(n)$ denotes the number of 3-regular cubic partitions of n, whose generating function is

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{f_3f_6}{f_1f_2}.$$

Equation (126) is the $\alpha = 0$ case of (124). $\square$

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