THE MALLOWS REPRESENTATION OF THE IRRATIONAL NUMBERS

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Abstract

We first introduce the Mallows tree as the labeled infinite ternary tree whose vertices are labeled by certain pairs of fractions. Then, using it, we introduce a way of representing irrational numbers from $(0,1) \setminus \mathbb{Q}$ as infinite sequences of L's, M's, and R's, which we call the Mallows representation of the irrational numbers. We present a number of results on the combinatorics and metric theory of Mallows representations, including a characterization of those $\alpha \in (0,1) \setminus \mathbb{Q}$ which have eventually periodic Mallows representations.

1. Introduction

Starting with the initial sequence $\langle \frac{0}{1}, \frac{1}{1} \rangle$, successively update it by inserting fractions as follows. In the first step, between $\frac{0}{1}$ and $\frac{1}{1}$, insert their *mediant* $\frac{0+1}{1+1}$ to obtain $\langle \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \rangle$. In the second step, between each adjacent pair of fractions from $\langle \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \rangle$, insert their mediants $\frac{0+1}{1+2}$ and $\frac{1+1}{2+1}$ to obtain $\langle \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \rangle$. In the third step, between each adjacent pair of fractions from $\langle \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \rangle$, insert their mediants $\frac{0+1}{1+3}, \frac{1+1}{3+2}, \frac{1+2}{2+3}$, and $\frac{2+1}{3+1}$ to obtain $\langle \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \rangle$, and so on. The *Stern–Brocot tree* [1, 8] is the labeled binary tree whose jth level ($j \geq 1$) is labeled by the fractions inserted at the jth step. (Let us mention that what is usually called the Stern–Brocot tree is something more general, of which the above one is the left subtree; see Remark 1.) It is known that this tree contains (as a label of some vertex) each fraction from $(0,1) \cap \mathbb{Q}$ precisely once. Also, at each level of the tree, the left-to-right order accords with the increasing order. For background information, we refer the reader to [2].

Take an irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$ arbitrarily. Then it should be either smaller or larger than $\frac{1}{2}$. Suppose that it is smaller than $\frac{1}{2}$. Then it should be either smaller or larger than the left (equivalently, smaller) child $\frac{1}{3}$ of $\frac{1}{2}$. Suppose that it is

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larger than $\frac{1}{3}$. Then it should be either smaller or larger than the right (equivalently, larger) child $\frac{2}{5}$ of $\frac{1}{3}$. By continuing in this way, we can associate an infinite sequence of L's (for "left") and R's (for "right") with each irrational number from $(0,1)\setminus\mathbb{Q}$. As this correspondence between irrational numbers and associated infinite sequences is one-to-one, the associated infinite sequences can be used to represent irrational numbers. Let us call the associated infinite sequence the *Stern–Brocot representation of the irrational number*.

In [5], Mallows introduced a variation of the Stern–Brocot sequence by inserting not one but two fractions in the above construction. Explicitly, in the first step, between $\frac{0}{1}$ and $\frac{1}{1}$, insert two fractions $\frac{0+1}{1+1}$ and $\frac{0+2\cdot 1}{1+2\cdot 1}$ to obtain $\langle \frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{1}{0} \rangle$. In the second step, between $\frac{0}{1}$ and $\frac{1}{2}$ (resp. $\frac{1}{2}$ and $\frac{2}{3}$, $\frac{2}{3}$ and $\frac{1}{1}$), insert $\frac{0+1}{1+2}$ and $\frac{0+2\cdot 1}{1+2\cdot 2}$ (resp. $\frac{2\cdot 1+2}{2\cdot 2+3}$ and $\frac{1+2}{2+3}$, $\frac{2+1}{3+1}$ and $\frac{2+2\cdot 1}{3+2\cdot 1}$) to obtain $\langle \frac{0}{1}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \rangle$, and so forth. Let us write MT_i $(i \geq 1)$ for the ordered set consisting of the fractions inserted at the ith step. Then each fraction from $(0,1)\cap\mathbb{Q}$ appears in one of MT_i $(i \geq 1)$ precisely once. Moreover, the left-to-right order accords with the increasing order on MT_i for each $i \geq 1$. For proofs, see [5] or [3].

