#A71 INTEGERS 25 (2025)

ON THE BROCARD–RAMANUJAN EQUATION WITH 7-FREE INTEGERS AND PRIME POWERS

Abderrahim Makki Naciri
Department of Mathematics, LIBMA laboratory, Cadi Ayyad University, Semlalia Faculty, Marrakesh, Morocco
a.makkinaciri@uca.ac.ma

Received: 1/15/25, Revised: 5/18/25, Accepted: 7/15/25, Published: 8/15/25

Abstract

This paper proves that for any integer $k \geq 2$, the Brocard–Ramanujan Diophantine equation $n!+1 = x^2$ has only finitely many integer solutions (n, x), assuming $x \pm 1$ is a k-free integer or has less than k prime divisors. Specifically, we identify all integer solutions when $x \pm 1$ is a 7-free integer or a prime power. We then extend this result and make a remark on the resolution of the Brocard–Ramanujan problem.

1. Introduction

The Brocard–Ramanujan problem (see [4] and [18]) concerns the Diophantine equation

$$n! + 1 = x^2. \tag{1}$$

Only three solutions are known: $(4, 5), (5, 11)$, and $(7, 71)$. Extensive computations up to 10^9 (see [2]) have yielded no additional solutions, suggesting these three pairs may be exhaustive. Despite numerous attempts and varied approaches, this problem remains unsolved. A review of the existing literature reveals that three major approaches have been used to address Equation (1). The first is the variant-based approach, which consists of studying modified versions of Equation (1) (see, for example, [1], [5], [6], [7], [10], [12], [19], [20]). The second consists of solving Equation (1) under the abc conjecture or its variants (see [5], [11], [15]). The third is the subset-based approach, which focuses on exploring solutions within specific subsets of $\mathbb{N}$. More specifically, it involves solving, for a given integer sequence (u_m), the equation $n! + 1 = u_m^2$, where n and m are the unknowns. Regarding this latter approach, recent studies have explored several variants of the Brocard–Ramanujan Diophantine equation. D. Marques [14] proved that the Fibonacci version, $n!+1 = F_m^2$, has no solution except for $(n, m) = (4, 5)$. J. J. Bravo

DOI: 10.5281/zenodo.16881781

et al. [3] demonstrated that the Tripell version admits no solution except $(4, 3)$. M. Ismail et al. [9] showed that the Narayana version has no solution. P. T. Young [21] obtained similar results for the Tribonacci and Tetranacci versions. Finally, the author [13] observed that Sylvester's version has no positive integer solutions. Additional versions involving recurrence sequences can be found in references [16] and [17].

This paper explores new subsets of $\mathbb{N}$ with a finite number of solutions to Equation (1). We begin by establishing the following theorem.

Theorem 1. *The following statements hold.*
(i) For any given integer $k \geq 2$, there are only finitely many integer solutions (n, x) to the Brocard–Ramanujan equation $n! + 1 = x^2$, where $x \pm 1$ is a k-free number. Specifically, this equation has at most the three pairs $(4, 5)$, $(5, 11)$, and $(7, 71)$ as solutions where $x \pm 1$ is a 7-free integer.
(ii) For any given integer $l \geq 2$, there are only finitely many integer solutions (n, x) to the Brocard–Ramanujan equation $n! + 1 = x^2$, where $x \pm 1$ has less than l prime divisors. Specifically, this equation has at most the pair $(4, 5)$ as a solution where $x \pm 1$ is a prime power.

In Section 4, we provide a generalization of Theorem 1 and make a remark on the resolution of the Brocard–Ramanujan problem.

