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CONGRUENCES MODULO 512 FOR THE OVERPARTITION FUNCTION

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Abstract

Let $\overline{p}(n)$ denote the number of overpartitions of n. Many scholars have investigated congruence properties modulo powers of 2 that hold for $\overline{p}(n)$. In this paper, we establish two infinite families of congruences modulo 512 for $\overline{p}(n)$ by employing some q-series techniques. For instance, one result proved in the present paper is that for any $\alpha \geq 0$ and $n \geq 0$,

$$\overline{p}\big(3^{8\alpha+7}(24n+5)\big) \equiv 0 \pmod{512}.$$

Finally, we conjecture that there are two infinite families of congruences modulo high powers of 2 satisfied by $\overline{p}(n)$.

1. Introduction

A *partition* of a positive integer n is a weakly decreasing sequence of positive integers whose sum equals n. To provide combinatorial proofs for some well-known q-series identities, Corteel and Lovejoy [4] introduced the concept of overpartitions. An *overpartition* of n is a partition of n in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n, and we define that $\overline{p}(0) = 1$. From [4], the generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}, \tag{1}$$

where we always assume that q is a complex number such that $|q| < 1$ and adopt the following standard notation:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

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In 2004, Mahlburg [12] proved that

$$\lim_{X\to\infty}\frac{\#\{0\le n<X:\overline{p}(n)\equiv 0 \pmod{64}\}}{X}=1.$$

He further conjectured that for any positive integer k,

$$\lim_{X\to\infty}\frac{\#\{0\le n<X:\overline{p}(n)\equiv 0 \pmod{2^k}\}}{X}=1. \tag{2}$$

The above conjecture, if true, would be a powerful arithmetic density property modulo powers of 2. Kim [10] and Xiong [14] later confirmed (2) for the cases $k=7$ and $k=8$. Although Mahlbug's conjecture is still open for any $k\ge 9$, it reveals that there are numerous congruence properties modulo powers of 2 for $\overline{p}(n)$.

Many scholars have studied congruence relations modulo 2 for the function $\overline{p}(n)$. For instance, Fortin et al. [7], and Hirschhorn et al. [8] have established the 2-, 3- and 4-dissections of the generating function of $\overline{p}(n)$, enabling the derivation of certain congruences modulo 4 and 8. In particular, they deduced the following three Ramanujan-like identities:

$$\sum_{n=0}^{\infty}\overline{p}(2n+1)q^n=2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q;q)_\infty^4(q^4;q^4)_\infty},$$
$$\sum_{n=0}^{\infty}\overline{p}(4n+3)q^n=8\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty^6}{(q;q)_\infty^8},$$
$$\sum_{n=0}^{\infty}\overline{p}(8n+7)q^n=64\frac{(q^2;q^2)_\infty^{22}}{(q;q)_\infty^{23}}. \tag{3}$$

Hirschhorn et al. [8] further conjectured that

$$\overline{p}(\ell n+r)=\begin{cases}0 \pmod 4 & \text{if } \ell\equiv\pm3 \pmod 8,\\ 0 \pmod 8 & \text{if } \ell\equiv\pm1 \pmod 8,\end{cases}$$

where ℓ is an odd prime and r is a quadratic nonresidue modulo ℓ. This conjecture was subsequently proved by Kim [11].

Additionally, Kim [11] got a complete characterization of $\overline{p}(n)$ modulo 8. Andersen [1] considered several Hecke-type congruences modulo small powers of 2 for $\overline{p}(n)$. Later, several authors have established Ramanujan-type congruences modulo 16, 32, and 64 for $\overline{p}(n)$; see, for example, [3, 5, 17]. Chen et al. [3] demonstrated that $\overline{p}(n)$ satisfies congruences modulo powers of 2, while Du et al. [6] derived several congruence families modulo small powers of 2 satisfied by $\overline{p}(n)$. Tang [13] also deduced numerous internal congruences and congruences modulo powers of 2 for $\overline{p}(n)$. Yang et al. [16] proved several infinite families of congruences modulo 256 that hold for $\overline{p}(n)$. They derived that for any $\alpha\ge 0$ and $n\ge 0$,

$$\overline{p}\big(3^{4\alpha+3}(24n+5)\big)\equiv\overline{p}\big(3^{4\alpha+3}(24n+13)\big)\equiv 0 \pmod{256}. \tag{4}$$

