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ARITHMETIC PROPERTIES OF THE COEFFICIENTS OF SOME MOCK THETA FUNCTIONS

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Abstract

In this paper, we study the arithmetic properties of the coefficients of some mock theta functions. We present a number of such properties, including infinite families of congruences. Some recurrence relations connecting the coefficients of the mock theta functions with certain restricted partition functions are also established.

1. Introduction

In 1920, Ramanujan introduced 17 functions in his last letter to G. H. Hardy [25, p. 534], which he called mock theta functions. Initially, Ramanujan divided his list of mock theta functions into odd orders as three, five, and seven. After Ramanujan, many new mock theta functions were defined and studied by different mathematicians. An account of these can be found in the papers by Andrews [4], Andrews and Hickerson [6], Gordon and McIntosh [14], and Hikami [17], as well as in [7, 10, 15, 16, 21, 26]. In this paper, we are interested in the following mock theta functions of order two, six, and eight:

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q^2;q^2)_n^2}, \tag{1}$$

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2} (-q;q)_n}{(q;q^2)_{n+1}}, \tag{2}$$

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$$\lambda(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n}, \tag{3}$$

$$v(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}, \tag{4}$$

where

$$(a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty:=\prod_{k=0}^{\infty}(1-aq^k).$$

For brevity, we will write, for any positive integer n,

$$\ell_n:=(q^n;q^n)_\infty, \quad \text{and} \quad (a_1,a_2,...,a_n;q)_\infty:=(a_1;q)_\infty(a_2;q)_\infty...(a_n;q)_\infty.$$

The function defined in Equation (4) is the eighth-order mock theta function defined in [14, pp. 322-323]. Agarwal and Sood [1] gave a combinatorial interpretation of $v(q)$ using split $(n+t)$-color partitions. Rana and Sareen [27] extended their results using signed partitions. The function defined in Equation (1) is the second-order mock theta function, which appeared in Ramanujan's lost notebook [25] (see also [4]). A combinatorial interpretation of $\mu(q)$ was given by Kaur and Rana in [20].

The functions defined in Equations (2)-(3) are the sixth-order mock theta functions. Kaur and Rana [19] proved some particular infinite families of congruences for the coefficients of the mock theta function $\lambda(q)$.

In this paper, we prove congruence properties for the coefficients of the mock theta functions defined in Equations (2) and (4). We also prove some recurrence relations connecting the coefficients of the mock theta functions and certain restricted partition functions. The results on mock theta functions $v(q)$ and $\mu(q)$, $\sigma(q)$, and $\lambda(q)$ are established in Sections 3, 4, and 5, respectively. Section 2 is devoted to recording some preliminary results, which will be used in the subsequent sections.

We end this section with some definitions. A *partition* of a positive integer n can be defined as a finite sequence of positive integers $(\delta_1,\delta_2,...,\delta_k)$ such that $\sum_{j=1}^{k}\delta_j=n, \quad \delta_j\geq\delta_{j+1}$, where the δ_j are called the *parts* or *summands* of the partition. The number of partitions of n is denoted by $p(n)$. The generating function for the partition function $p(n)$ is given by Euler [12] as

$$\sum_{n=0}^{\infty}p(n)q^n=\frac{1}{(q;q)_\infty}=\frac{1}{\ell_1}, \qquad p(0)=1. \tag{5}$$

Euler [3] provided the following recurrence relation for finding the values of the partition function $p(n)$:

$$p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-p(n-12)-p(n-15)+...$$
$$+(-1)^k p\Big(n-k(3k-1)/2\Big)+(-1)^k p\Big(n-k(3k+1)/2\Big)+..=\begin{cases}1, & if\ n=0\\ 0, & \text{otherwise,}\end{cases}$$

where the numbers of the form $k(3k \pm 1)/2$ are called pentagonal numbers. For more results on recurrence relations of different partition functions, one can see [13, 22, 23, 28, 29]. Ramanujan [24] offered the following congruences for $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

In a letter to Hardy, Ramanujan [9] introduced the general partition function $p_r(n)$ as

$$\sum_{n=0}^{\infty} p_r(n)q^n = \frac{1}{(q;q)_\infty^r} = \frac{1}{\ell_1^r}, \tag{6}$$

where r is any non-zero integer and $p_1(n)$ is the partition function $p(n)$ defined in (5). For $r \geq 1$, $p_r(n)$ is the *color partition function* of an integer $n \geq 1$ in which each part in the partitions of n is assumed to have r different colors, and all of them are considered distinct.