2. Preliminary Results

We begin by recalling classical number theoretic results. Let $\pi(n)$ denote the number of primes less than or equal to n. The Prime Number Theorem states that $\pi(n) \sim \frac{n}{\ln(n)}$ as $n \to \infty$. Furthermore, Chebyshev's theorem refinement yields

$$\frac{n}{\ln(n)} \leq \pi(n) \leq \frac{3}{2}\frac{n}{\ln(n)}, \ for \ all \ n \geq 2.$$

Stirling's formula states that $n!$ asymptotically behaves like $n^n \exp(-n)\sqrt{2\pi n}$ as $n \to \infty$. Moreover, for all $n \geq 1$, we have

$$\left(\frac{n}{e}\right)^n \leq n! \leq n^n.$$

For a prime p, let $\nu_p(n)$ denote the largest power of p dividing n. Legendre's formula states:

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right],$$

where $[x]$ is the integer part of x. This yields

$$\nu_p(n!) \leq \sum_{i=1}^{\infty} \frac{n}{p^i} = \frac{n}{p-1}.$$

Thus, if p^α divides $n!$, then $p \leq n$ and $\alpha \leq \frac{n}{p-1}$. All these classic results can be found in the reference [8].

For all positive integer x, we let $\omega(x)$ denote the number of distinct primes in the prime factorization of x and we let $K(x)$ denote the maximum exponent in the prime factorization of x. That is, if the prime factorization of x is

$$x = \prod_{i \in I} p_i^{\alpha_i},$$

then we have $\omega(x) = |I|$ and $K(x) = \max_{i \in I} \alpha_i$.

Lemma 1. *Let n and x be positive integers. If x divides $n!$, then*

$$x \leq \exp\left(\frac{3nK(x)}{2}\right).$$

Proof. Let $x = \prod_{i \in I} p_i^{\alpha_i}$ be the prime factorization of x. By definition of $K(x)$, we have $\alpha_i \leq K(x)$ for all $i \in I$. Thus, $x \leq \prod_{i \in I} p_i^{K(x)}$. Since x divides $n!$, every prime factor p_i satisfies $p_i \leq n$. Moreover, the number of distinct prime factors, $|I|$, is bounded by $\pi(n)$. Therefore we have

$$\begin{aligned} x &\leq n^{K(x)|I|} \\ &\leq n^{K(x)\pi(n)} \\ &\leq n^{\frac{3nK(x)}{2\ ln(n)}} \\ &= \exp\left(\frac{3nK(x)}{2}\right). \end{aligned}$$

□

Lemma 2. *Let n, x be positive integers. If x divides $n!$, then*

$$x \leq n^{\frac{n}{a}} \left(a\omega(x) + 1\right)^{n\omega(x)},$$

for any arbitrary positive integer a.

Proof. Let $x = \prod_{i \in I} p_i^{\alpha_i}$ be the prime factorization of x and assume that x divides $n!$. Note that $|I| = \omega(x)$, $p_i \leq n$ for all $i \in I$, and $\alpha_i \leq \frac{n}{p_i - 1}$ for all $i \in I$. For a positive integer a, define:

$$J = \left\{ i \in I \mid p_i \leq a\omega(x) + 1 \right\}.$$

So we have

$$\sum_{i\in J}\alpha_i \le \sum_{i\in I}\frac{n}{p_i-1} \le \sum_{i\in I} n = n\omega(x)$$

and

$$\sum_{i\in I\setminus J}\alpha_i \le \sum_{i\in I\setminus J}\frac{n}{p_i-1} \le \sum_{i\in I\setminus J}\frac{n}{a\omega(x)} \le \sum_{i\in I}\frac{n}{a\omega(x)} = \frac{n}{a}.$$

Therefore,

$$x = \prod_{i\in I} p_i^{\alpha_i} = \prod_{i\in J} p_i^{\alpha_i} \prod_{i\in I\setminus J} p_i^{\alpha_i} \le \Big(a\omega(x)+1\Big)^{\sum\limits_{i\in J}\alpha_i} n^{\sum\limits_{i\in I\setminus J}\alpha_i} \le \Big(a\omega(x)+1\Big)^{n\omega(x)} n^{\frac{n}{a}}.$$

□

3. Proof of Theorem 1

We now apply the preceding lemmas to prove Theorem 1.

