PERRIN SQUARES WHICH ARE AGAIN PERRIN

Karen J. Sacanamboy
Departamento de Matemáticas, Universidad del Cauca, Popayán, Colombia
sacanamboy@unicauca.edu.co

Eric F. Bravo
Departamento de Matemáticas, Universidad del Cauca, Popayán, Colombia
fbravo@unicauca.edu.co

Jhon J. Bravo
Departamento de Matemáticas, Universidad del Cauca, Popayán, Colombia
jbravo@unicauca.edu.co

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Abstract

The Perrin sequence $(R_n)_{n\in\mathbb{Z}}$ is defined by $R_0 = 3$, $R_1 = 0$, $R_2 = 2$, and by $R_{n+3} = R_{n+1} + R_n$ for $n \in \mathbb{Z}$. In this note, we solve the Diophantine equation $R_n = \pm R_m^2$ in integers m, n. We prove that $R_n \cap \pm R_m^2 = \{0, \pm 1, 4, 25\}$.

1. Introduction

The *Perrin numbers* $(R_n)_{n\geq 0}$ are defined by the Fibonacci-like recurrence relation

$$R_{n+3} = R_{n+1} + R_n \quad \text{for} \quad n \geq 0,$$

with initial conditions $R_0 = 3$, $R_1 = 0$, and $R_2 = 2$. For $n \geq 0$, the first few Perrin numbers are

$$3,\ 0,\ 2,\ 3,\ 2,\ 5,\ 5,\ 7,\ 10,\ 12,\ 17,\ 22,\ 29,\ 39,\ 51,\ 68,\ 90, \ldots.$$

The *Padovan numbers* $(P_n)_{n\geq 0}$ satisfy the same recurrence equation as Perrin numbers, but with different initial values. Therefore, both sequences share the same characteristic polynomial given by $P(X) = X^3 - X - 1$. For $n \geq 0$, the first few Padovan numbers are

$$1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16,\ 21,\ 28,\ 37,\ 49,\ 65,\ 86, \ldots.$$

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Both sequences can be extended to negative indices by $R_{-n} = R_{-n+3} - R_{-n+1}$. Since the constant term of P is -1, these sequences have integer members as well. For $n \leq -1$, the first few Perrin and Padovan numbers are

$$-1, 1, 2, -3, 4, -2, -1, 5, -7, 6, -1, -6, 12, -13, 7, 5, -18, 25, \ldots$$

and

$$0, 1, 0, 0, 1, -1, 1, 0, -1, 2, -2, 1, 1, -3, 4, -3, 0, 4, -7, 7, -3, \ldots,$$

respectively. In Chapter 8 of his book Math Hysteria [8], and originally in his Scientific American column [9], Ian Stewart asks if there are any Padovan numbers beyond $P_{15} = 49$ that are squares. He also notes that the squares that do occur in this short list are squares of Padovan numbers themselves, and he asks if that is always the case. Surprisingly, Stewart's two problems are equivalent. This follows from the fact that there are no Padovan numbers beyond $P_{15} = 49$ that are squares of Padovan numbers, which was proved by de Weger [11] by solving the equation

$$P_n = P_m^2$$

in positive integers n, m. Extending the Padovan sequence to negative indices, the following problems were posed by de Weger [11] in 2004: solve

$$P_{-n} = \pm P_m^2, \quad P_n = P_{-m}^2, \quad \text{and} \quad P_{-n} = \pm P_{-m}^2$$

in positive integers n, m. Recently, these three equations were solved by Bravo and Luca [1]. We solve the above four equations with Perrin numbers instead of Padovan numbers with the technique developed by Bravo et. al [3, 4]. We also deal with some special cases.

2. Results

Theorem 1. *The only solution of equation*

$$R_n = R_m^2 \tag{1}$$

in positive integers m, n is $R_1 = 0 = R_1^2$.

Theorem 2. *The only solutions of equation*

$$R_{-n} = \pm R_m^2 \tag{2}$$

in positive integers m, n are listed below:

$$R_{-5} = 4 = R_2^2 = R_4^2;$$
$$R_{-18} = 25 = R_5^2 = R_6^2.$$

Theorem 3. *There are no solutions in positive integers m, n of equation*

$$R_n = R_{-m}^2. \tag{3}$$

Theorem 4. *The only solutions of equation*

$$R_{-n} = \pm R_{-m}^2 \tag{4}$$

in positive integers m, n are given by:

$R_{-29} = R_{-11} = R_{-7} = R_{-1} = -1 = -R_{-1}^2 = -R_{-2}^2 = -R_{-7}^2 = -R_{-11}^2 = -R_{-29}^2$;
$R_{-2} = 1 = R_{-1}^2 = R_{-2}^2 = R_{-7}^2 = R_{-11}^2 = R_{-29}^2$;
$R_{-5} = 4 = R_{-3}^2 = R_{-6}^2 = R_{-20}^2$;
$R_{-18} = 25 = R_{-8}^2 = R_{-16}^2$.

3. The Perrin Sequence

We begin by recalling some properties of this ternary recurrence sequence. Denoting the zeros of P by α, β, γ, with α being the only real zero, the general term of $(R_k)_{k\in\mathbb{Z}}$ is of the form

$$R_k = \alpha^k + \beta^k + \gamma^k. \tag{5}$$

Here

$$\beta = \alpha^{-1/2}e^{i\theta} \quad \text{and} \quad \gamma = \alpha^{-1/2}e^{-i\theta} \quad \text{with} \quad \theta \in (0, \pi). \tag{6}$$

Numerically,

$$\alpha \in (1.32, 1.33) \quad \text{and} \quad |\beta| = |\gamma| = \alpha^{-1/2} \in (0.86, 0.87).$$

It follows then that the contribution of the zeros β and γ to Equation (5) is very small, namely

$$\left|R_k - \alpha^k\right| \leq 2\alpha^{-k/2} \quad \text{holds for all} \quad k \geq 0. \tag{7}$$

In addition, it can be shown by induction that

$$\alpha^{k-2} \leq R_k \leq \alpha^{k+1} \quad \text{for all} \quad k \geq 2. \tag{8}$$

Furthermore, using Equation (5) and the fact that $\alpha\beta\gamma = 1$, one can easily see that

$$R_k^2 = R_{2k} + 2R_{-k} \quad \text{for all} \quad k \in \mathbb{Z}. \tag{9}$$

Bravo, Bravo, and Luca [2, Lemma 11, Corollary 3] found the following facts about Perrin sequence.

Lemma 1. *For an integer $k \geq 6$, we have $\alpha^{\frac{k}{2}-3\times10^{15}\log k} < |R_{-k}| < 2.01\alpha^{k/2}$.*

Theorem 5. *The Perrin sequence has exactly* 1 *zero.*

We end this section of preliminaries on the Perrin sequence by mentioning that we can identify the automorphisms of the Galois group of the splitting field $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$ of P over $\mathbb{Q}$ wih the permutations of the zeros of P, since

$$\mathrm{Gal}\big(\mathbb{K}/\mathbb{Q}\big) \simeq \{(1)\,,(\alpha\beta)\,,(\alpha\gamma)\,,(\beta\gamma)\,,(\alpha\beta\gamma)\,,(\alpha\gamma\beta)\} \simeq S_3.$$

For example, the permutation $(\alpha\beta)$ corresponds to the automorphism $\sigma_{\alpha\beta} : \alpha \to \beta,\ \beta \to \alpha$.