To prove recurrence relations, we will also use the restricted partition functions $\bar{p}_r(n)$ and ${b_k}^r(n)$, where $\bar{p}_r(n)$ denotes the number of overpartitions of n with r colors and ${b_k}^r(n)$ denotes the number of k-regular partitions of n with r colors, respectively. The corresponding generating functions are given by

$$\sum_{n=0}^{\infty} \bar{p}_r(n)q^n = \left(\frac{\ell_2}{\ell_1^2}\right)^r, \tag{7}$$

$$\sum_{n=0}^{\infty} b_k^r(n)q^n = \left(\frac{\ell_k}{\ell_1}\right)^r. \tag{8}$$

Throughout the paper, it is assumed that $\bar{p}_r(0) = 1$ and $b_k^r(0) = 1$, and $\bar{p}_r(n) = 0$ and $b_k^r(n) = 0$ if n is not a non-negative integer.

2. Preliminaries

Ramanujan defined the general theta function $f(c, d)$ [8, p. 34, (18.1)] as

$$f(c,d) = \sum_{m=-\infty}^{\infty} c^{m(m+1)/2} d^{m(m-1)/2}, \quad |cd| < 1.$$

Three useful special cases [8, p. 35, Entry 18] of $f(c, d)$ are given by

$$\phi(q) := f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = \frac{\ell_2^5}{\ell_1^2 \ell_4^2}, \tag{9}$$

$$\psi(q) := f(q, q^3) = \sum_{m=0}^{\infty} q^{m(m+1)/2} = \frac{\ell_2^2}{\ell_1}, \tag{10}$$

and

$$\mathfrak{f}(-q) := f(-q, -q^2) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2} = \ell_1. \tag{11}$$

The product representations on the right-hand side of Equations (9)-(11) are the consequences of the Jacobi Triple Product identity given by

$$f(c, d) = (-c; cd)_\infty (-d; cd)_\infty (cd; cd)_\infty.$$

By using elementary q-operations, it is easily seen that

$$\phi(-q) = \sum_{m=-\infty}^{\infty} (-q)^{m^2} = \frac{(q; q)_\infty}{(-q; q)_\infty} = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty} = \frac{\ell_1^2}{\ell_2}. \tag{12}$$

In some of the proofs, we will also use Jacobi's identity [8, p. 39, Entry 24] given by

$$\ell_1^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2}. \tag{13}$$

Lemma 1 ([11, Theorem 2.1])**.** *If p is an odd prime, then*

$$\psi(q) = \sum_{m=0}^{(p-3)/2} q^{(m^2+m)/2} f\left(q^{\left(p^2+(2m+1)p\right)/2}, q^{\left(p^2-(2m+1)p\right)/2}\right) + q^{(p^2-1)/8} \psi(q^{p^2}).$$

Furthermore, $\dfrac{m^2+m}{2} \not\equiv \dfrac{p^2-1}{8} \pmod{p}$ *for* $0 \le m \le \dfrac{p-3}{2}$.

Lemma 2 ([11, Theorem 2.2])**.** *If $p \ge 5$ is a prime, then*

$$\ell_1 = \sum_{\substack{t=-(p-1)/2 \\ t \ne (\pm p-1)/6}}^{(p-1)/2} (-1)^t q^{(3t^2+t)/2} f\left(-q^{\left(3p^2+(6t+1)p\right)/2}, -q^{\left(3p^2-(6t+1)p\right)/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} \ell_{p^2},$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \dfrac{(p-1)}{6}, & \text{if } p \equiv 1 \pmod{6} \\ \dfrac{(-p-1)}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Furthermore, if $-\dfrac{p-1}{2} \le t \le \dfrac{p-1}{2}$ *and* $t \ne \dfrac{\pm p-1}{6}$, *then* $\dfrac{3t^2+t}{2} \not\equiv \dfrac{p^2-1}{24} \pmod{p}$.

