AN INTEGRAL REPRESENTATION OF k-JACOBSTHAL AND k-JACOBSTHAL–LUCAS NUMBERS

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Abstract

In this paper, we provide new integral representations of k-Jacobsthal and k-Jacobsthal–Lucas numbers. Using Binet's formulas for these numbers, we establish several identities that are provable by simple integral calculus. Our results not only generalize the integral representations of the Jacobsthal and Jacobsthal–Lucas numbers but also apply to all the companion numbers of k-Jacobsthal numbers.

1. Introduction

Several ways are available to represent integer sequences defined by a second-order linear recurrence relation without the recurrence relation, including Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas numbers, one of which is an integral representation; for examples in recent years, see [6, 10, 13, 16, 17, 18, 19, 20, 23, 24]. Recall that the *Jacobsthal numbers* J_n are defined by the recurrence relations

$$J_0 = 0, J_1 = 1, \quad \text{and} \quad J_n = J_{n-1} + 2J_{n-2}, \quad n \geq 2.$$

The *Jacobsthal–Lucas numbers* j_n are defined by the recurrence relations

$$j_0 = 2, j_1 = 1, \quad \text{and} \quad j_n = j_{n-1} + 2j_{n-2}, \quad n \geq 2.$$

The Jacobsthal and Jacobsthal–Lucas numbers are like the related Fibonacci and Lucas numbers; they are a specific type of Lucas sequences [15]; see more details

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in [12]. In 2024, İpek [13] presented integral representations of the Jacobsthal and Jacobsthal–Lucas numbers as follows:

$$J_{\ell n} = \frac{nJ_\ell}{2^n} \int_{-1}^{1} (j_\ell + 3J_\ell x)^{n-1} dx$$

and

$$j_{\ell n} = \frac{1}{2^n} \int_{-1}^{1} (j_\ell + 3(n+1)J_\ell x)(j_\ell + 3J_\ell x)^{n-1} dx,$$

where ℓ and n are non-negative integers.

There are some generalizations of Jacobsthal and Jacobsthal–Lucas numbers defined in different ways; see for instance [1, 2, 3, 4, 5, 8, 14, 22, 25]. In 2013, Jhala et al. [14] introduced and studied a generalization of the Jacobsthal numbers, the so-called *k-Jacobsthal numbers*, as follows. For k is a positive integer, the k-Jacobsthal numbers $J_{k,n}$ are defined by the recurrence relations

$$J_{k,0} = 0, J_{k,1} = 1, \quad \text{and} \quad J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2}, \quad n \geq 2. \tag{1}$$

In 2014, Campos et al. [5] defined the *k-Jacobsthal–Lucas numbers* $j_{k,n}$ by the recurrence relations

$$j_{k,0} = 2, j_{k,1} = k, \quad \text{and} \quad j_{k,n} = kj_{k,n-1} + 2j_{k,n-2}, \quad n \geq 2. \tag{2}$$

We can see that the classical Jacobsthal and Jacobsthal–Lucas numbers are obtained for $k = 1$.

In this paper, we follow in the footsteps of Stewart [23] and İpek [13], giving new integral representations of k-Jacobsthal and k-Jacobsthal–Lucas numbers. Using Binet's formulas for these numbers, we establish several identities, applying simple integral calculus to prove them. Furthermore, we study all the companion numbers of k-Jacobsthal numbers that preserve the recurrence relation with arbitrary initial conditions, and give some new and well-known identities. Finally, we provide the integral representations of these numbers associated with the k-Jacobsthal and k-Jacobsthal–Lucas numbers.

2. Preliminaries

In this section, we give some results that are required for the proofs of the main results. The following identities, which rely on Binet's formulas. The recurrence relations (1) and (2) generate a characteristic equation of the form

$$r^2 - kr - 2 = 0. \tag{3}$$

Since $k \geq 1$, this equation has the roots

$$r_1 = \frac{k + \sqrt{k^2+8}}{2} \quad \text{and} \quad r_2 = \frac{k - \sqrt{k^2+8}}{2}.$$

Therefore, Binet's formulas for the k-Jacobsthal and k-Jacobsthal–Lucas numbers are

$$J_{k,n} = \frac{1}{\Delta_k}\left(\sigma_k^n - \frac{(-2)^n}{\sigma_k^n}\right), \tag{4}$$

and

$$j_{k,n} = \sigma_k^n + \frac{(-2)^n}{\sigma_k^n}, \tag{5}$$

where $\Delta_k = \sqrt{k^2+8}$ and $\sigma_k = \frac{k+\Delta_k}{2}$; see also [14, Proposition 2.1] and [5, Proposition 2.1]. Equipped with Equations (4) and (5), one may readily verify the algebraic relations in the next two lemmas.