Proof of Theorem 1. (*i*) Let k be a positive integer. We simultaneously consider the cases where $x-1$ and $x+1$ are k-free, relying on the inequality

$$(x-\varepsilon)(x+\varepsilon) \le 2(x-\varepsilon)^2,\ \textit{for}\ \varepsilon = \pm 1\ \textit{and}\ x \ge 3.$$

Let $\varepsilon = \pm 1$. Suppose (n, x) satisfies Equation (1), where $x \ge 3$ and $x-\varepsilon$ is a k-free integer, i.e., $K(x-\varepsilon) < k$. Since $x-\varepsilon$ divides $n! = (x-\varepsilon)(x+\varepsilon)$, Lemma 1 yields

$$x-\varepsilon \le \exp\left(\frac{3nK(x-\varepsilon)}{2}\right) \le \exp\left(\frac{3nk}{2}\right),$$

and hence

$$n! = (x-\varepsilon)(x+\varepsilon) \le 2(x-\varepsilon)^2 \le 2\exp\Big(3nk\Big).$$

Using Stirling's inequality,

$$\left(\frac{n}{e}\right)^n \le n!,$$

implying

$$\left(\frac{n}{e}\right)^n \le 2\exp\Big(3nk\Big),$$

and hence

$$n \le 2\exp\Big(3k+1\Big). \tag{2}$$

Therefore, there are only finitely many solutions.

For solutions (n,x) of Equation (1), with $x \pm 1$ a 7-free integer, i.e., $k \leq 6$, Inequality (2) yields $n \leq 2\exp(19) < 10^9$. So, according to Berndt and Galway's computations [2], the possible solutions are $(4,5)$, $(5,11)$, and $(7,71)$.

(ii) Let l be a positive integer and $\varepsilon = \pm 1$. Suppose (n,x) satisfies Equation (1) , where $x \geq 3$ and $x-\varepsilon$ has fewer than l prime divisors, i.e., $\omega(x-\varepsilon) < l$. Applying Lemma 2, with $a=4$, we obtain

$$x-\varepsilon \leq n^{\frac{n}{4}}\Big(4\omega(x-\varepsilon)+1\Big)^{n\omega(x-\varepsilon)} \leq n^{\frac{n}{4}}\Big(4l+1\Big)^{nl},$$

and hence

$$n! = (x-\varepsilon)(x+\varepsilon) \leq 2(x-\varepsilon)^2 \leq 2n^{\frac{n}{2}}\big(4l+1\big)^{2nl}.$$

Now, applying Stirling's inequality yields

$$\left(\frac{n}{e}\right)^n \leq n! \leq 2n^{\frac{n}{2}}(4l+1)^{2nl},$$

which simplifies to

$$\left(\frac{n}{e}\right) \leq 2\sqrt{n}(4l+1)^{2l}.$$

Thus,

$$n \leq 4e^2(4l+1)^{4l}, \tag{3}$$

establishing that the number of solutions is finite.

For solutions (n,x) of Equation (1), with $x \pm 1$ a prime power, Inequality (3) yields $n \leq 4e^2 9^8 < 10^9$. Then the possible solution is $(4,5)$, according to Berndt and Galway's computations [2]. □

Remark. Consider the sequence (u_m) of Fermat (respectively, Mersenne) numbers. Since $u_m - 1$ (respectively, $u_m + 1$) is a prime power, the Diophantine equation $n! + 1 = u_m^2$ admits a unique solution $(n,m) = (4,1)$ (respectively, no solutions).

4. Generalization

For any pair (k,l) of positive integers ≥ 2, let us define

$$\mathcal{F}_k = \{x \in \mathbb{N} \mid K(x) < k\} \qquad \text{and} \qquad \mathcal{P}_l = \{x \in \mathbb{N} \mid \omega(x) < l\}.$$

Their product set, $\mathcal{F}_k\mathcal{P}_l$, consists of integers yz with $y \in \mathcal{F}_k$ and $z \in \mathcal{P}_l$. Note that $\mathcal{F}_k\mathcal{P}_l$ is larger than the union $\mathcal{F}_k \cup \mathcal{P}_l$.