In this article, we first introduce the *Mallows tree* as the labeled infinite ternary tree whose ith level $(i \geq 1)$, in left-to-right order, is labeled by 3^{i-1} pairs of fractions from MT_i. (Here, we are identifying the ordered set MT_i of $2\cdot 3^{i-1}$ fractions with the ordered set of 3^{i-1} pairs of fractions.) Then, in the same spirit as the Stern–Brocot representation, which represents irrational numbers from $(0,1)\setminus\mathbb{Q}$ as infinite sequences of L's and R's by using the Stern–Brocot tree, we introduce the *Mallows representation of irrational numbers*, which represents irrational numbers from $(0,1)\setminus\mathbb{Q}$ as infinite sequences of L's (for "left"), M's (for "middle"), and R's (for "right") by using the Mallows tree. After establishing the link between the Mallows representations and the Stern–Brocot representations in Section 3, we shall prove a number of assertions in the combinatorics of Mallows representations in Section 4, including the following ones: the Mallows representation of an $\alpha \in (0,1)\setminus\mathbb{Q}$ is eventually periodic if and only if α is quadratic (Corollary 2); the Mallows representation of an $\alpha \in (0,1)\setminus\mathbb{Q}$ is positively Poisson stable (resp. transitive) if and only if the continued fraction representation of α is positively Poisson stable (resp. transitive) (Corollaries 3 and 4); the Mallows representation of an $\alpha \in (0,1)\setminus\mathbb{Q}$ does not contain an occurrence of M if and only if for all $k \geq 1$, the $2k$th element of the continued fraction representation of α is even (Proposition 9). In Section 5, we shall substantiate various assertions in the metric theory of Mallows representations, including the ensuing ones: for almost every $\alpha \in (0,1)\setminus\mathbb{Q}$, any finite sequence containing either M or both L and R appears in its Mallows representation with frequency 0 (Theorem 6); for almost every $\alpha \in (0,1)\setminus\mathbb{Q}$, the ℓth matching fraction of its Mallows representation is 1 for all $\ell \geq 1$ (Theorem 7); the measure of the set of those $\alpha \in (0,1)\setminus\mathbb{Q}$ whose Mallows representations contain M at position i is bounded above by $3-2\sqrt{2}$ (Theorem 9).

2. Preliminaries

2.1. Notation and Terminology

Let us first set up notation and terminology on sequences. The length of a finite sequence w is written as $\mathrm{lh}(w)$. We shall use the symbol $\frown$ to represent the concatenation operator, e.g., $\langle 1,2\rangle^\frown\langle 3,4,5\rangle = \langle 1,2,3,4,5\rangle$. Also, w^n (resp. w^∞) stands for the concatenation of n (resp. countably many) copies of w. The set of all length ℓ (resp. infinite) sequences over a non-empty set Σ is denoted by Σ^ℓ (resp. Σ^∞). Given a sequence $\langle x_1, x_2, x_3, \ldots\rangle$, we say that x_i is *at position* i of this sequence. The expressions of the form $\langle x_i\rangle_{i=n}^{n+\ell-1}$ and $\langle x_i\rangle_{i=n}^{\infty}$ are understood to mean the finite sequence $\langle x_n, x_{n+1}, \ldots, x_{n+\ell-1}\rangle$ and the infinite sequence $\langle x_n, x_{n+1}, x_{n+2}, \ldots\rangle$, respectively.

Let us next fix our notation and terminology on continued fractions, most of which are taken from [4]. We write $[a_0; a_1, a_2, \ldots]$ and $[a_0; a_1, \ldots, a_\ell]$ for the following infinite and finite continued fractions, respectively:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \qquad \text{and} \qquad a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_\ell}}}.$$

We assume that a_0 is an integer and $a_1, a_2, \ldots$ are positive integers. We call $a_0, a_1, a_2, \ldots$ *elements* of the continued fraction. In this article, we shall use continued fractions to represent real numbers: For each irrational number α, there exists a unique infinite continued fraction whose value is equal to α, which we take as the *continued fraction representation of* α. For each rational number α, there exist precisely two finite continued fractions whose values are equal to α. Since they are of the form $[a_0; a_1, \ldots, a_\ell]$ and $[a_0; a_1, \ldots, a_\ell - 1, 1]$ ($\ell \geq 0$), we take the former as the *continued fraction representation of* α. Henceforth, for brevity, we shall abbreviate the continued fraction representation as the *CF-representation*. Also, we shall call elements of the CF-representation of α simply *elements of* α.

Lastly, we shall present our notation and terminology on trees. Throughout this article, a tree means a rooted infinite (either binary or ternary) tree. In such a tree, the concept of *level* is defined inductively as follows. The root is at level 1. If a vertex is at level ℓ, then its children are at level $\ell + 1$. We also use the ensuing concept. For any vertices v, v' in the rooted infinite binary tree T such that v is equal to or an ancestor of v', the *path from* v *to* v' is a finite sequence $\mathrm{Path}_{\mathrm{T}}(v, v')$

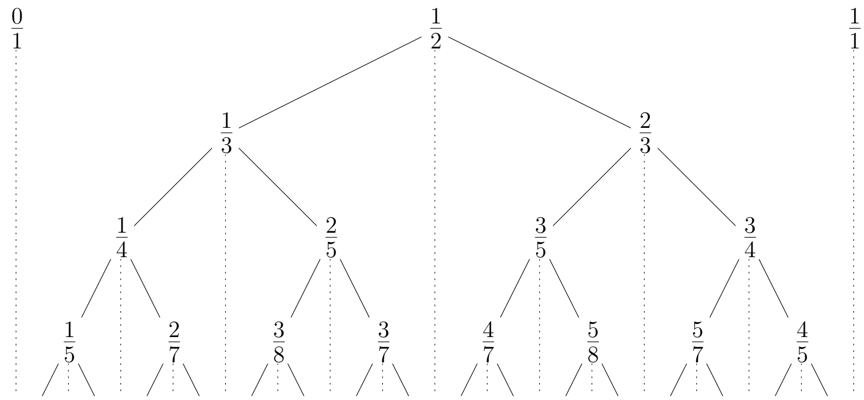

Figure 1: Top levels of the Stern–Brocot tree.