Lemma 3 ([2, Lemma 2.3])**.** *If p is an odd prime, then*

$$\ell_1^3 = \sum_{\substack{k=0 \\ k\neq (p-1)/2}}^{(p-1)} (-1)^k q^{k(k+1)/2} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn(pn+2k+1)/2} + p(-1)^{(p-1)/2} q^{(p^2-1)/8} \ell_{p^2}^3.$$

Furthermore, if $k \neq \dfrac{p-1}{2}$ *and* $0 \leq k \leq p-1$, *then* $\dfrac{k^2+k}{2} \not\equiv \dfrac{p^2-1}{8} \pmod{p}$.

Lemma 4. *We have*

$$\frac{\ell_2^2}{\ell_1} = \frac{\ell_6\ell_9^2}{\ell_3\ell_{18}} + q\frac{\ell_{18}^2}{\ell_9}, \tag{14}$$

$$\frac{\ell_4}{\ell_1} = \frac{\ell_{12}\ell_{18}^4}{\ell_3^3\ell_{36}^2} + q\frac{\ell_6^2\ell_9^3\ell_{36}}{\ell_3^4\ell_{18}^2} + 2q^2\frac{\ell_6\ell_{18}\ell_{36}}{\ell_3^3}. \tag{15}$$

Identity (14) is Equation (14.3.3) of [18]. The identity (15) follows from Equations (33.2.1) and (33.2.5) of [18].

In addition to the above identities, we need the following congruences, which are easy consequences of the binomial theorem. For any prime p, and positive integers n and t, we have

$$\ell_n^{tp} \equiv \ell_{np}^t \pmod{p}, \tag{16}$$

$$\ell_1^{2^t} \equiv \ell_2^{2^{t-1}} \pmod{2^t}. \tag{17}$$

In order to state our congruences, we will also use the Legendre symbol.

Let p be any odd prime and ω be any integer relatively prime to p. Then the Legendre symbol $\left(\dfrac{\omega}{p}\right)$ is defined by

$$\left(\frac{\omega}{p}\right) = \begin{cases} 1, & \text{if } \omega \text{ is a quadratic residue of } p \\ -1, & \text{if } \omega \text{ is a quadratic non-residue of } p. \end{cases}$$

We will also use the following notation: for any real number x,

$$\lfloor x \rfloor = k, \text{ where } k \leq x < k+1 \text{ and } k \text{ is an integer.}$$

3. Results on $v(q)$ and $\mu(q)$

Throughout the section, we set $\sum_{n=0}^{\infty} P_v(n)q^n = v(q)$, where $v(q)$ is as defined in Equation (4).

Theorem 1. *For any integer $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} P_v(2n+1)q^n = \frac{\ell_4^3}{\ell_1\ell_2}. \tag{18}$$

Proof. From Ramanujan's lost notebook [5, p. 280, Entry 12.5.1] and [21, p. 288], we note that

$$\mu(-q^2) + 4v(q) = \frac{(q^4;q^4)_\infty(-q^2;q^4)_\infty^3}{(q^2;q^4)_\infty^2(-q^4;q^4)_\infty^2} + 4q\frac{(q^8;q^8)_\infty(-q^4;q^4)_\infty}{(q^4;q^8)_\infty(q^2;q^4)_\infty}, \tag{19}$$

where $\mu(q)$ is defined as in Equation (1). Now, simplifying Equation (19), and then extracting the terms involving q^{2n+1}, dividing by q, and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} P_v(2n+1)q^n = \frac{(q^4;q^4)_\infty(-q^2;q^2)_\infty}{(q^2;q^4)_\infty(q;q^2)_\infty}. \tag{20}$$

Now, the desired result easily follows from Equation (20). □

Theorem 2. *For any integer $n \geq 0$, we have*

(i) $P_v(6n+5) \equiv 0 \pmod 3$.