Theorem 1 established that for any pair (k,l) of positive integers ≥ 2, the Brocard–Ramanujan equation has finitely many solutions (n,x), where x belongs to the set

$$S = \{x_1 \pm 1 \mid x_1 \in \mathcal{F}_k \cup \mathcal{P}_l\}.$$

This set can be considered a large one in the sense that its natural density is non-zero. Indeed, it is well known (see, for example, [8]) that the natural density of $\mathcal{F}_k$ alone equals $\frac{1}{\zeta(k)}$, where ζ is the Riemann function. To further enlarge S, we introduce multivariate polynomials where variables take values from $\mathcal{F}_k\mathcal{P}_l$, replacing $\mathcal{F}_k \cup \mathcal{P}_l$.

Theorem 2. *Let k, l, and m be positive integers, with $k, l \geq 2$. Let P be a polynomial in m variables with integers coefficients and define*

$$\mathcal{S} = \{x_1 \dots x_m P(x_1, \dots, x_m) \pm 1 \mid x_1, \dots, x_m \in \mathcal{F}_k\mathcal{P}_l\}.$$

Then the Brocard–Ramanujan Diophantine equation $n! + 1 = x^2$ has only finitely many integer solutions where x belongs to $\mathcal{S}$.

Proof. Let us write the polynomial

$$Q = X_1 \dots X_m P(X_1, \dots, X_m)\big(X_1 \dots X_m P(X_1, \dots, X_m) \pm 2\big)$$

in the form:

$$Q = \sum_{(i_1,\dots,i_m)\in L} a_{i_1,\dots,i_m} X_1^{i_1} \dots X_m^{i_m},$$

with L a finite subset of $\mathbb{N}^m$ and $a_{i_1,\dots,i_m} \in \mathbb{Z}$, for all $(i_1, \dots, i_m) \in L$. The total degree of Q is given by

$$d = \max_{(i_1,\dots,i_m)\in L} i_1 + i_2 + \dots + i_m.$$

Define

$$M = \sum_{(i_1,\dots,i_m)\in L} |a_{i_1,\dots,i_m}|$$

and note that for all positive integers $x_1, \dots, x_m$, the following inequality holds:

$$Q(x_1, \dots, x_m) \leq M(\max_{1\leq i\leq m} x_i)^d. \tag{4}$$

Now assume (n,x) is a positive integer pair satisfying $n!+1 = x^2$, with $x \in \mathcal{S}$, i.e., $x = x_1 \cdots x_m P(x_1, \dots, x_m) \pm 1$ for some $x_1, \dots, x_m \in \mathcal{F}_k\mathcal{P}_l$. We will demonstrate that n is bounded.

From $n! + 1 = x^2$, we obtain

$$n! = Q(x_1, \dots, x_m).$$

Using (4), we deduce

$$n! \leq M(\max_{1\leq i\leq m} x_i)^d. \tag{5}$$

On the other hand, let i_0 be an index such that $x_{i_0} = \max_{1\leq i\leq m} x_i$ and let $y \in \mathcal{F}_k$ and $z \in \mathcal{P}_l$ such that $x_{i_0} = yz$. By Lemma 1, since y divides $n!$, we have

$$y \leq \exp\Big(\frac{3kn}{2}\Big).$$

Applying Lemma 2, with $a = 2d$, yields

$$z \leq n^{\frac{n}{2d}}\big(2dl+1\big)^{ln}.$$

Combining these inequalities with (5), we obtain

$$n! \leq M\Big(\exp\Big(\frac{3kn}{2}\Big)n^{\frac{n}{2d}}\big(2dl+1\big)^{ln}\Big)^d.$$

Now applying Stirling's inequality yields

$$\Big(\frac{n}{e}\Big)^n \leq n! \leq M\Big(\exp\Big(\frac{3kn}{2}\Big)n^{\frac{n}{2d}}\big(2dl+1\big)^{ln}\Big)^d.$$

Taking into account that M is a positive integer and $M^{\frac{1}{n}} \leq M$, this inequality simplifies to

$$\frac{n}{e} \leq M\exp\Big(\frac{3kd}{2}\Big)\sqrt{n}\Big(2dl+1\Big)^{dl}.$$

Thus

$$n \leq M^2\exp\Big(3kd+2\Big)\Big(2dl+1\Big)^{2dl},$$

establishing that the number of solutions is finite. □

Remark on the Resolution of the Brocard–Ramanujan Problem. Let us now observe the set $\mathcal{S}$ from Theorem 2, where the equation $n!+1 = x^2$ admits only finitely many solutions. This set is parameterized by a multivariate polynomial, with variables running through the large set $\mathcal{F}_k\mathcal{P}_l$. So the potential size of $\mathcal{S}$ raises an intriguing question: For a suitable multivariate polynomial P, does $\mathcal{S}$ encompass all odd positive integers, except perhaps a few, for sufficiently large k and l? Affirming this would resolve the Brocard–Ramanujan problem, according to Theorem 2.