Shortly thereafter, Yao [18] derived several infinite families of congruences modulo 64 and 1024 for $\overline{p}(n)$. She proved that for any $\alpha \geq 0$ and $n \geq 0$,

$$\overline{p}\big(3^{16\alpha+15}(24n+5)\big) \equiv \overline{p}\big(3^{16\alpha+15}(24n+13)\big) \equiv 0 \pmod{1024}. \tag{5}$$

Recently, Xue and Yao [15] further established many explicit congruences modulo 2048 for $\overline{p}(n)$. They proved that for any $\alpha \geq 0$ and $n \geq 0$,

$$\overline{p}\big(3^{32\alpha+31}(24n+5)\big) \equiv \overline{p}\big(3^{32\alpha+31}(24n+13)\big) \equiv 0 \pmod{2048}. \tag{6}$$

Therefore, it is natural to ask whether there exist congruence families modulo 512 similar to (4)-(6) for $\overline{p}(n)$. The following theorem says that there is a positive answer for this question.

Theorem 1. *For any $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}(3^{8\alpha+7}(24n+5)) \equiv \overline{p}(3^{8\alpha+7}(24n+13)) \equiv 0 \pmod{512}. \tag{7}$$

Motivated by (4)-(7), we pose the following conjecture.

Conjecture 1. For any $m \geq 2$, $\alpha \geq 0$, and $n \geq 0$,

$$\overline{p}\big(3^{2^m\alpha+2^m-1}(24n+5)\big) \equiv \overline{p}\big(3^{2^m\alpha+2^m-1}(24n+13)\big) \equiv 0 \pmod{2^{m+6}}. \tag{8}$$

2. Proof of Theorem 1

In this section, we give a proof of Theorem 1. Before stating it, we collect the following important lemma. For the sake of convenience, we write

$$E(q) = (q;q)_\infty.$$

Lemma 1. *We have*

$$\frac{E(q^2)^2}{E(q)} = \frac{E(q^6)E(q^9)^2}{E(q^3)E(q^{18})} + q\frac{E(q^{18})^2}{E(q^9)}, \tag{9}$$

$$\frac{E(q^6)^3E(q^9)^6}{E(q^3)^3E(q^{18})^3} = \frac{E(q^6)^8E(q^9)}{E(q^3)^4E(q^{18})^2} - q^3\frac{E(q^{18})^6}{E(q^9)^3}, \tag{10}$$

$$\frac{E(q)}{E(q^2)^2} = \frac{E(q^3)^2E(q^9)^3}{E(q^6)^6} - q\frac{E(q^3)^3E(q^{18})^3}{E(q^6)^7} + q^2\frac{E(q^3)^4E(q^{18})^6}{E(q^6)^8E(q^9)^3}. \tag{11}$$

The identity (9) is from [2, p. 49, Corollary], while (10) and (11) come from [9, Lemmas 2.1 and 2.2].

Now, we turn to prove Theorem 1.

Proof of Theorem 1. In what follows, all the congruences are modulo 512 unless otherwise specified. By (3) and (9), we have

$$\sum_{n=0}^{\infty}\overline{p}(8n+7)q^n \equiv 64\frac{E(q^2)^{14}}{E(q)^7} = 64\left(\frac{E(q^6)E(q^9)^2}{E(q^3)E(q^{18})}+q\frac{E(q^{18})^2}{E(q^9)}\right)^7.$$

Picking out those terms in which the power of q is congruent to 1 modulo 3, and applying (10), we can see that

$$\sum_{n=0}^{\infty}\overline{p}\big(8(3n+1)+7\big)q^{3n+1} \equiv 64\left(-q\frac{E(q^6)^{16}E(q^9)}{E(q^3)^8E(q^{18})^2}-3q^4\frac{E(q^6)^8E(q^{18})^6}{E(q^3)^4E(q^9)^3}-3q^7\frac{E(q^{18})^{14}}{E(q^9)^7}\right), \tag{12}$$

then, dividing both sides of (12) by q and replacing q by $q^{1/3}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}(24n+15)q^n &\equiv 64\left(-\frac{E(q^2)^{16}E(q^3)}{E(q)^8E(q^6)^2}-3q\frac{E(q^2)^8E(q^6)^6}{E(q)^4E(q^3)^3}-3q^2\frac{E(q^6)^{14}}{E(q^3)^7}\right)\\ &= 64\left(-\frac{E(q^3)}{E(q^6)^2}\left(\frac{E(q^2)^2}{E(q)}\right)^8-3q\frac{E(q^6)^6}{E(q^3)^3}\left(\frac{E(q^2)^2}{E(q)}\right)^4-3q^2\frac{E(q^6)^{14}}{E(q^3)^7}\right). \end{aligned}\tag{13}$$