(ii) Let $p \geq 3$ be any prime such that $\left(\frac{-2}{p}\right) = -1$. Then for any integer $\alpha \geq 0$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} P_v\left(2\cdot p^{2\alpha}n + \frac{3\cdot p^{2\alpha}+1}{4}\right)q^n \equiv \psi(q)\psi(q^2) \pmod 2, \tag{21}$$

$$P_v\left(2\cdot p^{2\alpha+2}n + 2\cdot p^{2\alpha+1}j + \frac{3\cdot p^{2\alpha+2}+1}{4}\right) \equiv 0 \pmod 2. \tag{22}$$

(iii) Let $p \geq 5$ be any prime such that $\left(\frac{-18}{p}\right) = -1$. Then for integers $\alpha \geq 0$ and $1 \leq j \leq (p-1)$, we have

$$\sum_{n=0}^{\infty} P_v\left(6\cdot p^{2\alpha}n + \frac{19\cdot p^{2\alpha}+1}{4}\right)q^n \equiv 3\ell_1\ell_6^3 \pmod 6, \tag{23}$$

$$P_v\left(6\cdot p^{2\alpha+2}n + 6\cdot p^{2\alpha+1}j + \frac{19\cdot p^{2\alpha+2}+1}{4}\right) \equiv 0 \pmod 6. \tag{24}$$

Proof. (i) Using Equations (14) and (15) in Equation (18), we obtain

$$\sum_{n=0}^{\infty} P_v(2n+1)q^n = \frac{\ell_{12}^2\ell_{18}^6}{\ell_3^3\ell_6\ell_{36}^3} + q\frac{\ell_{12}\ell_6\ell_9^3}{\ell_3^4} + 3q^2\frac{\ell_{12}\ell_{18}^3}{\ell_3^3} + q^3\frac{\ell_6^2\ell_9^3\ell_{36}^3}{\ell_3^4\ell_{18}^3} + 2q^4\frac{\ell_6\ell_{36}^3}{\ell_3^3}. \tag{25}$$

Extracting the terms involving q^{3n+2} from Equation (25), dividing by q^2, and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} P_v(6n+5)q^n = 3\frac{\ell_4\ell_6^3}{\ell_1^3}. \tag{26}$$

Hence, the result (i) follows immediately from Equation (26).

(ii) Using Equation (16) in Equation (18), we obtain

$$\sum_{n=0}^{\infty} P_v(2n+1)q^n \equiv \psi(q)\psi(q^2) \pmod 2,$$

which is the $\alpha = 0$ case of Equation (21). Assume that Equation (21) is true for some $\alpha \geq 0$. Now, using Lemma 1 in Equation (21), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} P_v\left(2\cdot p^{2\alpha}n + \frac{3\cdot p^{2\alpha}+1}{4}\right)q^n \\
&\equiv \Big[\sum_{m=0}^{(p-3)/2} q^{(m^2+m)/2} f\left(q^{\left(p^2+(2m+1)p\right)/2}, q^{\left(p^2-(2m+1)p\right)/2}\right) + q^{(p^2-1)/8}\psi(q^{p^2})\Big] \\
&\quad\times \Big[\sum_{k=0}^{(p-3)/2} q^{(k^2+k)} f\left(q^{\left(p^2+(2k+1)p\right)}, q^{\left(p^2-(2k+1)p\right)}\right) + \\
&\qquad q^{(p^2-1)/4}\psi(q^{2p^2})\Big] \pmod 2.
\end{aligned} \tag{27}$$

Consider the congruence

$$\left(\frac{m^2+m}{2}\right) + \left(k^2+k\right) \equiv 3\left(\frac{p^2-1}{8}\right) \pmod p,$$

which is equivalent to

$$(2m+1)^2 + 2(2k+1)^2 \equiv 0 \pmod p. \tag{28}$$

For $\left(\frac{-2}{p}\right) = -1$, the congruence given in Equation (28) has only one solution, $m = k = \frac{(p-1)}{2}$. Therefore, extracting the terms involving $q^{pn+3(p^2-1)/8}$ from Equation (27), dividing by $q^{3(p^2-1)/8}$, and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} P_v\left(2\cdot p^{2\alpha+1}n + \frac{3\cdot p^{2\alpha+2}+1}{4}\right)q^n \equiv \psi(q^p)\psi(q^{2p}) \pmod 2. \tag{29}$$

Extracting the terms involving q^{pn} and replacing q^p by q in Equation (29), we obtain

$$\sum_{n=0}^{\infty} P_v\left(2\cdot p^{2\alpha+2}n + \frac{3\cdot p^{2\alpha+2}+1}{4}\right)q^n \equiv \psi(q)\psi(q^2) \pmod 2,$$

which is the $\alpha+1$ case of Equation (21). Hence, by the method of induction, we complete the proof of Equation (21). Again, extracting the terms involving q^{pn+j}, $1 \leq j \leq (p-1)$ from Equation (29), we arrive at Equation (22).