4. Linear Forms in Logarithms

For an algebraic number η of degree d over $\mathbb{Q}$ with minimal primitive polynomial

$$a_d X^d + a_{d-1}X^{d-1} + \cdots + a_0 = a_d(X - \eta^{(1)}) \cdots (X - \eta^{(d)}) \in \mathbb{Z}\,[X]\,,$$

we put

$$h(\eta) = \frac{1}{d}\left(\log(|a_d|) + \sum_{j=1}^{d} \max\left\{\log\left(\left|\eta^{(j)}\right|\right), 0\right\}\right)$$

for the *logarithmic height of* $\eta := \eta^{(1)}$. In particular, if $\eta = p/q \in \mathbb{Q}$ is in lowest terms with $q \geq 1$, then $h(\eta) = \log\max\big\{|p|, q\big\}$ and $h(\eta^{p/q}) = |p/q|h(\eta)$. These and the following basic properties of this height will be used later without reference:

$$h(\eta) = h\big(\eta^{(j)}\big), \quad h(\eta_1 + \eta_2) \leq h(\eta_1) + h(\eta_2) + \log 2, \quad h(\eta_1\eta_2^{\pm 1}) \leq h(\eta_1) + h(\eta_2).$$

A proof of some of these properties can be found in Waldschmidt [10, Property 3.3].

We will only need the special case of two logarithms of the theorem of E. M. Matveev [7]. So we quote here his result in this case. Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, let η_1 and η_2 be non-zero elements of $\mathbb{K}$, and let b_1 and b_2 be integers. Set

$$\Lambda = \eta_1^{b_1}\eta_2^{b_2} - 1 \quad \text{and} \quad B \geq \max\left\{|b_1|, |b_2|\right\}.$$

Let A_1 and A_2 be real numbers such that

$$A_j \geq \max\{Dh(\eta_j),\ |\log\eta^{(j)}|,\ 0.16\} \quad \text{for} \quad j = 1, 2.$$

With this notation, the main result of Matveev [7] implies the following estimate.

Theorem 6. *If* $\Lambda \neq 0$ *and* $\mathbb{K} \subseteq \mathbb{C}$*, then we have*

$$|\Lambda| > \exp\big(-9.20483 \times 10^{11} D^2 A_1 A_2 (1 + \log D)(1 + \log(2B))\big).$$

5. Reduction Tools

We now remind to the reader of the Baker-Davenport reduction method from Dujella and Pethő [5, Lemma 5(a)], which turns out to be useful in order to reduce the bounds arising from applying Theorem 6.

Lemma 2. *Let A, B, κ, μ be real numbers with $A > 0$ and $B > 1$. Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\epsilon := \|\mu q\| - M\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution of the inequality*

$$0 < |s\kappa - r + \mu| < AB^{-w}$$

in positive integers r, s, w, with

$$s \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

We now recall some basic results from the theory of continued fractions in order to prove Lemma 3 below. For each continued fraction $[a_0, a_1, \ldots, a_n]$ we define $p_0, p_1, \ldots, p_n$ and $q_0, q_1, \ldots, q_n$ via

$$p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \quad p_k = a_k p_{k-1} + p_{k-2} \quad (2 \leq k \leq n),$$

$$q_0 = 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2} \quad (2 \leq k \leq n).$$

With the previous notation we have:

Theorem 7 ([6], Theorem 150)**.** *The functions p_n and q_n satisfy*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

Theorem 8 ([6], Theorem 155)**.** *$q_n \geq q_{n-1}$ for $n \geq 1$, with inequality when $n > 1$.*

Let $a_n' = [a_n, a_{n+1}, \ldots]$ be the n*th complete quotient* of the continued fraction $x = [a_0, a_1, \ldots]$. Then it can be proved that (see [6, pág. 179])

$$x = \frac{a_{n+1}' p_n + p_{n-1}}{a_{n+1}' q_n + q_{n-1}}. \tag{10}$$

Theorem 9 ([6], Theorem 168)**.** *If $[a_0, a_1, a_2, \ldots] = x$, then*

$$a_0 = [x], \quad a_n = [a_n'] \quad (n \geq 0).$$

The theorem that follows implies that p_n/q_n is the fraction, among all fractions whose denominator does not exceed q_n, that provides the best approximation to x.

Theorem 10 ([6], Theorem 182)**.** *If $n > 1$, $0 < q \leq q_n$, and $p/q \neq p_n/q_n$, then*

$$|p_n - q_n x| < |p - qx|.$$

Lemma 2 cannot be applied when μ is a linear combination of 1 and κ since then $\epsilon = -M \|\kappa q\| < 0$ for any choice of q and M. In this case, we use the following result.

Lemma 3. *For $n \geq 0$, let p_n/q_n be the convergents of the continued fraction $x = [a_0, a_1, \ldots]$. Let M be a positive integer and put*

$$a_M = \max\{a_j : j \in [0, N+1]\},$$

where N is a non-negative integer such that $M \in [q_N, q_{N+1})$. If $p, q \in \mathbb{Z}$ with $q > 0$, then

$$\left| x - \frac{p}{q} \right| > \frac{1}{(a_M + 2)q^2} \quad \textit{for all} \quad q < M.$$

Proof. By Equation (10) and Theorem 7 we have

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{(a'_{n+1} q_n + q_{n-1}) q_n}.$$

From this, and Theorems 8 and 9, we obtain

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{(a_{n+1} + 2) q_n^2}.$$

Since $q < M$, there exists $s \leq N$ such that $q_s \leq q < q_{s+1}$. Hence, by Theorem 10, we get

$$|xq - p| > |xq_s - p_s| > \frac{1}{(a_{s+1} + 2) q_s} \geq \frac{1}{(a_M + 2) q}.$$

Dividing the resulting inequality by q we arrive at the desired result. □

6. The Proof Of Theorem 1

Proof. Here we solve Equation (1) in a completely elementary way. The fact that $|R_n - \alpha^n|$ tends to 0 if n grows is the heart of the proof in the case $n - 2m \neq 0$. In the other case, we use the zero multiplicity of the Perrin sequence to show that Equation (1) has no solution. Let us see.