Lemma 1. *Let k and n be non-negative integers with $k \neq 0$. Then the following hold:*

(i) $j_{k,n} + \Delta_k J_{k,n} = 2\sigma_k^n$;

(ii) $j_{k,n} - \Delta_k J_{k,n} = 2\frac{(-2)^n}{\sigma_k^n}$;

(iii) $j_{k,n}^2 - \Delta_k^2 J_{k,n}^2 = (-2)^{n+2}$.

Lemma 2. *Let k, m, and n be non-negative integers with $k \neq 0$. Then the following hold:*

(i) $2J_{k,m+n} = J_{k,m}j_{k,n} + J_{k,n}j_{k,m}$;

(ii) $2j_{k,m+n} = j_{k,m}j_{k,n} + \Delta_k^2 J_{k,m}J_{k,n}$.

Remark 1. Lemmas 1(iii) and 2(i) are presented in [26].

3. The Integral Representations of k-Jacobsthal and k-Jacobsthal–Lucas Numbers

In this section, thanks to the technique of [23], we obtain new integral representations of k-Jacobsthal and k-Jacobsthal–Lucas numbers. We start with the integral representation of k-Jacobsthal numbers $J_{k,\ell n}$ based on the numbers $J_{k,\ell}$ and $j_{k,\ell}$.

Theorem 1. *Let k, ℓ, and n be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2+8}$. The k-Jacobsthal numbers $J_{k,\ell n}$ are represented by*

$$J_{k,\ell n} = \frac{nJ_{k,\ell}}{2^n}\int_{-1}^{1}(j_{k,\ell} + \Delta_k J_{k,\ell}x)^{n-1}dx. \tag{6}$$

Proof. For $n = 0$ or $\ell = 0$, we are done. Let us assume that $\ell, n > 0$. Let $u(x) = j_{k,\ell} + \Delta_k J_{k,\ell} x$. Then $du = \Delta_k J_{k,\ell} dx$. Using integration by substitution leads to

$$\begin{aligned}\int_{-1}^{1} (j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^{n-1} dx &= \frac{1}{n\Delta_k\, J_{k,\ell}} \left[(j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^n\right]_{-1}^{1} \\ &= \frac{1}{n\Delta_k\, J_{k,\ell}} \left[(j_{k,\ell} + \Delta_k\, J_{k,\ell})^n - (j_{k,\ell} - \Delta_k\, J_{k,\ell})^n\right].\end{aligned}$$

Applying (i) and (ii) of Lemma 1 with n replaced by ℓ, we get

$$\begin{aligned}\int_{-1}^{1} (j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^{n-1} dx &= \frac{1}{n\Delta_k\, J_{k,\ell}} \left[\left(2\sigma_k^\ell\right)^n - \left(2\frac{(-2)^\ell}{\sigma_k^\ell}\right)^n\right] \\ &= \frac{2^n}{nJ_{k,\ell}} \left[\frac{1}{\Delta_k}\left(\sigma_k^{\ell n} - \frac{(-2)^{\ell n}}{\sigma_k^{\ell n}}\right)\right].\end{aligned}$$

It follows from Equation (4), after replacing n by ℓn, that

$$\int_{-1}^{1} (j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^{n-1} dx = \frac{2^n}{nJ_{k,\ell}} J_{k,\ell n}.$$

Then Equation (6) has been proved. □

The integral representations of the k-Jacobsthal numbers for even and odd orders are shown as follows:

Theorem 2. *Let k and n be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2 + 8}$.*

(i) The k-Jacobsthal numbers $J_{k,2n}$ are represented by

$$J_{k,2n} = \frac{kn}{2^n} \int_{-1}^{1} (k^2 + 4 + k\Delta_k\, x)^{n-1} dx. \tag{7}$$

(ii) The k-Jacobsthal numbers $J_{k,2n+1}$ are represented by

$$J_{k,2n+1} = \frac{1}{2^{n+1}} \int_{-1}^{1} (nk^2 + k^2 + 4 + k(n+1)\Delta_k\, x)(k^2 + 4 + k\Delta_k\, x)^{n-1} dx.$$

Proof. (i) Notice that $J_{k,2} = k$ and $j_{k,2} = k^2 + 4$. Setting $\ell = 2$ in (6), we have

$$J_{k,2n} = \frac{kn}{2^n} \int_{-1}^{1} (k^2 + 4 + k\Delta_k\, x)^{n-1} dx.$$

(ii) Reindexing n by $n + 1$ in (7), we get

$$J_{k,2n+2} = \frac{k(n+1)}{2^{n+1}} \int_{-1}^{1} (k^2 + 4 + k\Delta_k\, x)^{n} dx. \tag{8}$$