(iii) Using Equation (17) in Equation (26), we obtain

$$\sum_{n=0}^{\infty} P_v(6n+5)q^n \equiv 3\ell_1\ell_6^3 \pmod 6,$$

which is the $\alpha = 0$ case of Equation (23). Now, proceeding in the same way as in (ii) of Theorem 2, we arrive at Equations (23) and (24). □

Theorem 3. *For any integer $n \geq 0$, we have*

$$P_v(2n+1) = \sum_{m=0}^{\infty} b_4^1\Big(n - m(m+1)\Big),$$

where $\sum_{n=0}^{\infty} P_v(n)q^n = v(q)$, and $v(q)$ and $b_4^1(n)$ are as defined in (4) and (8), respectively.

Proof. From Equation (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} P_v(2n+1)q^n &= \frac{\ell_4}{\ell_1} \cdot \frac{\ell_4^2}{\ell_2} \\ &= \left(\sum_{n=0}^{\infty} b_4^1(n)q^n\right)\psi(q^2) \\ &= \left(\sum_{n=0}^{\infty} b_4^1(n)q^n\right)\left(\sum_{m=0}^{\infty} q^{m(m+1)}\right) \\ &= \left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_4^1\Big(n-m(m+1)\Big)\right)q^n. \qquad (30)\end{aligned}$$

On comparing the coefficients of q^n in Equation (30), we arrive at the desired result. □

Theorem 4. *For any integer $n \geq 0$, we have*

$$\begin{aligned}P_v(6n+5) = 3b_6^3(n) &+ 3\sum_{m=1}^{\infty}(-1)^m b_6^3\Big(n-2m(3m+1)\Big) \\ &+ 3\sum_{m=1}^{\infty}(-1)^m b_6^3\Big(n-2m(3m-1)\Big),\end{aligned}$$

where $\sum_{n=0}^{\infty} P_v(n)q^n = v(q)$, and $v(q)$ and $b_6^3(n)$ are as defined in (4) and (8), respectively.

Proof. From Equation (26), we note that

$$\sum_{n=0}^{\infty} P_v(6n+5)q^n = 3\left(\frac{\ell_6}{\ell_1}\right)^3 \ell_4. \tag{31}$$

Employing Equations (8) and (11) in Equation (31), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} P_v(6n+5)q^n &= 3\left(\sum_{n=0}^{\infty} b_6^3(n)q^n\right)\left(\sum_{m=-\infty}^{\infty}(-1)^m q^{2m(3m-1)}\right)\\ &= 3\left(\sum_{n=0}^{\infty} b_6^3(n)q^n\right)\\ &\qquad \times\left(1+\sum_{m=1}^{\infty}(-1)^m q^{2m(3m+1)}+\sum_{m=1}^{\infty}(-1)^m q^{2m(3m-1)}\right)\\ &= 3\sum_{n=0}^{\infty} b_6^3(n)q^n + 3\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(-1)^m b_6^3\Big(n-2m(3m+1)\Big)q^n\\ &\qquad + 3\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(-1)^m b_6^3\Big(n-2m(3m-1)\Big)q^n. \end{aligned} \tag{32}$$

On comparing the coefficients of q^n in Equation (32), we arrive at the desired result. □

Remark 1. Let $\mu(q)$ be as defined in Equation (1) and $\sum_{n=0}^{\infty} P_\mu(n)q^n = \mu(q)$. Then from Equations (6) and (19), we have

$$\sum_{n=0}^{\infty} P_\mu(n)q^n \equiv \frac{1}{\ell_1^3} = \sum_{n=0}^{\infty} p_3(n)q^n \pmod 4,$$

which implies $P_\mu(n) \equiv p_3(n) \pmod 4$.

4. Results on $\sigma(q)$

In this section, we set $\sum_{n=0}^{\infty} P_\sigma(n)q^n = \sigma(q)$, where $\sigma(q)$ is as defined in Equation (2).