`Case 1`: $n-2m \neq 0$. Suppose that Equation (1) holds. Then using Equation (7) we obtain

$$\begin{aligned}\left|\alpha^n - \alpha^{2m}\right| &= \left|(\alpha^n - R_n) + (R_m^2 - \alpha^{2m})\right| \\ &\leq |\alpha^n - R_n| + |R_m - \alpha^m||R_m + \alpha^m| \\ &\leq |\alpha^n - R_n| + |R_m - \alpha^m|(2\alpha^m + |R_m - \alpha^m|) \\ &\leq 2\alpha^{-n/2} + 4\alpha^{m/2} + 4\alpha^{-m}.\end{aligned}$$

Dividing the resulting inequality above by α^{2m} and using the fact that $n \geq m$, we get that

$$\left|\alpha^{n-2m} - 1\right| \leq 2\alpha^{-5m/2} + 4\alpha^{-3m/2} + 4\alpha^{-3m}. \tag{11}$$

Since $n-2m \neq 0$, the minimum of the left-hand side of Equation (11) is reached at $n-2m=-1$, namely

$$\left|\alpha^{n-2m} - 1\right| \geq \left|\alpha^{-1} - 1\right| = 0.24512\ldots,$$

while when $m \geq 7$, the right-hand side of Equation (11) is smaller than that, namely

$$2\alpha^{-5m/2} + 4\alpha^{-3m/2} + 4\alpha^{-3m} \leq 0.23430\ldots.$$

Thus, $m \leq 6$. Since $(R_n)_{n\geq 0}$ is strictly increasing for $n \geq 6$, by quick inspection we find that 0 is the only coincidence between R_n and R_m^2 for $n \geq 0$ and $0 \leq m \leq 6$.

`Case 2`: $n-2m=0$. In this case, using Equation (9) with $k=m$, Equation (1) becomes

$$R_{-m} = 0,$$

which has no solution by Theorem 5. This completes the proof of Theorem 1. □

7. The Proof Of Theorem 2

Proof. Suppose (m,n) is a solution of Equation (2) with $m \geq 2$ and $n \geq 6$. First, combining Equation (8) and Lemma 1 in Equation (2) we get

$$\alpha^{2m-4} \leq R_m^2 = |\pm R_m^2| = |R_{-n}| < 2.01\alpha^{n/2}.$$

Taking logarithms in the resulting inequality, we obtain

$$n - 4m \geq -12. \tag{12}$$

Putting $\varepsilon := \pm 1$ and using Equation (5) and Equation (6) with $z := e^{-i\theta}$ in Equation (2), we obtain

$$\alpha^{n/2}z^n - \varepsilon\alpha^{2m} + \alpha^{n/2}z^{-n} = 2\varepsilon\alpha^m(\beta^m + \gamma^m) + \varepsilon(\beta^m + \gamma^m)^2 - \alpha^{-n}.$$

Dividing both sides of the above equality by $\alpha^{n/2}$, taking absolute values, and using Equation (12), we get

$$\begin{aligned} \left|z^n - \varepsilon\alpha^{(4m-n)/2} + z^{-n}\right| &\le \frac{2\alpha^m(2|\beta|^m) + (2|\beta|^m)^2 + \alpha^{-n}}{\alpha^{n/2}} \\ &\le \frac{4\alpha^{3/2} + 4\alpha^{-(8m+n)/8} + \alpha^{-9n/8}}{\alpha^{3n/8}} \\ &< \frac{9}{\alpha^{3n/8}}. \end{aligned} \tag{13}$$

But $|z| = 1$, so

$$\left|z^n - \varepsilon\alpha^{(4m-n)/2} + z^{-n}\right| = \left|z^{2n} - \varepsilon\alpha^{(4m-n)/2}z^n + 1\right| = \left|(z^n - \lambda_1)(z^n - \lambda_2)\right|, \tag{14}$$

where λ_1, λ_2 are the zeros of the quadratic polynomial

$$X^2 - \varepsilon\alpha^{(4m-n)/2}X + 1. \tag{15}$$

Consequently, from Equation (13) and Equation (14) we obtain

$$\left|(z^n - \lambda_1)(z^n - \lambda_2)\right| < \frac{9}{\alpha^{3n/8}}.$$

We may assume without loss of generality that $|z^n - \lambda_1| \le |z^n - \lambda_2|$. Moreover, $|z| = 1$, hence

$$|\lambda_1 z^{-n} - 1| < \frac{3}{\alpha^{3n/16}}. \tag{16}$$

We now find a lower bound for $|\lambda_1 z^{-n} - 1|$ by using Theorem 6. To do this, we put

$$\eta_1 := \lambda_1 = \frac{\varepsilon\alpha^{(4m-n)/2} + \sqrt{\alpha^{4m-n} - 4}}{2}, \quad \eta_2 := z, \quad b_1 := 1, \quad \text{and} \quad b_2 := -n.$$

Suppose $\Lambda_1 := \eta_1^{b_1}\eta_2^{b_2} - 1 = 0$. Then from Equation (14) we get

$$\beta^{-n} = \varepsilon\alpha^{2m} - \gamma^{-n}.$$

Conjugating the above relation by the automorphism $\sigma_{\alpha\beta\gamma}$, and then taking absolute values on both sides of the resulting equality, we get

$$\alpha^{n/2} \le \alpha^{-m} + \alpha^{-n},$$

which is not possible for any $n \ge 2$ and $m \ge 2$. In consequence $\Lambda_1 \ne 0$. Let's take $\mathbb{K} := \mathbb{Q}\left(\sqrt{\alpha}, \beta, \sqrt{\alpha^{4m-n} - 4}\right)$ and let $\mathbb{L} := \mathbb{Q}\left(\sqrt{\alpha}, \beta\right)$. Then we have $\mathbb{K} = \mathbb{L}\left(\sqrt{\alpha^{4m-n} - 4}\right)$ and therefore $D = \left[\mathbb{K} : \mathbb{Q}\right] = \left[\mathbb{K} : \mathbb{L}\right]\left[\mathbb{L} : \mathbb{Q}\right] \le 2(12) = 24$. We also take $B := n$ because $n \ge 6$. Moreover, since $z = (\gamma/\beta)^{1/2}$ by Equation (6), we get $h(\eta_2) = h(\gamma/\beta)/2 \le h(\beta) = (\log\alpha)/3$. Then, we take $A_2 := 8\log\alpha$. It remains to choose A_1. It is a straightforward exercise to check that if λ_1, λ_2 are the zeros of the quadratic polynomial $X^2 + bX + c \in \mathbb{C}[x]$, then

$$h(\lambda_i) \leq h(b) + h(c) + \log 2, \quad i = 1, 2. \tag{17}$$

By taking $b = -\varepsilon\alpha^{(4m-n)/2}$ and $c = 1$, by Equation (17) we have

$$h(\eta_1) \leq \tfrac{1}{2}|n - 4m|h(\alpha) + \log 2. \tag{18}$$

Combining Equation (2) with Lemma 1 and Equation (8), one has

$$\alpha^{\frac{n}{2} - 3\times 10^{15}\log n} < |R_{-n}| = \big| \pm R_m^2 \big| = R_m^2 \leq \alpha^{2m+2}.$$