Using $J_{k,2n+2} = kJ_{k,2n+1} + 2J_{k,2n}$ with Equations (7) and (8), we obtain

$$\begin{aligned} J_{k,2n+1} &= \frac{1}{k}\left(J_{k,2n+2} - 2J_{k,2n}\right) \\ &= \frac{n+1}{2^{n+1}} \int_{-1}^{1} (k^2+4+k\Delta_k\, x)^n dx - \frac{2n}{2^n} \int_{-1}^{1} (k^2+4+k\Delta_k\, x)^{n-1} dx \\ &= \frac{1}{2^{n+1}} \int_{-1}^{1} \big((n+1)(k^2+4+k\Delta_k\, x) - 4n\big)\, (k^2+4+k\Delta_k\, x)^{n-1} dx \\ &= \frac{1}{2^{n+1}} \int_{-1}^{1} (nk^2+k^2+4+k(n+1)\Delta_k\, x)(k^2+4+k\Delta_k\, x)^{n-1} dx. \end{aligned}$$

This completes the proof. □

Setting $k = 1$ in Theorems 1 and 2, we have the following two results of [13].

Corollary 1 ([13])**.** *Let ℓ and n be non-negative integers. The Jacobsthal numbers $J_{\ell n}$ are represented by*

$$J_{\ell n} = \frac{nJ_\ell}{2^n} \int_{-1}^{1} (j_\ell + 3J_\ell x)^{n-1} dx.$$

Corollary 2 ([13])**.** *Let ℓ and n be non-negative integers.*

(i) The Jacobsthal numbers J_{2n} are represented by

$$J_{2n} = \frac{n}{2^n} \int_{-1}^{1} (5+3\,x)^{n-1} dx.$$

(ii) The Jacobsthal numbers J_{2n+1} are represented by

$$J_{2n+1} = \frac{1}{2^{n+1}} \int_{-1}^{1} (n+5+3(n+1)\,x)(5+3\,x)^{n-1} dx.$$

Next, we obtain integral representations for the k-Jacobsthal–Lucas numbers $j_{k,\ell n}$ based on the two numbers $J_{k,\ell}$ and $j_{k,\ell}$.

Theorem 3. *Let k, ℓ, and n be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2+8}$. The k-Jacobsthal–Lucas numbers $j_{k,\ell n}$ are represented by*

$$j_{k,\ell n} = \frac{1}{2^n} \int_{-1}^{1} (j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell} x)(j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^{n-1} dx. \tag{9}$$

Proof. For $n = 0$ or $\ell = 0$, it is easy to see that Equation (9) holds. We assume now that $\ell, n > 0$ and will solve Equation (9) using integration by parts. Let u and v be such that

$$u(x) = j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell} x \quad \text{and} \quad dv = (j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^{n-1} dx.$$

Then $du = (n+1)\Delta_k J_{k,\ell} dx$ and so

$$v = \int (j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^{n-1} dx = \frac{1}{n\Delta_k\, J_{k,\ell}} \left(j_{k,\ell} + \Delta_k\, J_{k,\ell} x\right)^n .$$

It follows that

$$\begin{aligned} I &= \frac{1}{2^n} \int_{-1}^{1} (j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell} x)(j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^{n-1} dx \\ &= \frac{1}{2^n n \Delta_k\, J_{k,\ell}} \left[(j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell} x) \left(j_{k,\ell} + \Delta_k\, J_{k,\ell} x\right)^n \right]_{-1}^{1} \\ &\quad - \frac{n+1}{n 2^n} \int_{-1}^{1} (j_{k,\ell} + \Delta_k J_{k,\ell} x)^n dx. \end{aligned} \tag{10}$$

Replacing n by $n+1$ in Equation (6), we may turn it into

$$J_{k,\ell n+\ell} = \frac{(n+1)J_{k,\ell}}{2^{n+1}} \int_{-1}^{1} (j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^n dx$$

and so

$$\frac{2J_{k,\ell n+\ell}}{nJ_{k,\ell}} = \frac{n+1}{n2^n} \int_{-1}^{1} (j_{k,\ell} + \Delta_k\, J_{k,\ell} x)^n dx.$$

This together with Equation (10) gives

$$\begin{aligned} I = & \frac{1}{2^n n \Delta_k\, J_{k,\ell}} \left[(j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell}) \left(j_{k,\ell} + \Delta_k\, J_{k,\ell}\right)^n \right] \\ & - \frac{1}{2^n n \Delta_k\, J_{k,\ell}} \left[(j_{k,\ell} - (n+1)\Delta_k\, J_{k,\ell}) \left(j_{k,\ell} - \Delta_k\, J_{k,\ell}\right)^n \right] - \frac{2J_{k,\ell n+\ell}}{nJ_{k,\ell}}. \end{aligned}$$