Theorem 5. *For any integer $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} P_\sigma(2n+1)q^n = \frac{\ell_2^2\ell_6^2}{\ell_1^2\ell_3}. \tag{33}$$

Proof. Ramanujan listed the linear relation connecting the sixth-order mock theta functions in [15]:

$$\nu(q^2) - \sigma(-q) = q\frac{\ell_4^2\ell_{12}^2}{\ell_2^2\ell_6}, \tag{34}$$

where $\nu(q)$ is the sixth-order mock theta function defined by

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q;q)_{2n+1}}{(q;q^2)_{n+1}}.$$

Replacing q by $-q$ in Equation (34), we obtain

$$\sigma(q) = \nu(q^2) + q\frac{\ell_4^2\ell_{12}^2}{\ell_2^2\ell_6}. \tag{35}$$

Extracting the terms involving q^{2n+1} from Equation (35), dividing by q, and replacing q^2 by q, we arrive at the desired result. □

Theorem 6. *Let $p \geq 5$ be any prime such that $\left(\frac{-2}{p}\right) = -1$ and j be any integer with $1 \leq j \leq (p-1)$. Then for integers $n, \alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} P_\sigma\left(2\cdot p^{2\alpha}n + \frac{11\cdot p^{2\alpha}+1}{12}\right)q^n \equiv \ell_2\psi(q^3) \pmod{2}, \tag{36}$$

$$P_\sigma\left(2\cdot p^{2\alpha+2}n + 2\cdot p^{2\alpha+1}j + \frac{11\cdot p^{2\alpha+2}+1}{12}\right) \equiv 0 \pmod{2}. \tag{37}$$

Proof. Employing Equations (10) and (16) in Equation (33), we obtain

$$\sum_{n=0}^{\infty} P_\sigma(2n+1)q^n \equiv \ell_2\psi(q^3) \pmod{2},$$

which is the $\alpha = 0$ case of Equation (36). Now, proceeding in the same way as in (ii) of Theorem 2, we arrive at Equations (36) and (37). □

Theorem 7. *For any integer $n \geq 0$, we have*

$$P_\sigma(2n+1) = \sum_{m=0}^{\infty} b_2^2\left(n - \frac{3m^2+3m}{2}\right),$$

where $\sum_{n=0}^{\infty} P_\sigma(n)q^n = \sigma(q)$, and $\sigma(q)$ and $b_2^2(n)$ are as defined in (2) *and* (8)*, respectively.*

Proof. From Equation (33), we note that

$$\sum_{n=0}^{\infty} P_\sigma(2n+1)q^n = \left(\frac{\ell_2}{\ell_1}\right)^2 \psi(q^3). \tag{38}$$

Employing Equations (8) and (10) in Equation (38), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} P_\sigma(2n+1)q^n &= \left(\sum_{n=0}^{\infty} b_2^2(n)q^n\right)\left(\sum_{m=0}^{\infty} q^{3m(m+1)/2}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} b_2^2\left(n-\frac{3m^2+3m}{2}\right)\right)q^n. \end{aligned}\tag{39}$$

On comparing the coefficients of q^n in Equation (39), we arrive at the desired result. □

5. Results on $\lambda(q)$

The following three identities from [19] will be useful in this section:

$$\sum_{n=0}^{\infty} P_\lambda(2n)q^n = \frac{\ell_2^3\ell_3^2}{\ell_1^3\ell_6}, \tag{40}$$

$$\sum_{n=0}^{\infty} P_\lambda(6n+2)q^n = 3\frac{\ell_3^5}{\ell_6}\left(\frac{\ell_2}{\ell_1^2}\right)^3, \tag{41}$$

$$\sum_{n=0}^{\infty} P_\lambda(6n+4)q^n = \frac{\ell_2^2\ell_3^2\ell_6^2}{\ell_1^5}, \tag{42}$$

where

$$\sum_{n=0}^{\infty} P_\lambda(n)q^n = \lambda(q), \text{ and } \lambda(q) \text{ is as defined in Equation (3).}$$

Theorem 8. *For any integer $n \geq 0$, we have*

$$P_\lambda(2n) = b_2^3(n) + 2\sum_{k=1}^{\infty}(-1)^k b_2^3(n-3k^2),$$

where $b_2^3(n)$ is as defined in (8).