Next, substituting (9) into (13), and selecting those terms of the form of q^{3n+2}, it turns into

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}\big(24(3n+2)+15\big)q^{3n+2} &\equiv 64\left(-\frac{E(q^3)}{E(q^6)^2}\left(4q^2\frac{E(q^6)^{16}}{E(q^3)^8}-3q^8\frac{E(q^{18})^{16}}{E(q^9)^8}\right)\right.\\ &\quad\left.-3q\frac{E(q^6)^6}{E(q^3)^3}\left(4q\frac{E(q^6)^8}{E(q^3)^4}-3q^4\frac{E(q^{18})^8}{E(q^9)^4}\right)-3q^2\frac{E(q^6)^{14}}{E(q^3)^7}\right)\\ &\equiv 64\left(5q^2\frac{E(q^6)^{14}}{E(q^3)^7}+q^5\frac{E(q^6)^6E(q^{18})^8}{E(q^3)^3E(q^9)^4}+3q^8\frac{E(q^{18})^{16}E(q^3)}{E(q^9)^8E(q^6)^2}\right).\end{aligned}$$

After simplification, we deduce that

$$\sum_{n=0}^{\infty}\overline{p}(72n+63)q^n \equiv 64\left(5\frac{E(q^2)^{14}}{E(q)^7}+q\frac{E(q^2)^6E(q^6)^8}{E(q)^3E(q^3)^4}+3q^2\frac{E(q)E(q^6)^{16}}{E(q^2)^2E(q^3)^8}\right). \tag{14}$$

Applying (9)-(12), picking out those terms of the form of q^{3n+1}, and substituting

(11) into (14), upon simplification, one can arrive at

$$\sum_{n=0}^{\infty}\overline{p}\big(72(3n+1)+63\big)q^{3n+1} \equiv 64\left(4q\frac{E(q^6)^{16}E(q^9)}{E(q^3)^8E(q^{18})^2}+4q^4\frac{E(q^6)^8E(q^{18})^6}{E(q^3)^4E(q^9)^3}+q^7\frac{E(q^{18})^{14}}{E(q^9)^7}\right),$$

or, equivalently,

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}\big(216n+135\big)q^{n} &\equiv 64\left(4\frac{E(q^2)^{16}E(q^3)}{E(q)^8E(q^6)^2}+4q\frac{E(q^2)^8E(q^6)^6}{E(q)^4E(q^3)^3}+q^2\frac{E(q^6)^{14}}{E(q^3)^7}\right)\\ &\equiv 64\left(4\frac{E(q^3)}{E(q^6)^2}\left(\frac{E(q^2)^2}{E(q)}\right)^8+4q\frac{E(q^6)^6}{E(q^3)^3}\left(\frac{E(q^2)^2}{E(q)}\right)^4+q^2\frac{E(q^6)^{14}}{E(q^3)^7}\right).\end{aligned}$$

Following a similar strategy as above, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{p}\big(216(3n+2)+135\big)q^{3n+2} &\equiv 64\left(4\frac{E(q^3)}{E(q^6)^2}\left(4q^2\frac{E(q^6)^{16}}{E(q^3)^8}-3q^8\frac{E(q^{18})^{16}}{E(q^9)^8}\right)\right.\\ &\quad\left.+4q\frac{E(q^6)^6}{E(q^3)^3}\left(4q\frac{E(q^6)^8}{E(q^3)^4}-3q^4\frac{E(q^{18})^8}{E(q^9)^4}\right)+q^2\frac{E(q^6)^{14}}{E(q^3)^7}\right)\\ &\equiv 64\left(q^2\frac{E(q^6)^{14}}{E(q^3)^7}+4q^5\frac{E(q^6)^6E(q^{18})^8}{E(q^3)^3E(q^9)^4}+4q^8\frac{E(q^3)E(q^{18})^{16}}{E(q^6)^2E(q^9)^8}\right).\end{aligned}$$

That is,

$$\sum_{n=0}^{\infty}\overline{p}(648n+567)q^{n} \equiv 64\left(\frac{E(q^2)^{14}}{E(q)^7}+4q\frac{E(q^2)^6E(q^6)^8}{E(q)^3E(q^3)^4}+4q^2\frac{E(q)E(q^6)^{16}}{E(q^2)^2E(q^3)^8}\right).$$