In view of Lemma 1(i)(ii) and Lemma 2(i), we have

$$\begin{aligned} I &= \frac{1}{2^n n \Delta_k\, J_{k,\ell}} \left[2^n \sigma_k^{\ell n} \left(j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell}\right) \right] \\ &\quad - \frac{1}{2^n n \Delta_k\, J_{k,\ell}} \left[2^n \frac{(-2)^{\ell n}}{\sigma_k^{\ell n}} \left(j_{k,\ell} - (n+1)\Delta_k\, J_{k,\ell}\right) \right] - \frac{2J_{k,\ell n+\ell}}{nJ_{k,\ell}} \\ &= \frac{1}{n\, J_{k,\ell}} \left[\frac{1}{\Delta_k} \left(\sigma_k^{\ell n} - \frac{(-2)^{\ell n}}{\sigma_k^{\ell n}} \right) j_{k,\ell} + (n+1) J_{k,\ell} \left(\sigma_k^{\ell n} + \frac{(-2)^{\ell n}}{\sigma_k^{\ell n}} \right) \right] - \frac{2J_{k,\ell n+\ell}}{nJ_{k,\ell}} \\ &= \frac{1}{n\, J_{k,\ell}} \left(J_{k,\ell n} j_{k,\ell} + (n+1) J_{k,\ell} j_{k,\ell n} \right) - \frac{2J_{k,\ell n+\ell}}{nJ_{k,\ell}} \\ &= j_{k,\ell n} + \frac{1}{n\, J_{k,\ell}} \left(J_{k,\ell n} j_{k,\ell} + J_{k,\ell} j_{k,\ell n} \right) - \frac{2J_{k,\ell n+\ell}}{nJ_{k,\ell}} \\ &= j_{k,\ell n}, \end{aligned}$$

which completes the proof. □

Setting $k=1$ in Theorem 3, we have the following result of [13].

Corollary 3 ([13])**.** *Let ℓ and n be non-negative integers. The Jacobsthal–Lucas numbers $j_{\ell n}$ are represented by*

$$j_{\ell n} = \frac{1}{2^n}\int_{-1}^{1}(j_\ell + 3(n+1)J_\ell x)(j_\ell + 3J_\ell x)^{n-1}dx.$$

Finally, we establish integral representations for general forms $J_{k,\ell n+r}$ and $j_{k,\ell n+r}$ by using both $J_{k,\ell n}$ and $j_{k,\ell n}$ shown in the following two theorems.

Theorem 4. *Let k, ℓ, n, and r be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2+8}$. The k-Jacobsthal numbers $J_{k,\ell n+r}$ are represented by*

$$\begin{aligned} &J_{k,\ell n+r} \\ &\quad = \frac{1}{2^{n+1}}\int_{-1}^{1}(nJ_{k,\ell}j_{k,r} + J_{k,r}j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell}J_{k,r}x)\,(j_{k,\ell} + \Delta_k\, J_{k,\ell}x)^{n-1}dx. \end{aligned}$$

Proof. Using (i) of Lemma 2 with m and n replaced by ℓn and r, respectively, we get

$$J_{k,\ell n+r} = \frac{1}{2}J_{k,\ell n}j_{k,r} + \frac{1}{2}J_{k,r}j_{k,\ell n}.$$

An application of Theorems 1 and 3 leads us to

$$\begin{aligned} &J_{k,\ell n+r} \\ &\quad = \frac{1}{2}\left(\frac{nJ_{k,\ell}}{2^n}\int_{-1}^{1}(j_{k,\ell} + \Delta_k\, J_{k,\ell}x)^{n-1}dx\right)j_{k,r} \\ &\qquad + \frac{1}{2}J_{k,r}\left(\frac{1}{2^n}\int_{-1}^{1}(j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell}x)(j_{k,\ell} + \Delta_k\, J_{k,\ell}x)^{n-1}dx\right) \\ &\quad = \frac{1}{2^{n+1}}\int_{-1}^{1}(nJ_{k,\ell}j_{k,r} + J_{k,r}j_{k,\ell} + (n+1)\Delta_k\, J_{k,\ell}J_{k,r}x)\,(j_{k,\ell} + \Delta_k\, J_{k,\ell}x)^{n-1}dx. \end{aligned}$$

This completes the proof. □

Remark 2. Notice that the integral representations of the k-Jacobsthal numbers for even and odd orders given in Theorem 2 are recovered from Theorem 4 by setting $(\ell, r) = (2, 0)$ and $(\ell, r) = (2, 1)$, respectively.

