A NOTE ON THREE CONSECUTIVE POWERFUL NUMBERS

Tsz Ho Chan
Department of Mathematics, Kennesaw State University, Marietta, Georgia
tchan4@kennesaw.edu

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Abstract

This note concerns the non-existence of three consecutive powerful numbers. We use Pell equations, elliptic curves, and second-order recurrences to show that there are no such triplets with the middle term a perfect cube and each of the other two having only a single prime factor raised to an odd power.

1. Introduction and Main Results

A positive integer n is *powerful* or *squareful* if $p^2 \mid n$ for all primes p such that $p \mid n$. It is well-known that any powerful number can be factored uniquely as $n = a^2 b^3$ for some positive integer a and squarefree number b. Here a number n is *squarefree* if $p^2 \nmid n$ for all primes p. The following is an initial list of powerful numbers:

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 72, \ldots.$$

Notice that 8 and 9 are consecutive powerful numbers. Indeed, there are infinitely many such pairs. For example, the solutions of the Pell equation

$$x^2 - 2y^2 = 1$$

give consecutive powerful pairs $2y^2, x^2$ as y is even by considering perfect squares modulo 4. The interested readers may consult [5] and [8] for more discussions on this topic. Next, one can ask if there are three consecutive powerful numbers.

Conjecture 1 (Erdős, Mollin, Walsh)**.** No three consecutive powerful numbers exist.

This appears to be very hard. Some relevant references are [4], [7] and [6]. Conditional on the *abc*-conjecture, one can show that there are only finitely many triples of consecutive powerful numbers. In this note, we consider the special case of three consecutive powerful numbers of the form $x^3 - 1$, x^3, $x^3 + 1$ and prove the following.

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Theorem 1. *There are no three consecutive powerful numbers of the form*

$$x^3 - 1 = p^3y^2, \;\; x^3, \;\; x^3 + 1 = q^3z^2 \tag{1}$$

with primes p, q and positive integers x, y, z.

Corollary 1. *The Diophantine equation $64x^6 - 1 = p^3q^3y^2$ has no solution with integers x, y and primes p, q.*

The author hopes that this note will stimulate further studies of powerful numbers and Conjecture 1.

Throughout this paper, all variables are integers. The letters p and q stand for prime numbers. The symbol $a \mid b$ means that a divides b, and $a \nmid b$ means that a does not divide b. The symbol $p^k \parallel a$ means that $p^k \mid a$ but $p^{k+1} \nmid a$.

2. Some Basic Observations

Lemma 1. *The difference between any two perfect squares cannot be* 2.

Proof. One simply observes that $(n+1)^2 - n^2 = 2n + 1 > 2$ when $n \geq 1$, and $(0+1)^2 - 0^2 = 1 \neq 2$. □

Lemma 2. *If $x \equiv 1 \pmod 3$, then $3 \parallel x^2 + x + 1$.*

Proof. Suppose $x \equiv 1 \pmod 3$. Say $x = 3x' + 1$ for some integer x'. Then

$$x^2 + x + 1 = (3x' + 1)^2 + (3x' + 1) + 1 = 9x'^2 + 9x' + 3 = 3(3x'^2 + 3x' + 1)$$

which is divisible by 3 but not 9. □

Lemma 3. *If $x \equiv 2 \pmod 3$, then $3 \parallel x^2 - x + 1$.*

Proof. This follows from Lemma 2 by the substitution $x \mapsto -x$. □

Lemma 4. *Suppose $a \neq 0$, c and e are some fixed integers. If $y^2 = ax^4 + cx^2 + e$ has an integer solution (x, y) with $x \neq 0$, so does the elliptic curve $Y^2 = X^3 + cX^2 + aeX$.*

Proof. One simply multiplies both sides of $y^2 = ax^4 + cx^2 + e$ by a^2x^2 and gets $(axy)^2 = (ax^2)^3 + c(ax^2)^2 + ae(ax^2)$. This yields a non-zero integer solution for the above elliptic curve with $X = ax^2 \neq 0$ and $Y = axy$. □

Lemma 5. *The Pell equation $x^2 - 3y^2 = 1$ has all positive integer solutions (x_k, y_k) generated by $x_k + y_k\sqrt{3} = (2 + 1 \cdot \sqrt{3})^k$ for $k \in \mathbb{N}$. Moreover, the solutions satisfy the recursions: $x_1 = 2$, $x_2 = 7$, $x_k = 4x_{k-1} - x_{k-2}$; $y_1 = 1$, $y_2 = 4$, $y_k = 4y_{k-1} - y_{k-2}$ for $k > 2$.*