Utilizing (9)-(12), we obtain

$$\sum_{n=0}^{\infty}\overline{p}\big(648(3n+1)+567\big)q^{3n+1} \equiv 64\left(3q\frac{E(q^6)^{16}E(q^9)}{E(q^3)^8E(q^{18})^2}+q^4\frac{E(q^6)^8E(q^{18})^6}{E(q^3)^4E(q^9)^3}-3q^7\frac{E(q^{18})^{14}}{E(q^9)^7}\right).$$

Hence,

$$\sum_{n=0}^{\infty}\overline{p}(1944n+1215)q^{n} \equiv 64\left(3\frac{E(q^2)^{16}E(q^3)}{E(q)^8E(q^6)^2}+q\frac{E(q^2)^8E(q^6)^6}{E(q)^4E(q^3)^3}-3q^2\frac{E(q^6)^{14}}{E(q^3)^7}\right).$$

In the same vein, we derive that

$$\sum_{n=0}^{\infty} \overline{p}\big(1944(3n+2)+1215\big)q^{3n+2} \equiv 64\bigg(-3q^2\frac{E(q^6)^{14}}{E(q^3)^7} - 3q^5\frac{E(q^6)^6E(q^{18})^8}{E(q^3)^3E(q^9)^4} - q^8\frac{E(q^3)E(q^{18})^{16}}{E(q^6)^2E(q^9)^8}\bigg),$$

from which we obtain that

$$\sum_{n=0}^{\infty} \overline{p}(5832n+5103)q^n \equiv 64\bigg(-3\frac{E(q^2)^{14}}{E(q)^7} - 3q\frac{E(q^2)^6E(q^6)^8}{E(q)^3E(q^3)^4} - q^2\frac{E(q)E(q^6)^{16}}{E(q^2)^2E(q^3)^8}\bigg).$$

Using (9), (10), and (12), we further arrive at

$$\sum_{n=0}^{\infty} \overline{p}\big(5832(3n+1)+5103\big)q^{3n+1} \equiv 64q^7\frac{E(q^{18})^{14}}{E(q^9)^7}.$$

It follows that

$$\sum_{n=0}^{\infty} \overline{p}(17496n+10935)q^n = \sum_{n=0}^{\infty} \overline{p}\big(8\cdot 3^7n+5\cdot 3^7\big)q^n \equiv 64q^2\frac{E(q^6)^{14}}{E(q^3)^7}. \tag{15}$$

Finally,

$$\sum_{n=0}^{\infty} \overline{p}\big(17496(3n+2)+10935\big)q^n = \sum_{n=0}^{\infty} \overline{p}\big(8\cdot 3^8n+7\cdot 3^8\big)q^n \equiv 64\frac{E(q^2)^{14}}{E(q)^7}.$$

Thus, one can obtain that for any $n \geq 0$,

$$\overline{p}\big(8\cdot 3^8n+7\cdot 3^8\big) \equiv \overline{p}(8n+7). \tag{16}$$

Based on (16) and induction, we can deduce that for any $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \overline{p}\big(8\cdot 3^{8\alpha}n+7\cdot 3^{8\alpha}\big)q^n \equiv \sum_{n=0}^{\infty} \overline{p}(8n+7)q^n \equiv 64\frac{E(q^2)^{14}}{E(q)^7}. \tag{17}$$

Combining (15) and (17) gives

$$\sum_{n=0}^{\infty} \overline{p}\big(8\cdot 3^{8\alpha+7}n+5\cdot 3^{8\alpha+7}\big)q^n \equiv 64q^2\frac{E(q^6)^{14}}{E(q^3)^7}. \tag{18}$$

Since there are no terms in which the powers of q are congruent to 0 or 1 modulo 3 on the right-hand side of (18), we readily get

$$\overline{p}\big(8\cdot 3^{8\alpha+7}(3n)+5\cdot 3^{8\alpha+7}\big) \equiv 0 \pmod{2^9},$$
$$\overline{p}\big(8\cdot 3^{8\alpha+7}(3n+1)+5\cdot 3^{8\alpha+7}\big) \equiv 0 \pmod{2^9}.$$

This completes the proof. □

3. Closing Remarks

In this paper, we establish two infinite families of congruences modulo 512 for the overpartition function $\overline{p}(n)$ by utilizing some q-series techniques. Moreover, we conjecture that there exist two infinite families of congruences modulo high powers of s satisfied by $\overline{p}(n)$. Obviously, if (8) is true, then for any $k \geq 1$, the following inequality also holds:

$$\lim_{X\to\infty} \frac{\#\{0 \leq n < X : \overline{p}(n) \equiv 0 \pmod{2^k}\}}{X} > 0.$$

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