Taking logarithms in the resulting inequality above we get

$$n - 4m < 6.01 \times 10^{15} \log n.$$

Hence, from Equation (18) and the above inequality we can take $A_1 := 6.768 \times 10^{15} \log n$. Now Theorem 6 implies that

$$|\lambda_1 z^{-n} - 1| > \exp\big(- 6.74539 \times 10^{31} \log^2 n\big),$$

where we used the fact that $1 + \log(2n) < 2\log n$ for all $n \geq 6$. Combining Equation (16) and the inequality immediately above, and then taking logarithms in the resulting inequality, we obtain

$$n < 9.27 \times 10^{36}. \tag{19}$$

Next we reduce the upper bound of $n - 4m$. Indeed, using Equation (5) we get

$$\begin{aligned} |R_{-k}| &= |\beta|^{-k}\left|1 + \left(\frac{\gamma}{\beta}\right)^{-k}\right|\left|1 + \frac{\alpha^{-k}}{\beta^{-k} + \gamma^{-k}}\right| \\ &> \alpha^{k/2}\left| - \left(\frac{\beta}{\gamma}\right)^{k} - 1\right|\left|1 - \left|\frac{\alpha^{-k}}{\beta^{-k} + \gamma^{-k}}\right|\right| \\ &> 2\alpha^{k/2}\left|\left(\frac{\beta}{\gamma}\right)^{k} + 1\right|. \end{aligned} \tag{20}$$

In the above we have also used that $\big|\alpha^{-k}/\big(\beta^{-k} + \gamma^{-k}\big)\big| < (\alpha - 1)^{-1}$ for all $k \geq 1$, which follows by Theorem 5 since

$$\big|\beta^{-k} + \gamma^{-k}\big| = \big|R_{-k} - \alpha^{-k}\big| \geq |R_{-k}| - \alpha^{-k} \geq 1 - \alpha^{-k} > (\alpha - 1)\alpha^{-k}.$$

Combining Equation (2) with Equation (20) with $k = n$, and using the fact that $R_m^2 \leq \alpha^{2m+2}$ for all $m \geq 4$ by Equation (8), we obtain

$$\left|\left(\frac{\beta}{\gamma}\right)^{n} + 1\right| < \frac{0.88}{\alpha^{(n-4m)/2}}. \tag{21}$$

But $(\beta/\gamma)^n = e^{2in\theta}$ by Equation (6) and so Equation (21) becomes

$$\left|e^{i(2n\theta+\pi)} - 1\right| < \frac{0.88}{\alpha^{(n-4m)/2}}. \tag{22}$$

On the other hand,

$$\left|e^{i(2n\theta+\pi)} - 1\right| = \left|\cos(2n\theta+\pi) + i\sin(2n\theta+\pi) - 1\right| \geq \left|\sin(2n\theta+\pi)\right|. \tag{23}$$

If we put $r := \lfloor (2n\theta+\pi)/\pi \rceil$, then $2n\theta + \pi - r\pi \in [-\pi/2, \pi/2]$. Therefore

$$\left|\sin(2n\theta+\pi)\right| = \left|\sin(2n\theta+\pi-r\pi)\right| \geq 2\left|\frac{2n\theta}{\pi} + 1 - r\right|, \tag{24}$$

where we used that

$$|\sin y| \geq (2/\pi)|y| \quad \text{for all} \quad y \in [-\pi/2, \pi/2]\,. \tag{25}$$

Thus, we can conclude from Equation (22), Equation (23), and Equation (24) that

$$\left|\frac{2\theta}{\pi} - \frac{r-1}{n}\right| < \frac{0.44}{n\alpha^{(n-4m)/2}}. \tag{26}$$

Next we apply Lemma 3. To do so, we put $x := 2\theta/\pi$. Note that θ/π is an irrational number because otherwise $\beta^q = \gamma^q$ for some $q \in \mathbb{Z}^+$, which is not possible since conjugating the above relation by the automorphism $\sigma_{\alpha\gamma}$, and then taking absolute values on both sides of the resulting equality, we obtain $1 > |\beta|^q = |\alpha|^q > 1$. Therefore, x is an irrational number since it is the multiplication of a non-zero rational number and an irrational number. Next we calculate the continued fraction expansion $[a_0, a_1, \ldots] = [1, 1, 1, 4, 3, 6, 12, 1, 3, 3, 1, 7102, \ldots]$ of x, and its convergents

$$\left\{\frac{p_j}{q_j} : j = 0, 1, \ldots\right\} = \left\{1,\ 2,\ \frac{3}{2},\ \frac{14}{9},\ \frac{45}{29},\ \frac{284}{183},\ \frac{3453}{2225}, \ldots\right\}.$$

We also put $M := 9.27 \times 10^{36}$, which is an upper bound of n according to Equation (19). We can check that $M \in (q_{68}, q_{69})$ and so $a_M = \max\{a_j : j \in [0, 69]\} = a_{11} = 7102$. Then, by Lemma 3 we have that

$$\left|\frac{2\theta}{\pi} - \frac{r-1}{n}\right| > \frac{1}{7104n^2}. \tag{27}$$

Combining Equation (26) and Equation (27) and taking into account that $n < 9.27 \times 10^{36}$ by Equation (19), we obtain $\alpha^{(n-4m)/2} < 2.9 \times 10^{40}$ and thus $n - 4m \leq 662$.

We can improve this upper bound for $n - 4m$ and hence that of n by repeating the arguments before Equation (19). For this we again apply Theorem 6 to the left-hand side of Equation (16) with the same parameters as last time except that now A_1 can be 762. This time we get $n < 7.25 \times 10^{21}$. Now we apply again

Lemma 3 to the left-hand side of Equation (26) but this time with $M = 7.25 \times 10^{21}$. Now $q_{41} < M < q_{42}$ so $a_M = \max\{a_j : j \in [0, 42]\} = a_{11} = 7102$ and we arrive again at Equation (27). Combining Equation (26) and Equation (27) with this new bound for n, we get that $n - 4m \leq 415$. By repeating the whole process again, we get $n < 4.56 \times 10^{21}$ and $n - 4m \leq 411$. Thus, from Equation (12) we get $n - 4m \in [-12, 411]$.

Now we reduce the upper bound of n. If $n - 4m \in [-12, -5]$, then the zeros of Equation (15) satisfy $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or $|\lambda_1| < 1$ and $|\lambda_2| > 1$. In any case we obtain by Equation (16) that

$$||\lambda_1| - 1| \leq |\lambda_1 z^{-n} - 1| < 3\alpha^{-3n/16}$$

and then

$$n < \frac{16 \log\left(3/(||\lambda_1| - 1|)\right)}{3 \log \alpha}.$$

The maximum value of the right-hand side of the above inequality is reached at $n - 4m = -5$ and $\varepsilon = -1$. In this case $|\lambda_1| = 0.86884\ldots$ and therefore $n < 59.363$. Hence $n \leq 59$ for all $n - 4m \in [-12, -5]$.