Acknowledgement. The author sincerely thanks the anonymous referee and Managing Editor, Bruce Landman, for their helpful suggestions, which made this article clearer.

References

[1] D. Berend and J. E. Harmse, On polynomial-factorial Diophantine equations, *Trans. Amer. Math. Soc* **358** (2005), 1741-1779.

[2] B. C. Berndt and W. F. Galway, On the Brocard–Ramanujan Diophantine equation $n! + 1 = m^2$, *Ramanujan J* **4** (2000), 41-42.

[3] J. J. Bravo, M. Díaz, and J. L. Ramírez, On a variant of the Brocard–Ramanujan equation and an application, *Publ. Math. Debrecen* **98** (1-2) (2021), 243-253.

[4] H. Brocard, Question 1532, *Nouv. Ann. Math* **4** (1885), 291.

[5] A. Dabrowski, On the Diophantine equation $n! + A = y^2$, *Nieuw Arch. Wisk* **14** (1996), 321-324.

[6] A. Dabrowski and M. Ulas, Variations on the Brocard–Ramanujan equation, *J. Number Theory* **133** (2013), 1168-1185.

[7] P. Erdos and R. Obláth, Uber diophantische Gleichungen der Form $n! = x^p \pm y^p$ und $n! \pm m! = x^p$, *Acta Szeged* **8** (1937), 241-255.

[8] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Number* (5th ed.), Oxford Univ. Press, New York, 1979.

[9] M. Ismail, S. E. Rihane, and M. Anwar, Narayana sequence and the Brocard–Ramanujan equation, *Notes Number Theory Disc. Math* **29** (3) (2023), 462-473.

[10] O. Kihel, F. Luca, and A. Togbé, Variants of the Diophantine equation $n! + 1 = y^2$, *Port. Math* 67 (2010), 1-11.

[11] F. Luca, The Diophantine equation $P(x) = n!$ and a result of M. Overholt, *Glas. Mat. Ser* **37**(2) (2002), 269-273.

[12] A. Makki Naciri, On the variant $Q(n!) = P(x)$ of the Brocard–Ramanujan Diophantine equation. *Ramanujan J* **65** (2024) 1791-1798.

[13] A. Makki Naciri, A note on Sylvester's sequence, *Integers* **24** (2024), #A117, 7 pp.

[14] D. Marques, The Fibonacci version of the Brocard–Ramanujan Diophantine equation, *Port. Math* **68** (2011) 185-189.

[15] M. Overholt, The Diophantine equation $n! + 1 = m^2$, *Bull. London Math. Soc* **25** (1993), 104.

[16] I. Pink and M. Szikszai, A Brocard–Ramanujan-type equation with Lucas and associated Lucas sequences, *Glas. Mat. Ser* **52** (72) (2017), 11-21.

[17] P. Pongsriiam, Fibonacci and Lucas numbers associated with Brocard–Ramanujan equation, *Commun. Korean Math. Soc* **32** (2017), 511-522.

[18] S. Ramanujan, Question 469, *J. Indian Math. Soc* **5** (1913), 59.

[19] W. Takeda, On the finiteness of solutions for polynomial-factorial Diophantine equations, *Forum Math* **2** (33) (2020), 361-374.

[20] M. Ulas, Some observations on the Diophantine equation $y^2 = x! + A$ and related results, *Bull. Aust. Math. Soc*, **86** (3) (2012), 377-388.

[21] P. T. Young, On the Brocard–Ramanujan equation with tribonacci and tetranacci numbers, *Integers* **24** (2024), #A92, 8 pp.