Setting $k=1$ in Theorem 4, we have the following result of [13].

Corollary 4 ([13])**.** *Let ℓ, n, and r be non-negative integers. The Jacobsthal numbers $J_{\ell n+r}$ are represented by*

$$J_{\ell n+r} = \frac{1}{2^{n+1}}\int_{-1}^{1}(nJ_\ell j_r + J_r j_\ell + 3(n+1)\,J_\ell J_r x)\,(j_\ell + 3\,J_\ell x)^{n-1}dx.$$

Theorem 5. *Let k, ℓ, n, and r be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2+8}$. The k-Jacobsthal–Lucas numbers $j_{k,\ell n+r}$ are represented by*

$$j_{k,\ell n+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} \left(n\Delta_k^2 J_{k,\ell} J_{k,r} + j_{k,\ell} j_{k,r} + (n+1)\Delta_k J_{k,\ell} j_{k,r} x\right) \left(j_{k,\ell} + \Delta_k J_{k,\ell} x\right)^{n-1} dx.$$

Proof. Using (ii) of Lemma 2 with m and n replaced by ℓn and r, respectively, we get

$$j_{k,\ell n+r} = \frac{1}{2} j_{k,\ell n} j_{k,r} + \frac{1}{2}\left(\Delta_k^2 J_{k,\ell n} J_{k,r}\right).$$

This together with Theorems 1 and 3 completes the proof. □

Using the same idea as in Theorem 2, or setting $(\ell, r) = (2, 0)$ and $(\ell, r) = (2, 1)$ in Theorem 5, we also have the following integral representations of the k-Jacobsthal–Lucas numbers for even and odd orders.

Theorem 6. *Let k and n be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2+8}$.*

(i) The k-Jacobsthal–Lucas numbers $j_{k,2n}$ can be represented by the integral

$$j_{k,2n} = \frac{1}{2^n} \int_{-1}^{1} (k^2 + 4 + k(n+1)\Delta_k x)(k^2 + 4 + k\Delta_k x)^{n-1} dx.$$

(ii) The k-Jacobsthal–Lucas numbers $j_{k,2n+1}$ can be represented by the integral

$$j_{k,2n+1} = \frac{k}{2^{n+1}} \int_{-1}^{1} \left(n\Delta_k^2 + k^2 + 4 + k(n+1)\Delta_k x\right) \left(k^2 + 4 + k\Delta_k x\right)^{n-1} dx.$$

Setting $k = 1$ in Theorem 5, we have the following result of [13].

Corollary 5 ([13])**.** *For ℓ, n, and r are non-negative integers, the Jacobsthal–Lucas numbers $j_{\ell n+r}$ can be represented by the integral*

$$j_{\ell n+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} (9nJ_\ell J_r + j_\ell j_r + 3(n+1)J_\ell j_r x)(j_\ell + 3J_\ell x)^{n-1} dx.$$

4. The Companion k-Jacobsthal Numbers

In this section, we study all the companion numbers of k-Jacobsthal numbers that preserve the recurrence relation with arbitrary initial conditions and establish some new and well-known identities. Further, we give integral representations of these numbers associated with the k-Jacobsthal and k-Jacobsthal–Lucas numbers. We define the *companion k-Jacobsthal numbers* $\mathcal{J}_{k,n} = \mathcal{J}_{k,n}(a,b)$ by

$$\mathcal{J}_{k,0} = a, \mathcal{J}_{k,1} = b, \quad \text{and} \quad \mathcal{J}_{k,n} = k\mathcal{J}_{k,n-1} + \mathcal{J}_{k,n-2}, \quad n \geq 2,$$

where a and b are arbitrary non-negative integers. Note that $\mathcal{J}_{k,n}$ corresponds to a special case of Horadam numbers [11]. The first companion k-Jacobsthal numbers are:

$$\begin{aligned}
\mathcal{J}_{k,0} &= a \\
\mathcal{J}_{k,1} &= b \\
\mathcal{J}_{k,2} &= bk + 2a \\
\mathcal{J}_{k,3} &= bk^2 + 2ak + 2b \\
\mathcal{J}_{k,4} &= bk^3 + 2ak^2 + 4bk + 4a \\
\mathcal{J}_{k,5} &= bk^4 + 2ak^3 + 6bk^2 + 8ak + 4b \\
\mathcal{J}_{k,6} &= bk^5 + 2ak^4 + 8bk^3 + 12ak^2 + 12bk + 8a \\
\mathcal{J}_{k,7} &= bk^6 + 2ak^5 + 10bk^4 + 16ak^3 + 24bk^2 + 24ak + 8b.
\end{aligned}$$

Some particular cases of the previous definition are:

1. the k-Jacobsthal numbers [14] $J_{k,n} = \mathcal{J}_{k,n}(0,1)$;
2. the k-Jacobsthal–Lucas numbers [5] $j_{k,n} = \mathcal{J}_{k,n}(2,k)$;
3. the associated k-Jacobsthal numbers [22] $A_{k,n} = \mathcal{J}_{k,n}(1,1)$;
4. the associated k-Jacobsthal–Lucas numbers [22] $B_{k,n} = \mathcal{J}_{k,n}(2,k+4)$;
5. the Jacobsthal-like numbers [21] $V_n = \mathcal{J}_{1,n}(2,2)$.