Proof. This is a standard result in the theory of Pell equation that all integer solutions are generated by some fundamental (minimal) solution. See [1, Theorem 5.3] for example. The recursions easily follow from the observation

$$(2+\sqrt{3})^k - 4(2+\sqrt{3})^{k-1} + (2+\sqrt{3})^{k-2} = (2+\sqrt{3})^{k-2}[(2+\sqrt{3})^2 - 4(2+\sqrt{3}) + 1] = 0$$

as $2+\sqrt{3}$ is a root to the quadratic equation $x^2 - 4x + 1 = 0$. □

Lemma 6. *The generalized Pell equation $x^2 - 3y^2 = -2$ has all positive integer solutions (x_k, y_k) generated by $x_k + y_k\sqrt{3} = (1 + 1 \cdot \sqrt{3})(2 + 1 \cdot \sqrt{3})^k$ for $k \in \mathbb{N}$. Moreover, the solutions satisfy the recursions: $x_1 = 1$, $x_2 = 5$, $x_k = 4x_{k-1} - x_{k-2}$; $y_1 = 1$, $y_2 = 3$, $y_k = 4y_{k-1} - y_{k-2}$ for $k > 2$.*

Proof. One can easily see that $x = 1 = y$ is the smallest positive integer solution (i.e., $x_1 + y_1\sqrt{3} = 1 + 1 \cdot \sqrt{3}$). Then one can generate all the integer solutions by combining this with the solutions in Lemma 5. See [2, Theorem 3.3] for example. The recursions follow from a similar observation as in Lemma 5. □

3. Proof of Theorem 1

We are now ready to prove Theorem 1.

Proof of Theorem 1. First, note that any three consecutive powerful numbers must be of the form $4n-1$, $4n$, $4n+1$ as $2 \parallel 4n+2$. Recall that the middle number is a cube. So, $4n = x^3$ and x is even. Suppose that, contrary of Theorem 1, we have

$$(x-1)(x^2+x+1) = p^3y^2 \quad \text{and} \quad (x+1)(x^2-x+1) = q^3z^2. \tag{2}$$

Note that $x > 1$ as $p, y > 0$. By the fact that $\gcd(a, b) = \gcd(a, b-a)$, we have $\gcd(x-1, x^2+x+1) = \gcd(x-1, (x-1)(x+2)+3) = \gcd(x-1, 3) = 1$ or 3. Hence,

$$\gcd(x-1, x^2+x+1) = \begin{cases} 3, & \text{if } x \equiv 1 \pmod 3, \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

Suppose $x \not\equiv 1 \pmod 3$. Then $\gcd(x-1, x^2+x+1) = 1$ by Equation (3). It follows that if $p \mid x-1$, then $p^3 \mid x-1$ and x^2+x+1 is a perfect square by Equation (2). This is impossible as $x^2 < x^2+x+1 < (x+1)^2$. Hence, $p^3 \mid x^2+x+1$. In summary, we have

$$x \not\equiv 1 \pmod 3 \text{ implies } p^3 \mid x^2+x+1 \text{ and } x-1 \text{ is a perfect square.} \tag{4}$$

Applying the substitution $x \mapsto -x$ to the above argument, we also have

$$\gcd(x+1, x^2-x+1) = \begin{cases} 3, & \text{if } x \equiv 2 \pmod 3, \\ 1, & \text{otherwise,} \end{cases} \tag{5}$$

and

$$x \not\equiv 2 \pmod 3 \text{ implies } q^3 \mid x^2 - x + 1 \text{ and } x + 1 \text{ is a perfect square.} \qquad (6)$$

`Case 1:` $x \equiv 0 \pmod 3$. From Equations (3) and (5), we have

$$\gcd(x-1, x^2+x+1) = 1 = \gcd(x+1, x^2-x+1).$$

By Equations (4) and (6), both $x-1$ and $x+1$ are perfect squares which contradicts Lemma 1.
`Case 2:` $x \equiv 1 \pmod 3$. From Equations (3) and (5), we have

$$\gcd(x-1, x^2+x+1) = 3 \text{ and } \gcd(x+1, x^2-x+1) = 1.$$

Hence, $q^3 \mid x^2 - x + 1$ and $x + 1 = u^2$ for some integer $u > 0$ by Equation (6).

Suppose $p = 3$. We have $(x-1)(x^2+x+1) = 3^3y^2$ and $3 \parallel x^2+x+1$ by Lemma 2. Thus, $9 \mid x - 1$ and $x - 1 = 9x'^2 = (3x')^2$ for some integer x'. This contradicts Lemma 1.