For $n - 4m \in [-4, 411]$, Equation (15) has complex zeros with modulus 1. We write $\lambda_1 = e^{i\phi_{n-4m}}$ where $\phi_{n-4m} \in (0, 2\pi)$. Then from Equation (16) and the fact that $z = e^{-i\theta}$ we have

$$\left|e^{i(n\theta + \phi_{n-4m})} - 1\right| < 3\alpha^{-3n/16}. \tag{28}$$

Now putting $r := \lfloor (n\theta + \phi_{n-4m})/\pi \rceil$ we get $n\theta + \phi_{n-4m} - r\pi \in [-\pi/2, \pi/2]$. Therefore from Equation (25) we get

$$\left|e^{i(n\theta + \phi_{n-4m})} - 1\right| \geq \left|\sin(n\theta + \phi_{n-4m})\right| \geq 2\left|\frac{n\theta}{\pi} - r + \frac{\phi_{n-4m}}{\pi}\right|. \tag{29}$$

From Equation (28) and Equation (29) we conclude that

$$|n\kappa - r + \mu_{n-4m}| < AB^{-n}, \tag{30}$$

where

$$\kappa := \theta/\pi, \quad \mu_{n-4m} := \phi_{n-4m}/\pi, \quad A := 1.5, \quad \text{and} \quad B := \alpha^{3/16}.$$

Here, we also take $M := 4.56 \times 10^{21}$, which is an upper bound of n, and apply Lemma 2 to Equation (30) for each $n - 4m \in [-4, 411] \setminus \{-3, 8\}$. We find computationally that $q_{45} = 86202026967371307850593$ is the denominator of the first convergent of the continued fraction expansion of κ such that $q_{45} > 2.736 \times 10^{22} = 6M$. In addition, we get that the minimum value of ϵ is $> 1.56966 \times 10^{-3}$ (which is achieved at $n - 4m = 371$ with $\varepsilon = \pm 1$), and the maximum value of $\log(Aq_{45}/\epsilon)/\log B$ is less than 1170.52, which is reached at $n-4m=312$ with $\varepsilon=\pm 1$. Thus, $n\leq 1170$ for all $n-4m\in[-4,411]\setminus\{-3,8\}$.

In the cases $n-4m=-3,8$, ϵ is always negative since the argument ϕ_{n-4m} of the complex number

$$\lambda_1=\frac{\varepsilon\alpha^{(4m-n)/2}+\sqrt{\alpha^{4m-n}-4}}{2}$$

satisfies

$$\phi_{n-4m}=\begin{cases}\pi-\theta & \text{if} \quad n-4m=-3, \quad \text{and} \quad \varepsilon=1;\\ \theta & \text{if} \quad n-4m=-3, \quad \text{and} \quad \varepsilon=-1;\\ 2\pi-2\theta & \text{if} \quad n-4m=8, \quad \text{and} \quad \varepsilon=1;\\ 2\theta-\pi & \text{if} \quad n-4m=8, \quad \text{and} \quad \varepsilon=-1.\end{cases}$$

Equation (30) in the case when $n-4m=-3$ and $\varepsilon=1$ is transformed into

$$\left|\frac{\theta}{\pi}-\frac{r-1}{n-1}\right|<\frac{1.5\alpha^{-3n/16}}{n-1}. \tag{31}$$

Now we apply Lemma 3 to the left-hand side of Equation (31). For this, we calculate the continued fraction expansion $[a_0,a_1,\ldots]$ and the convergents p_j/q_j of $x:=\theta/\pi$. Considering that $n-1<4.56\times 10^{21}:=M$ we get $M\in(q_{39},q_{40})$ and therefore $a_M=\max\{a_j:j\in[0,40]\}=a_{11}=3550$. Thus,

$$\left|\frac{\theta}{\pi}-\frac{r-1}{n-1}\right|>\frac{1}{3552(n-1)^2}.$$

Putting together the inequality immediately above with Equation (31) and using again that $n-1<4.56\times 10^{21}$, we obtain

$$n<\frac{16\log\left(2.42957\times 10^{25}\right)}{3\log\alpha}<1108.63.$$

In all other cases we obtain the same upper bound for n, so $n\leq 1108$ for $n-4m=-3,8$. Therefore, $n\leq 1170$ for all $n-4m\in[-4,411]$. Then by Equation (12), we obtain $m\leq 295$. By a computational search we complete the proof of Theorem 2 by finding that the only common values between R_{-n} and $\pm R_m^2$ for $n\leq 1170$ and $m\leq 295$ are those recorded in the statement of Theorem 2. □

8. The Proof Of Theorem 3

Proof. Suppose that (m,n) is a solution of Equation (3) with $m\geq 6$ and $n\geq 2$. Using Equation (5) and Equation (6) with $z:=e^{2i\theta}$ in Equation (3), we obtain

$$\alpha^m(z^m+2-\alpha^{n-m}+z^{-m})=\beta^n+\gamma^n-\alpha^{-2m}-2\alpha^{-m}\left(\beta^{-m}+\gamma^{-m}\right). \tag{32}$$

But

$$\begin{aligned}\alpha^m(z^m+2-\alpha^{n-m}+z^{-m}) &= \alpha^m z^{-m}(z^{2m}+(2-\alpha^{n-m})z^m+1)\\ &= \alpha^m z^{-m}(z^m-\lambda_1)(z^m-\lambda_2),\end{aligned} \tag{33}$$

where λ_1, λ_2 are the zeros of the quadratic polynomial

$$X^2+(2-\alpha^{n-m})X+1. \tag{34}$$

So from Equation (32) and Equation (33) we get that

$$\alpha^m z^{-m}(z^m-\lambda_1)(z^m-\lambda_2) = \beta^n+\gamma^n-\alpha^{-2m}-2\alpha^{-m}\left(\beta^{-m}+\gamma^{-m}\right).$$

Dividing the above equation by α^m and taking absolute values we obtain

$$|(z^m-\lambda_1)(z^m-\lambda_2)| \le \frac{2\alpha^{-n/2}+\alpha^{-2m}+4\alpha^{-m/2}}{\alpha^m} < \frac{3.3}{\alpha^m}.$$

We can assume without loss of generality that $|z^m-\lambda_1| \le |z^m-\lambda_2|$. In addition, $|z|=1$, and therefore

$$|\lambda_1 z^{-m}-1| < \frac{1.82}{\alpha^{m/2}}. \tag{35}$$

Now we apply Theorem 6 with the parameters

$$\eta_1 := \lambda_1 = \frac{\alpha^{n-m}-2+\sqrt{\alpha^{2(n-m)}-4\alpha^{n-m}}}{2}, \quad \eta_2 := z,$$

$b_1 := 1$, and $b_2 := -m$. Here $\eta_1, \eta_2 \in \mathbb{K} := \mathbb{Q}\big(\alpha, \beta, \sqrt{\alpha^{2(n-m)}-4\alpha^{n-m}}\big)$. If we put $\mathbb{L} := \mathbb{Q}(\alpha,\beta)$, then we have $\mathbb{K} = \mathbb{L}\big(\sqrt{\alpha^{2(n-m)}-4\alpha^{n-m}}\big)$. So,

$$D = \big[\mathbb{K}:\mathbb{Q}\big] = \big[\mathbb{K}:\mathbb{L}\big]\big[\mathbb{L}:\mathbb{Q}\big] \le 2\cdot 6 = 12.$$