Theorem 7 (Binet's formulas)**.** *Let k and n be non-negative integers with $k \neq 0$, $\Delta_k = \sqrt{k^2+8}$ and $\sigma_k = \frac{k+\Delta_k}{2}$. The companion k-Jacobsthal numbers $\mathcal{J}_{k,n}$ are given by*

$$\mathcal{J}_{k,n} = \left(\frac{2b - ak + a\Delta_k}{2\Delta_k}\right)\sigma_k^n + \left(\frac{ak - 2b + a\Delta_k}{2\Delta_k}\right)\frac{(-2)^n}{\sigma_k^n}. \tag{11}$$

Proof. The roots of the characteristic equation (3) are

$$r_1 = \frac{k+\Delta_k}{2} \quad \text{and} \quad r_2 = \frac{k-\Delta_k}{2}.$$

Note that $r_2 < 0 < r_1$, $r_1 r_2 = -2$, $r_1 + r_2 = k$, and $r_1 - r_2 = \Delta_k$. Therefore, the general term of $\mathcal{J}_{k,n}$ can be expressed in the form:

$$\mathcal{J}_{k,n} = C_1 r_1^n + C_2 r_2^n$$

for some coefficients C_1 and C_2. Since $\mathcal{J}_{k,0} = a$ and $\mathcal{J}_{k,1} = b$, we get

$$C_1 + C_2 = a \quad \text{and} \quad C_1 r_1 + C_2 r_2 = b.$$

It can be shown that

$$C_1 = \frac{b - ar_2}{r_1 - r_2} = \frac{2b - ak + a\Delta_k}{2\Delta_k} \quad \text{and} \quad C_2 = \frac{ar_1 - b}{r_1 - r_2} = \frac{ak + a\Delta_k - 2b}{2\Delta_k}.$$

Let $\sigma_k = \frac{k+\Delta_k}{2}$. Then $r_1 = \sigma_k$ and $r_2 = \frac{-2}{\sigma_k}$. Therefore, Equation (11) has been proved. □

Theorem 8 (Asymptotic behavior)**.** *Let k be a positive integer. Then*

$$\lim_{n\to\infty} \frac{\mathcal{J}_{k,n+1}}{\mathcal{J}_{k,n}} = \sigma_k.$$

Proof. By using Equation (11), we have

$$\lim_{n\to\infty} \frac{\mathcal{J}_{k,n+1}}{\mathcal{J}_{k,n}} = \lim_{n\to\infty} \frac{\left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\sigma_k^{n+1} + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(-2)^{n+1}}{\sigma_k^{n+1}}}{\left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\sigma_k^{n} + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(-2)^{n}}{\sigma_k^{n}}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\sigma_k + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(-2)^{n}}{\sigma_k^{2n}} \cdot \frac{(-2)}{\sigma_k}}{\left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right) + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(-2)^{n}}{\sigma_k^{2n}}}. \tag{12}$$

Since σ_k is the root of Equation (3), we have $\sigma_k^2 = k\sigma_k + 2 > 2$ and so $\left|\frac{-2}{\sigma_k^2}\right| < 1$. Then

$$\lim_{n\to\infty} \frac{(-2)^n}{\sigma_k^{2n}} = \lim_{n\to\infty} \left(\frac{-2}{\sigma_k^2}\right)^n = 0.$$

This together with Equation (12) gives

$$\lim_{n\to\infty} \frac{\mathcal{J}_{k,n+1}}{\mathcal{J}_{k,n}} = \sigma_k.$$

This completes the proof. □

The companion k-Jacobsthal numbers are associated with the k-Jacobsthal and k-Jacobsthal–Lucas numbers in the following results.