Suppose $p \neq 3$. If $p \mid x^2+x+1$, then $p^3 \mid x^2+x+1$ by Equation (3). This forces $x - 1 = 3v^2$ or $x - 1 = (3v)^2$ for some integer v. The latter contradicts Lemma 1. The former yields $u^2 - 3v^2 = 2$ which is also impossible as $u^2 \equiv 0$ or $1 \pmod 3$.

Therefore, we must have $p \mid x - 1$ and, hence, $p^3 \mid x - 1$ by Equation (3). From Equation (2) and Lemma 3, we have $x^2 + x + 1 = 3v^2$ and $x - 1 = 3p^3w^2$ for some integers $v, w > 0$ with $\gcd(v, 3) = 1 = \gcd(v, w)$. Substituting $x = u^2 - 1$ into $x^2 + x + 1$, we obtain

$$3v^2 = u^4 - u^2 + 1 \quad \text{or} \quad y^2 = 3u^4 - 3u^2 + 3$$

where $y = 3v$. By Lemma 4, the elliptic curve $Y^2 = X^3 - 3X^2 + 9X$ has a non-zero integer solution. This contradicts $(X, Y) = (0, 0)$ being the only integer solution by the SageMath command

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E = EllipticCurve([0,-3,0,9,0]); E.integral_points(),
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for example.
`Case 3:` $x \equiv 2 \pmod 3$. From Equations (3) and (5), we have

$$\gcd(x-1, x^2+x+1) = 1, \text{ and } \gcd(x+1, x^2-x+1) = 3.$$

Hence, $p^3 \mid x^2 + x + 1$ and $x - 1 = u^2$ for some integer $u > 0$ with $3 \nmid u$ by Equation (4).

Suppose $q = 3$. We have $(x+1)(x^2-x+1) = 3^3y^2$ and $3 \parallel x^2-x+1$ by Lemma 3. Thus, $9 \mid x + 1$ and $x + 1 = 9x'^2 = (3x')^2$ for some integer x'. This contradicts Lemma 1.

Suppose $q \neq 3$. We have either $q^3 \mid x+1$ or $q^3 \mid x^2-x+1$ by Equation (5). If the former is true, then $x+1 = 3q^3v^2$ and $x^2-x+1 = 3w^2$ for some integers $v, w > 0$ with $\gcd(w,3) = 1 = \gcd(v,w)$. Substituting $x = u^2+1$ into x^2-x+1, we obtain

$$3w^2 = u^4+u^2+1 \quad \text{or} \quad y^2 = 3u^4+3u^2+3$$

with $y = 3w$. By Lemma 4, the elliptic curve $Y^2 = X^3+3X^2+9X$ has a non-zero integer solution. The SageMath command

E = EllipticCurve([0,3,0,9,0]); E.integral_points()

yields $(X,Y) = (0,0)$ and $(3,\pm 9)$ as the only such solutions. It follows from the proof in Lemma 4 that $X = 3u^2 = 3$ and $u = 1$. Hence, $x = u^2+1 = 2$ but $x^3-1 = 7$ is not powerful.

Therefore, we must have $q^3 \mid x^2-x+1$. By Equation (5), we have $x+1 = 3v^2$ and $x^2-x+1 = 3q^3w^2$ for some integers v and w. Combining these with $x-1 = u^2$, we get the generalized Pell equation $u^2-3v^2 = -2$. By Lemma 6, $u = u_k$ satisfies

$$u_1 = 1, \quad u_2 = 5, \quad \text{and} \quad u_k = 4u_{k-1} - u_{k-2} \quad \text{for} \quad k > 2. \tag{7}$$

By substituting $x = u^2+1$ into x^2-x+1, we obtain

$$3q^3w^2 = (u^2+1)^2 - (u^2+1) + 1 = u^4+u^2+1 = (u^2+u+1)(u^2-u+1).$$

Since x is even and $x-1 = u^2$, it follows that u is odd and

$$\gcd(u^2+u+1, u^2-u+1) = \gcd(u^2+u+1, 2u) = \gcd(u^2+u+1, u) = 1. \tag{8}$$

`Subcase 1:` $u \equiv 1 \pmod 3$. Then $3 \mid u^2+u+1$. Suppose $q \mid u^2+u+1$. We have $u^2+u+1 = 3q^3w_1^2$ and $u^2-u+1 = w_2^2$ by Equation (8). This is impossible as $(u-1)^2 < u^2-u+1 < u^2$ unless $u = 1$. However, $u = 1$ yields $x = 2$ and $x^3-1 = 7$ which is not powerful. Therefore, we must have $q \mid u^2-u+1$. So, $u^2+u+1 = 3w_1^2$ and $u^2-u+1 = q^3w_2^2$ by Equation (8). After some algebra, one arrives at

$$(2w_1)^2 - 3\Big(\frac{2u+1}{3}\Big)^2 = 1.$$

Then Lemma 5 gives $2w_1 + \frac{2u+1}{3}\sqrt{3} = g_l + h_l\sqrt{3} = (2+\sqrt{3})^l$ where

$$h_1 = 1, \quad h_2 = 4, \quad \text{and} \quad h_l = 4h_{l-1} - h_{l-2} \quad \text{for} \quad l > 2. \tag{9}$$