Since $m \ge 6$ we take $B := m$. Using Equation (8) and Lemma 1 in Equation (3), we get

$$|n-m| < 6\times 10^{15}\log m. \tag{36}$$

By taking $b = 2-\alpha^{n-m}$ and $c=1$ we have $h(\eta_1) < |n-m|h(\alpha)+3\log 2 < 5.63\times 10^{14}\log m$ due to Equation (17), Equation (36) and the fact that $h(\alpha) = (\log\alpha)/3$. Therefore, we can take $A_1 := 6.76\times 10^{15}\log m$. Moreover, we can choose $A_2 := 8\log\alpha$ since $h(\eta_2) = h(\beta/\gamma) \le (2\log\alpha)/3$. Let us see that $\Lambda_2 := \eta_1^{b_1}\eta_2^{b_2}-1 \ne 0$. If it were not the case, then from Equation (33) we obtain

$$\beta^{-2m} = \alpha^n - \big(2\alpha^m+\gamma^{-2m}\big),$$

and so conjugating it by the automorphism $\sigma_{\alpha\beta\gamma}$ leads to

$$\alpha^m \le \alpha^{-n/2}+2\alpha^{-m/2}+\alpha^{-2m},$$

which is not true for $n \geq 2$ and $m \geq 3$. Therefore $\Lambda_2 \neq 0$. In summary, we get

$$|\lambda_1 z^{-m} - 1| > \exp\left(-1.40492 \times 10^{31} \log^2 m\right),$$

where we have used again the fact that $1 + \log(2m) < 2\log m$ for all $m \geq 6$. Combining the immediately preceding inequality with Equation (35) we establish the following result.

$$m < 6.8 \times 10^{35}. \tag{37}$$

Now we reduce $m - n$ as we did with $n - 4m$ in the proof of Theorem 2. In effect, using that $R_n \leq \alpha^{n+1}$ for all $n \geq 2$ (by Equation (8)) on the left-hand side of Equation (3) and apply Equation (20) with $k = m$ on the right-hand side, we obtain

$$4\alpha^m \left|\left(\frac{\beta}{\gamma}\right)^m + 1\right|^2 < |R_{-m}|^2 = |R_n| = R_n \leq \alpha^{n+1}.$$

This time we obtain

$$\left|\left(\frac{\beta}{\gamma}\right)^m + 1\right| < \frac{0.58}{\alpha^{(m-n)/2}}$$

instead of Equation (21). Using again that $(\beta/\gamma)^m = e^{2im\theta}$ by Equation (6), Euler's formula, and Equation (25) we find that the above inequality implies that

$$\left|\frac{2\theta}{\pi} - \frac{r-1}{m}\right| < \frac{0.29}{m\alpha^{(m-n)/2}}, \tag{38}$$

where now $r := \lfloor (2m\theta + \pi)/\pi \rceil$. We next use Lemma 3 again with the irrational number $x := 2\theta/\pi$. This time with $M = 6.8 \times 10^{35}$ from Equation (37) we have that $M \in (q_{64}, q_{65})$ and then $a_M = \{a_j : j \in [0, 65]\} = a_{11} = 7102$. Therefore

$$\left|\frac{2\theta}{\pi} - \frac{r-1}{m}\right| > \frac{1}{7104m^2}. \tag{39}$$

Comparing Equation (38) and Equation (39), we get from Equation (37) that

$$m - n < \frac{2\log\left(0.29 \cdot 7104 \cdot 6.8 \times 10^{35}\right)}{\log \alpha} < 641.096.$$

We can further reduce the above upper bound for $m - n$ and hence that for m by repeating the arguments before Equation (37). To do this we again apply Theorem 6 to the left-hand side of Equation (35) with the same parameters as last time, except that now A_1 can be 746. This time we get $m < 5.26 \times 10^{20}$. Now we apply Lemma 3 again to the left-hand side of Equation (38) but this time with $M = 5.26 \times 10^{20}$. Now $q_{39} < M < q_{40}$ so $a_M = \max\{a_j : j \in [0, 40]\} = a_{11} = 7102$ and we arrive again at Equation (39). Combining Equation (38) and Equation (39) with this new bound for m, we obtain that $m - n \leq 393$. Repeating the whole process again we obtain $m < 3.26 \times 10^{20}$ and $m - n \leq 390$.

On the other hand, combining Equation (8) and Lemma 1 in Equation (3) we get $m-n \geq -6$, so $m-n \in [-6, 390]$.

Now we reduce the upper bound of m. If $m-n \in \{-6,-5\}$, then Equation (34) has real zeros such that $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or $|\lambda_1| < 1$ and $|\lambda_2| > 1$. In any case we obtain from Equation (35) that

$$m < \frac{2\log\left(1.82/(||\lambda_1|-1|)\right)}{\log \alpha} < 14.2592.$$

Therefore $m \leq 14$ for $m-n \in \{-6,-5\}$. For $m-n \in [-4, 390]$, Equation (34) has complex zeros with modulus 1. We write $\lambda_1 = e^{i\delta_{m-n}}$ where $\delta_{m-n} \in (0, 2\pi)$. Then from Equation (35) and the fact that $z = e^{2i\theta}$ we have

$$\left|e^{i(-2m\theta+\delta_{m-n})} - 1\right| < \frac{1.82}{\alpha^{m/2}}.$$

Using Euler's formula and Equation (25) on the left-hand side of the above inequality we get

$$|m\kappa - r + \mu_{m-n}| < AB^{-m}, \tag{40}$$

where $r := \lfloor(-2m\theta + \delta_{m-n})/\pi\rceil$,

$$\kappa := -2\theta/\pi, \quad \mu_{m-n} := \delta_{m-n}/\pi, \quad A := 0.91, \quad \text{and} \quad B := \alpha^{1/2}.$$

We also take $M = 3.26\times10^{20}$ since $m < M$. Applying Lemma 2 to Equation (40) for each $m-n \in [-4, 390]\setminus\{-3, 8\}$ we find by means of a computational search that the maximum value of $\log(Aq_{44}/\epsilon)/\log B$ is attained at $m-n = 385$ and is < 407.601, where $q_{44} = 3082994406887617567959 32$ is the denominator of the first convergent of the continued fraction expansion of κ such that $q_{44} > 1.956 \times 10^{21} = 6M$, and the minimum value of ϵ is $\|\mu_{117}q_{44}\| - M\,\|\kappa q_{44}\| > 2.02215 \times 10^{-3}$. Thus, $m \leq 407$ for all $m-n \in [-4, 390] \setminus \{-3, 8\}$.

In the cases $m-n = -3, 8$, the argument δ_{m-n} of the complex number

$$\lambda_1 = \frac{\alpha^{n-m} - 2 + \sqrt{\alpha^{2(n-m)} - 4\alpha^{n-m}}}{2}$$

is a linear combination of θ and π, so ϵ is always negative. Indeed,

$$\delta_{m-n} = \begin{cases} 2\pi - 2\theta & \text{if} \quad m-n = -3; \\ 4\pi - 4\theta & \text{if} \quad m-n = 8. \end{cases}$$

If $m-n = -3$, Equation (40) becomes

$$\left| -\frac{2\theta}{\pi} - \frac{r-2}{m+1} \right| < \frac{0.91\alpha^{-m/2}}{m+1}. \tag{41}$$

Using Lemma 3 again with the irrational number $x = -2\theta/\pi$ and $M = 3.26 \times 10^{20}$ we get $M \in (q_{38}, q_{39})$ so $a_M = \max\{a_j : j \in [0, 39]\} = -1$. Then

$$\left| -\frac{2\theta}{\pi} - \frac{r-2}{m+1} \right| > \frac{1}{(m+1)^2}.$$

Therefore, taking into account that $m + 1 < 3.26 \times 10^{20}$ we get

$$m < \frac{2 \log \left(2.9666 \times 10^{20}\right)}{\log \alpha} < 335.272.$$

When $m - n = 8$, we get Equation (41) with $r - 4$ and $m + 2$ instead of $r - 2$ and $m + 1$, respectively. Taking into account that $m + 2 < 3.26 \times 10^{20}$, we arrive at the same absolute upper bound for m as in the case when $m - n = -3$. Thus, $m \leq 335$ for $m - n = -3, 8$.