Theorem 9. *Let k and n be non-negative integers with $k \neq 0$. Then*

$$\mathcal{J}_{k,n} = \frac{a}{2} j_{k,n} + \frac{2b - ak}{2} J_{k,n}. \tag{13}$$

Proof. It follows from (i) and (ii) of Lemma 1 and Equation (11) that

$$\begin{aligned}\mathcal{J}_{k,n} &= \left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\sigma_k^n + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(-2)^n}{\sigma_k^n}\\ &= \left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\left(\frac{j_{k,n}+\Delta_k\, J_{k,n}}{2}\right) + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\left(\frac{j_{k,n}-\Delta_k\, J_{k,n}}{2}\right)\\ &= \frac{a}{2} j_{k,n} + \frac{2b-ak}{2}\, J_{k,n}.\end{aligned}$$

This completes the proof. □

Theorem 10 (Generating functions)**.** *Let k be a positive integer. The generating function for the companion k-Jacobsthal numbers is*

$$\sum_{n=0}^{\infty} \mathcal{J}_{k,n} x^n = \frac{a+(b-ak)x}{1-kx-2x^2}.$$

Proof. Notice that the generating functions for the k-Jacobsthal numbers [14] and k-Jacobsthal–Lucas numbers [5] are

$$\sum_{n=0}^{\infty} J_{k,n} x^n = \frac{x}{1-kx-2x^2} \quad \text{and} \quad \sum_{n=0}^{\infty} j_{k,n} x^n = \frac{2-kx}{1-kx-2x^2}.$$

Then, by Equation (13),

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{J}_{k,n} x^n &= \frac{a}{2}\sum_{n=0}^{\infty} j_{k,n} x^n + \frac{2b-ak}{2}\sum_{n=0}^{\infty} J_{k,n} x^n\\ &= \frac{a}{2}\left(\frac{2-kx}{1-kx-2x^2}\right) + \frac{2b-ak}{2}\left(\frac{x}{1-kx-2x^2}\right)\\ &= \frac{a+(b-ak)x}{1-kx-2x^2}.\end{aligned}$$

This completes the proof. □

Theorem 11 (Catalan's identity)**.** *Let k, n, and r be positive integers. Then*

$$\mathcal{J}_{k,n-r}\mathcal{J}_{k,n+r} - \mathcal{J}_{k,n}^2 = (2b-ak+a\Delta_k)(ak-2b+a\Delta_k)(-2)^{n-r-2} J_{k,r}^2.$$

Proof. Let $C_1 = \frac{2b-ak+a\Delta_k}{2\Delta_k}$ and $C_2 = \frac{ak-2b+a\Delta_k}{2\Delta_k}$. By using Equation (11), we have

$$\begin{aligned}\mathcal{J}_{k,n-r}\mathcal{J}_{k,n+r} &= \left(C_1\sigma_k^{n-r} + C_2\frac{(-2)^{n-r}}{\sigma_k^{n-r}}\right)\left(C_1\sigma_k^{n+r} + C_2\frac{(-2)^{n+r}}{\sigma_k^{n+r}}\right)\\ &= C_1^2\sigma_k^{2n} + C_2^2\frac{(-2)^{2n}}{\sigma_k^{2n}} + C_1C_2(-2)^{n-r}\left(\sigma_k^{2r} + \frac{(-2)^{2r}}{\sigma_k^{2r}}\right)\end{aligned}$$

and

$$\mathcal{J}_{k,n}^2 = \left(C_1 \sigma_k^n + C_2 \frac{(-2)^n}{\sigma_k^n} \right)^2 = C_1^2 \sigma_k^{2n} + C_2^2 \frac{(-2)^{2n}}{\sigma_k^{2n}} + C_1 C_2 (-2)^{n-r} \left(2(-2)^r \right).$$

Then

$$\begin{aligned}
&\mathcal{J}_{k,n-r}\mathcal{J}_{k,n+r} - \mathcal{J}_{k,n}^2 \\
&\qquad = C_1 C_2 (-2)^{n-r} \left(\sigma_k^{2r} + \frac{(-2)^{2r}}{\sigma_k^{2r}} - 2(-2)^r \right) \\
&\qquad = \left(\frac{2b - ak + a\Delta_k}{2\Delta_k} \right) \left(\frac{ak - 2b + a\Delta_k}{2\Delta_k} \right) (-2)^{n-r} \left(\sigma_k^r - \frac{(-2)^r}{\sigma_k^r} \right)^2 \\
&\qquad = (2b - ak + a\Delta_k)(ak - 2b + a\Delta_k) \left(\frac{(-2)^{n-r}}{(-2)^2} \right) \left[\frac{1}{\Delta_k} \left(\sigma_k^r - \frac{(-2)^r}{\sigma_k^r} \right) \right]^2 \\
&\qquad = (2b - ak + a\Delta_k)(ak - 2b + a\Delta_k)(-2)^{n-r-2} J_{k,r}^2.
\end{aligned}$$

This completes the proof. □

Finally, new integral representations for the companion k-Jacobsthal numbers associated with the k-Jacobsthal and k-Jacobsthal–Lucas numbers are presented as follows.