Proof. From Equation (40), we note that

$$\sum_{n=0}^{\infty} P_\lambda(2n)q^n = \left(\frac{\ell_2}{\ell_1}\right)^3 \phi(-q^3). \tag{43}$$

Employing Equations (8) and (12) in Equation (43), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} P_\lambda(2n)q^n &= \left(\sum_{n=0}^{\infty} b_2^3(n)q^n\right)\left(\sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2}\right)\\ &= \left(\sum_{n=0}^{\infty} b_2^3(n)q^n\right)\left(1+2\sum_{k=1}^{\infty}(-1)^k q^{3k^2}\right)\end{aligned}$$

$$= \sum_{n=0}^{\infty} b_2^3(n)q^n + 2\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k b_2^3(n-3k^2)q^n. \tag{44}$$

On comparing the coefficients of q^n in Equation (44), we arrive at the desired result. □

Theorem 9. *For any integer $n \geq 0$, we have*

$$\begin{aligned}P_\lambda(6n+2) &= 3\sum_{m=0}^{\infty}(-1)^m(2m+1)\bar{p}_3\left(n-\frac{3m(m+1)}{2}\right)\\ &\quad + 6\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{m+k}(2m+1)\bar{p}_3\left(n-3k^2-\frac{3m(m+1)}{2}\right),\end{aligned}$$

where $\bar{p}_3(n)$ is as defined in (7).

Proof. From Equation (41), we note that

$$\sum_{n=0}^{\infty} P_\lambda(6n+2)q^n = 3\left(\frac{\ell_2}{\ell_1^2}\right)^3 \phi(-q^3)\ell_3^3. \tag{45}$$

Employing Equations (7), (12), and (13) in Equation (45), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} P_\lambda(6n+2)q^n &= 3\left(\frac{\ell_2}{\ell_1^2}\right)^3\left(1+2\sum_{k=1}^{\infty}(-1)^k q^{3k^2}\right)\\ &\quad\times\left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{3m(m+1)/2}\right)\\ &= 3\left(\sum_{n=0}^{\infty}\bar{p}_3(n)q^n\right)\left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{3m(m+1)/2}\right)\\ &\quad + 6\left(\sum_{n=0}^{\infty}\bar{p}_3(n)q^n\right)\left(\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{m+k}(2m+1)q^{3k^2+3m(m+1)/2}\right)\end{aligned}$$

$$= 3\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m(2m+1)\bar{p}_3\left(n-\frac{3m(m+1)}{2}\right)q^n$$
$$+ 6\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{m+k}(2m+1)\bar{p}_3\left(n-3k^2-\frac{3m(m+1)}{2}\right)q^n. \tag{46}$$

On comparing the coefficients of q^n in Equation (46), we arrive at the desired result. □

Theorem 10. *For any integer $n \geq 0$, we have*

$$P_\lambda(6n+4) = \sum_{m=0}^{\infty}\sum_{k=0}^{\lfloor n-\frac{m(m+1)}{2}\rfloor} b_3^2\Big(n-\frac{m(m+1)}{2}-k\Big)b_6^2(k),$$

where $b_3^2(n)$ and $b_6^2(n)$ are as defined in (8).

Proof. From Equation (42), we note that

$$\sum_{n=0}^{\infty}P_\lambda(6n+4)q^n = \left(\frac{\ell_3}{\ell_1}\right)^2\left(\frac{\ell_6}{\ell_1}\right)^2\left(\frac{\ell_2^2}{\ell_1}\right). \tag{47}$$

Using Equations (8) and (10) in Equation (47), we obtain

$$\sum_{n=0}^{\infty}P_\lambda(6n+4)q^n = \left(\sum_{n=0}^{\infty}b_3^2(n)q^n\right)\left(\sum_{k=0}^{\infty}b_6^2(k)q^k\right)\left(\sum_{m=0}^{\infty}q^{m(m+1)/2}\right)$$
$$= \left(\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n\rfloor}b_3^2(n-k)b_6^2(k)q^n\right)\left(\sum_{m=0}^{\infty}q^{m(m+1)/2}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\lfloor n-\frac{m(m+1)}{2}\rfloor} b_3^2\Big(n-\frac{m(m+1)}{2}-k\Big)b_6^2(k)q^n. \tag{48}$$

On comparing the coefficients of q^n in Equation (48), we arrive at the desired result. □

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