Thus, $u_k = u = \frac{3h_l - 1}{2}$ for some indices $k, l \geq 1$.

From Equations (7) and (9), one can show by induction that u_k and h_l are positive increasing sequences (for example, $u_2 > u_1 > 0$ and the induction hypothesis $u_{k-1} > u_{k-2} > 0$ implies $u_k = 4u_{k-1} - u_{k-2} > u_{k-1} + 3(u_{k-1} - u_{k-2}) > u_{k-1} > 0$). Hence,

$$u_k = 4u_{k-1} - u_{k-2} > 4u_{k-1} - u_{k-1} = 3u_{k-1}, \text{ and, similarly, } h_l > 3h_{l-1}. \tag{10}$$

Moreover, one can form the new sequence $v_k := u_k - h_k$ which satisfies

$$v_1 = 0, \quad v_2 = 1, \quad \text{and} \quad v_k = 4v_{k-1} - v_{k-2} \quad \text{for} \quad k > 2.$$

As $v_3 = 4$, one can see that $v_k = h_{k-1} > 0$ for $k \geq 2$. Hence, $u_k = h_k + h_{k-1} > h_k$ for all $k \geq 2$. By Equation (10) and the inequality $\frac{4n}{3} < \frac{3n-1}{2}$ when $n \geq 4$, we have

$$u_k = h_k + h_{k-1} < \frac{4h_k}{3} < \frac{3h_k - 1}{2} < \frac{3u_k - 1}{2} < 3u_k < u_{k+1}$$

for all $k \geq 2$. Therefore, $u_k = \frac{3h_l - 1}{2}$ is possible only when $k = l = 1$. This gives $x = u_1^2 + 1 = 2$ but $x^3 - 1 = 7$ is not powerful.
`Subcase 2:` $u \equiv -1 \pmod 3$. This is very similar to subcase 1 with $3 \mid u^2 - u + 1$ and $(2w_1)^2 - 3(\frac{2u-1}{3})^2 = 1$ instead. It also yields a contradiction. □

4. Proof of Corollary 1

Using Theorem 1, we can now prove Corollary 1.

Proof of Corollary 1. Suppose the equation $64x^6 - 1 = ((2x)^3 - 1)((2x)^3 + 1) = p^3q^3y^2$ has a solution with some integers x, y, and primes p, q. By the fact that $\gcd(a, b) = \gcd(a, b - a)$, we have

$$\gcd((2x)^3 - 1, (2x)^3 + 1) = \gcd((2x)^3 - 1, 2) = 1. \tag{11}$$

Suppose $pq \mid (2x)^3 - 1$. By Equation (11), we must have $(2x)^3 - 1 = p^3q^3y_1^2$ and $(2x)^3 + 1 = y_2^2$ for some integers y_1 and y_2. However, the elliptic curve $Y^2 = X^3 + 1$ has $(X, Y) = (-1, 0)$, $(0, \pm 1)$ and $(2, \pm 3)$ as its only integer solutions by SageMath for example. Thus, $x = 0$ or 1. However, neither $0^6 - 1 = -1$ nor $2^6 - 1 = 63$ are of the form $p^3q^3y_1^2$.

Suppose $pq \mid (2x)^3 + 1$. By Equation (11), we must have $(2x)^3 + 1 = p^3q^3y_1^2$ and $(2x)^3 - 1 = y_2^2$ for some integers y_1 and y_2. This contradicts $(X, Y) = (1, 0)$ being the only solution to the elliptic curve $Y^2 = X^3 - 1$ (see [3, Theorem 3.2] for example).

Therefore, p divides exactly one of $(2x)^3 - 1$ or $(2x)^3 + 1$, and q divides the other one. Without loss of generality, say $(2x)^3 - 1 = p^3y_1^2$ and $(2x)^3 + 1 = q^3y_2^2$. This contradicts Theorem 1. Consequently, $64x^6 - 1 = p^3q^3y^2$ cannot have any integer solution. □

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