Therefore, $m \leq 407$ holds for all $m - n \in [-4, 390]$. Since $m - n \geq -6$, we get $n \leq 413$. Through a computational search we finish the proof of Theorem 3 by finding that there are no coincidences between R_n and R_{-m}^2 for $n \leq 413$ and $m \leq 407$. □

9. The Proof Of Theorem 4

Proof. Here we solve Equation (4) in two cases. Let us see.

`Case 1`: $2m - n = 0$ and $\varepsilon = 1$. In this case, using Equation (9) with $k = -m$ we transform Equation (4) into $R_m = 0$, which implies that $m = 1$ by Theorem 5. Thus, $(m, n) = (1, 2)$ is the only solution of Equation (4) in this case.

`Case 2`: $2m - n \neq 0$ and $\varepsilon = \pm 1$, or $2m - n = 0$ and $\varepsilon = -1$. From now on we assume that Equation (4) holds and that $m, n \geq 6$. By using Lemma 1 in Equation (4), we get the inequalities

$$\alpha^{\frac{n}{2} - 3 \times 10^{15} \log n} < |R_{-n}| = \left| \pm R_{-m}^2 \right| < 4.0401 \alpha^m$$

and

$$\alpha^{m - 6 \times 10^{15} \log m} < \left| \pm R_{-m}^2 \right| = |R_{-n}| < 2.01 \alpha^{n/2}.$$

Taking logarithms in the two resulting inequalities above, we obtain

$$|2m - n| < 1.2 \times 10^{16} \log(\max\{m, n\}). \tag{42}$$

Now combining Equation (5) and Equation (6) with $z := e^{-2i\theta}$ in Equation (4) with $\varepsilon := \pm 1$, we get

$$(\beta^{2m-n} - \varepsilon)\alpha^m z^m - 2\varepsilon\alpha^m + (\gamma^{2m-n} - \varepsilon)\alpha^m z^{-m} = \varepsilon\alpha^{-2m} + 2\varepsilon\alpha^{-m}(\beta^{-m} + \gamma^{-m}) - \alpha^{-n}.$$

Dividing the above equation by $(\beta^{2m-n}-\varepsilon)\alpha^m$ and taking absolute values, we get

$$\left|z^m - \frac{2\varepsilon}{\beta^{2m-n}-\varepsilon} + \frac{\gamma^{2m-n}-\varepsilon}{\beta^{2m-n}-\varepsilon} z^{-m}\right| \le \frac{\alpha^{-2m}+4\alpha^{-m/2}+\alpha^{-n}}{|\alpha^{-(2m-n)/2}-1|\alpha^m} < \frac{14}{\alpha^m}, \qquad (43)$$

Now we can rewrite Equation (43) by multiplying its left-hand side by $|z^m| = 1$ as

$$\big|(z^m-\lambda_1)(z^m-\lambda_2)\big| < \frac{14}{\alpha^m},$$

where λ_1, λ_2 are the zeros of the quadratic polynomial

$$X^2 - \frac{2\varepsilon}{\beta^{2m-n}-\varepsilon}X + \frac{\gamma^{2m-n}-\varepsilon}{\beta^{2m-n}-\varepsilon}. \qquad (44)$$

We may assume without loss of generality that $|z^m-\lambda_1| \le |z^m-\lambda_2|$. Moreover, $|z| = 1$, and therefore

$$|\lambda_1 z^{-m} - 1| < \frac{3.75}{\alpha^{m/2}}. \qquad (45)$$

Next we apply Theorem 6 to $\Lambda_3 = \lambda_1 z^{-m} - 1$. This is not zero, otherwise we get

$$\varepsilon\beta^{-2m} - \beta^{-n} = \gamma^{-n} - \varepsilon(2\alpha^m + \gamma^{-2m}),$$

which, conjugating by the automorphism $\sigma_{\alpha\gamma}$ and taking absolute values, implies

$$\left|\alpha^m - \alpha^{n/2}\right| \le \alpha^{-n} + 2\alpha^{-m/2} + \alpha^{-2m},$$

which is impossible for all $n \ge 6$ and $m \ge 5$. In consequence, $\Lambda_3 \ne 0$. We put

$$\eta_1 := \lambda_1 = \frac{\varepsilon}{\beta^{2m-n}-\varepsilon} + \sqrt{\frac{1}{(\beta^{2m-n}-\varepsilon)^2} - \frac{\gamma^{2m-n}-\varepsilon}{\beta^{2m-n}-\varepsilon}}, \quad \eta_2 := z,$$

and $b_2 := -m$. Here $\mathbb{K} = \mathbb{Q}\big(\alpha, \beta, \sqrt{(\beta^{2m-n}-\varepsilon)^{-2} - (\beta^{2m-n}-\varepsilon)^{-1}(\gamma^{2m-n}-\varepsilon)}\,\big)$ so $D \le 12$. Since $m \ge 6$ we have $B = m$. On the side of the logarithmic heights, $h(\eta_2) = h(\gamma/\beta) \le (2/3)\log\alpha$, so $A_2 = 4\log\alpha$ is a correct choice. Again using Equation (17) with $b = -2\varepsilon/(\beta^{2m-n}-\varepsilon)$ and $c = (\gamma^{2m-n}-\varepsilon)/(\beta^{2m-n}-\varepsilon)$ we have $h(\eta_1) \le (\log\alpha)|2m-n| + 5\log 2$, so $A_1 = 2.03 \times 10^{16}\log(\max\{m,n\})$ is an allowed choice by Equation (42). Theorem 6 then tells us that

$$|\lambda_1 z^{-m} - 1| > \exp\big(-1.05473 \times 10^{31}\log(\max\{m,n\})(1+\log(2m))\big).$$

Comparing this with Equation (45) and using again that $1 + \log(2m) < 2\log m$ for all $m \ge 6$, we get

$$m < 1.51 \times 10^{32}\log(\max\{m,n\})\log m. \qquad (46)$$

If $n \in [m, 4m]$ then Equation (46) implies that $m < 1.51 \times 10^{32}\log(4m)\log m$ and therefore $m < 1.05 \times 10^{36}$ and $n \le 4m < 4.2 \times 10^{36}$. In the case when $m > n$, by