Theorem 12. *Let k, ℓ, and n be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2+8}$. The companion k-Jacobsthal numbers $\mathcal{J}_{k,\ell n}$ are represented by*

$$\mathcal{J}_{k,\ell n} = \frac{1}{2^{n+1}} \int_{-1}^{1} (a j_{k,\ell} + (2b - ak) n J_{k,\ell} + a(n+1)\Delta_k \, J_{k,\ell} x)(j_{k,\ell} + \Delta_k \, J_{k,\ell} x)^{n-1} dx.$$

Proof. Applying the integral representations of $J_{k,\ell n}$ and $j_{k,\ell n}$ from Theorems 1 and 3 to Equation (11), we obtain

$$\begin{aligned}
\mathcal{J}_{k,\ell n} &= \frac{a}{2} j_{k,\ell n} + \frac{2b - ak}{2} J_{k,\ell n} \\
&= \frac{a}{2} \left(\frac{1}{2^n} \int_{-1}^{1} (j_{k,\ell} + (n+1)\Delta_k \, J_{k,\ell} x)(j_{k,\ell} + \Delta_k \, J_{k,\ell} x)^{n-1} dx \right) \\
&\quad + \frac{2b - ak}{2} \left(\frac{n J_{k,\ell}}{2^n} \int_{-1}^{1} (j_{k,\ell} + \Delta_k \, J_{k,\ell} x)^{n-1} dx \right) \\
&= \frac{1}{2^{n+1}} \int_{-1}^{1} (a j_{k,\ell} + (2b - ak) n J_{k,\ell} + a(n+1)\Delta_k \, J_{k,\ell} x)(j_{k,\ell} + \Delta_k \, J_{k,\ell} x)^{n-1} dx.
\end{aligned}$$

This completes the proof. □

Remark 3. As in Theorems 7 and 12, we have the following results.

1. If $a = 0$, then $\mathcal{J}_{k,n} = bJ_{k,n}$ and

$$\mathcal{J}_{k,\ell n} = \frac{bnJ_{k,\ell}}{2^n} \int_{-1}^{1} (j_{k,\ell} + \Delta_k \, J_{k,\ell} x)^{n-1} dx.$$

2. If $ak = 2b$, then $\mathcal{J}_{k,n} = \frac{a}{2} j_{k,n}$ and

$$\mathcal{J}_{k,\ell n} = \frac{a}{2^{n+1}} \int_{-1}^{1} (j_{k,\ell} + (n+1)\Delta_k \, J_{k,\ell} x)(j_{k,\ell} + \Delta_k \, J_{k,\ell} x)^{n-1} dx.$$

3. When $a \neq 0$, and when $\mathcal{J}_{k,\ell}$ and $J_{k,\ell}$ are known, then we can use

$$j_{k,\ell} = \frac{2}{a} \left(\mathcal{J}_{k,\ell} - \frac{2b - ak}{2} \, J_{k,n} \right).$$

4. When $ak \neq 2b$, and when $\mathcal{J}_{k,\ell}$ and $j_{k,\ell}$ are known, then we can use

$$J_{k,\ell} = \frac{2}{2b - ak} \left(\mathcal{J}_{k,\ell} - \frac{a}{2} \, j_{k,n} \right).$$

Remark 4. Notice that the integral representations of the associated k-Jacobsthal numbers $A_{k,n}$ and associated k-Jacobsthal–Lucas numbers $B_{k,n}$ are obtained by setting $(a, b) = (1, 1)$ and $(a, b) = (2, k + 4)$, respectively.

Setting $k = 1$ in Theorem 12, we have the following corollary.

Corollary 6. *Let ℓ and n be non-negative integers. The companion 1-Jacobsthal numbers $\mathcal{J}_{1,\ell n}$ are represented by*

$$\mathcal{J}_{1,\ell n} = \frac{1}{2^{n+1}} \int_{-1}^{1} (aj_\ell + 3(2b - a)nJ_\ell + a(n+1) \, J_\ell x)(j_\ell + \Delta_k \, J_\ell x)^{n-1} dx.$$

Remark 5. As in Corollary 6, the integral representations of Jacobsthal-like numbers V_n are deduced by setting $(a, b) = (2, 2)$.

Remark 6. The integral representations for general forms $\mathcal{J}_{k,\ell n+r}$ are established by applying Theorems 4, 5, and 7.

5. Conclusions

In this paper, we discuss the integral representations of the k-Jacobsthal and k-Jacobsthal–Lucas numbers. It is worth mentioning that the new integral representations apply to all the companion numbers of k-Jacobsthal numbers that preserve the recurrence relation with arbitrary initial conditions.

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