Equation (42) we have $m < 2m-n < 1.2\times 10^{16}\log m$ and therefore $m < 4.88\times 10^{17}$. Similarly, when $n > 4m$ it follows from Equation (42) that $n/2 < 1.2\times 10^{16}\log n$ and so $n < 9.94\times 10^{17}$. In either case, we have

$$\max\{m,n\} < 4.2\times 10^{36}. \tag{47}$$

Now we reduce the bound of $|n-2m|$. Using that $|R_{-k}| < 2.01\alpha^{k/2}$ holds for all $k\geq 6$ (by Lemma 1) and Equation (20) in Equation (4), we get

$$2\alpha^{n/2}\left|\left(\frac{\beta}{\gamma}\right)^n + 1\right| < |R_{-n}| = |\pm R_{-m}^2| < 4.0401\alpha^m$$

and

$$4\alpha^m\left|\left(\frac{\beta}{\gamma}\right)^m + 1\right|^2 < |\pm R_{-m}^2| = |R_{-n}| < 2.01\alpha^{n/2}.$$

Therefore

$$\left|\left(\frac{\beta}{\gamma}\right)^{\max\{m,n\}} + 1\right| < \frac{4.0401}{\alpha^{|n-2m|/4}}.$$

Again using that $(\beta/\gamma)^{\max\{m,n\}} = e^{2i\max\{m,n\}\theta}$ by Equation (6), Euler's formula, and Equation (25), we get that the above inequality implies that

$$\left|\frac{2\theta}{\pi} - \frac{r-1}{\max\{m,n\}}\right| < \frac{2.02005}{\max\{m,n\}\alpha^{|n-2m|/4}}, \tag{48}$$

where now $r := \lfloor (2\max\{m,n\}\theta+\pi)/\pi\rceil$. Using Lemma 3 again with the irrational number $x := 2\theta/\pi$ and $M = 4.2\times 10^{36}$ (see Equation (47)), we get $M\in(q_{67},q_{68})$ so $a_M = a_{11} = 7102$ and then

$$\left|\frac{2\theta}{\pi} - \frac{r-1}{\max\{m,n\}}\right| > \frac{1}{7104\max\{m,n\}^2}. \tag{49}$$

Thus, from Equation (48), Equation (49), and Equation (47) we get $|n-2m|\leq 1335$. We can further reduce this bound for $|n-2m|$. To do this, we reduce the bound of $\max\{m,n\}$ given in Equation (47) by again applying Theorem 6 to Equation (45). In this application we use the same parameters as last time, only now A_1 can be 4547. Now we obtain then that $\max\{m,n\} < 6.56\times 10^{21}$. Next, we apply Lemma 3 again to the left-hand side of Equation (48) but with $M = 6.56\times 10^{21}$. This time $N = 40$ and then $a_M = a_{11} = 7102$, so we arrive again at Equation (49). Combining again Equation (48) and Equation (49) but using that $\max\{m,n\} < 6.56\times 10^{21}$, we get that $|n-2m|\leq 850$. If we repeat for the third time the whole process we get $\max\{m,n\} < 4.28\times 10^{21}$ and $|n-2m|\leq 844$.

Next, we reduce the upper bound of m. For $2m-n\in[-844,844]$, Equation (44) has complex zeros with modulus 1. We write $\lambda_1 = e^{i\zeta_{2m-n}}$ with $\zeta_{2m-n}\in(0,2\pi)$. Then from Equation (45) and the fact that $z = e^{-2i\theta}$, we obtain

$$\left|e^{i(2m\theta+\zeta_{2m-n})} - 1\right| < \frac{3.75}{\alpha^{m/2}}.$$

Using Euler's formula and Equation (25) on the left-hand side of the above inequality we arrive at

$$|m\kappa - r + \mu_{2m-n}| < AB^{-m}, \tag{50}$$

where now $r := \lfloor (2m\theta + \zeta_{2m-n})/\pi \rceil$,

$$\kappa := 2\theta/\pi, \quad \mu_{2m-n} := \zeta_{2m-n}/\pi, \quad A := 1.875, \quad \text{and} \quad B := \alpha^{1/2}.$$

We also take $M = 4.28 \times 10^{21}$ since $m \le \max\{m, n\} < M$. Applying Lemma 2 to Equation (50) for each $2m - n \in [-844, 844]$ except when $2m - n \in \{-3, -1\}$ and $\varepsilon = -1$, and when $2m - n = 5$ and $\varepsilon = 1$, we get computationally that $q_{72} = 45364104060616757805315065148766863569 > 2.568 \times 10^{22} = 6M$ is the denominator of the first convergent of the continued fraction expansion of κ which guarantees that the smallest value of ϵ is $> 6.83742 \times 10^{-16}$ (which is obtained at $2m - n = 839$ and $\varepsilon = -1$) and the largest value of m is < 872.719 and is reached at $2m - n = 839$ and $\varepsilon = -1$. Since $2m - n \ge -844$, we obtain that $n \le 2588$ for all $2m - n \in [-844, 844]$ except when $2m - n \in \{-3, -1\}$ and $\varepsilon = -1$, and when $2m - n = 5$ and $\varepsilon = 1$.

At these last values of $2m - n$ and ε, we have that $\epsilon < 0$ since the argument ζ_{2m-n} of the complex number

$$\lambda_1 = \frac{\varepsilon}{\beta^{2m-n} - \varepsilon} + \sqrt{\frac{1}{(\beta^{2m-n} - \varepsilon)^2} - \frac{\gamma^{2m-n} - \varepsilon}{\beta^{2m-n} - \varepsilon}}$$

satisfies that

$$\zeta_{2m-n} = \begin{cases} 2\theta - 2\pi & \text{if} \quad 2m - n = -3 \quad \text{and} \quad \varepsilon = -1; \\ \pi - 2\theta & \text{if} \quad 2m - n = -1 \quad \text{and} \quad \varepsilon = -1; \\ 4\pi - 4\theta & \text{if} \quad 2m - n = 5 \quad \text{and} \quad \varepsilon = 1. \end{cases}$$

If $2m - n = -3$ and $\varepsilon = -1$, then Equation (50) becomes

$$\left| \frac{2\theta}{\pi} - \frac{r + 2}{m + 1} \right| < \frac{1.875\alpha^{-m/2}}{m + 1}. \tag{51}$$

We finally use Lemma 3 with $x = 2\theta/\pi$. Since $m + 1 \le \max\{m, n\} + 1 < 4.28 \times 10^{21} := M$ we can see computationally that $M \in (q_{39}, q_{40})$ giving $a_M = a_{11} = 7102$. Therefore, a lower bound for the left-hand side of Equation (51) is $1/7104(m + 1)^2$ and hence

$$m < \frac{2 \log\left(5.70096 \times 10^{25}\right)}{\log \alpha} < 421.801,$$

where we used again that $m + 1 < 4.28 \times 10^{21}$. In the other two special cases we manage to convert Equation (50) into Equation (51) with $r - 1$ and $r - 4$ instead of $r + 2$, and $m - 1$ and $m - 2$ instead of $m + 1$. So the obtained upper bound for

m is the same as the obtained in the first special case. Since $2m - n \geq -844$, we get $n \leq 1686$ for $2m - n \in \{-3, -1\}$ and $\varepsilon = -1$, and for $2m - n = 5$ and $\varepsilon = 1$.

Therefore, $n \leq 2588$ and $m \leq 872$ holds for all $2m - n \in [-844, 844]$. We now computationally search $R_{-n} \cap \pm R_{-m}^2$ for $n \leq 2588$ and $m \leq 872$ obtaining the remaining solutions listed in Theorem 4. This ends the proof of Theorem 4. □

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