over the binary set $\{\mathtt{L}, \mathtt{R}\}$ defined recursively by

$$\mathrm{Path}_{\mathrm{T}}(v, v') = \begin{cases} \langle\rangle & \text{if } v = v', \\ \langle \mathtt{L} \rangle ^\frown \mathrm{Path}_{\mathrm{T}}(v'', v') & \text{if the left child } v'' \text{ of } v \text{ is} \\ & \quad \text{equal to or an ancestor of } v', \\ \langle \mathtt{R} \rangle ^\frown \mathrm{Path}_{\mathrm{T}}(v'', v') & \text{if the right child } v'' \text{ of } v \text{ is} \\ & \quad \text{equal to or an ancestor of } v'. \end{cases}$$

2.2. The Stern–Brocot Tree and the Stern–Brocot Representation

Let us start with the definition of the Stern–Brocot tree.

Definition 1. Define ordered sets SBT_j $(j \geq 1)$ of 2^{j-1} fractions inductively as follows. Let $\mathrm{SBT}_1 = \left\{\frac{1}{2}\right\}$. Suppose that we have defined $\mathrm{SBT}_1, \mathrm{SBT}_2, \ldots, \mathrm{SBT}_j$, and let $\frac{n_1}{m_1} < \frac{n_2}{m_2} < \cdots < \frac{n_{2^j+1}}{m_{2^j+1}}$ be the elements of $\left\{\frac{0}{1}, \frac{1}{1}\right\} \cup \mathrm{SBT}_1 \cup \mathrm{SBT}_2 \cup \cdots \cup \mathrm{SBT}_j$. Then SBT_{j+1} is the ordered set $\left\{\frac{n_1+n_2}{m_1+m_2}, \frac{n_2+n_3}{m_2+m_3}, \ldots, \frac{n_{2^j}+n_{2^j+1}}{m_{2^j}+m_{2^j+1}}\right\}$.

The *Stern–Brocot tree* (SBT) is the labeled binary tree such that the labels of its jth level $(j \geq 1)$, in left-to-right order, is SBT_j (see Figure 1).

Since the fraction $\frac{n_1+n_2}{m_1+m_2}$ is called the *mediant* of $\frac{n_1}{m_1}$ and $\frac{n_2}{m_2}$, the above way of constructing SBT_{j+1} from $\left\{\frac{0}{1}, \frac{1}{1}\right\} \cup \mathrm{SBT}_1 \cup \mathrm{SBT}_2 \cup \cdots \cup \mathrm{SBT}_j$ is referred to as the *mediant construction*.

Remark 1. What we have defined above is in fact different from what is usually called the Stern–Brocot tree [1, 8]. Usually, the Stern–Brocot tree is defined first by setting $\mathrm{SBT}_1 = \left\{\frac{1}{1}\right\}$ and then by applying the above mediant construction to $\left\{\frac{0}{1}, \frac{1}{0}\right\} \cup \mathrm{SBT}_1 \cup \mathrm{SBT}_2 \cup \cdots \cup \mathrm{SBT}_j$. Thus, ours is the left subtree of the usual version of the Stern–Brocot tree. We adopted the above definition not only because we do not use the usual version of the Stern–Brocot tree in this article but also because, by doing so, we do not have to write the phrase "the left subtree of" repeatedly.

The ensuing properties are well-known (see [2]).

Proposition 1. (i) *For each fraction* $\frac{n}{m}$ *from* $(0,1)\cap\mathbb{Q}$*, there exists precisely one vertex which has label* $\frac{n}{m}$*.*

(ii) *On each* SBT_j $(j\geq 1)$*, the left-to-right order accords with the increasing order.*

(iii) *The path from the root* v *to a vertex* v' *is given by the following formula:*

$$\mathrm{Path}_{\mathrm{SBT}}(v,v')=\begin{cases}\langle\mathtt{L}\rangle^{a_1-1}\frown\langle\mathtt{R}\rangle^{a_2}\frown\langle\mathtt{L}\rangle^{a_3}\frown\langle\mathtt{R}\rangle^{a_4}\frown\cdots\frown\langle\mathtt{L}\rangle^{a_\ell-1} & \text{if } \ell>1 \text{ is odd,}\\ \langle\mathtt{L}\rangle^{a_1-1}\frown\langle\mathtt{R}\rangle^{a_2}\frown\langle\mathtt{L}\rangle^{a_3}\frown\langle\mathtt{R}\rangle^{a_4}\frown\cdots\frown\langle\mathtt{R}\rangle^{a_\ell-1} & \text{if } \ell>1 \text{ is even,}\\ \langle\mathtt{L}\rangle^{a_1-2} & \text{if } \ell=1,\end{cases}$$

where $a_1,a_2,\ldots,a_\ell$ *are the elements of the CF-representation* $[0;a_1,a_2,\ldots,a_\ell]$ *of the label of* v'*.*

Since Part (i) of the above proposition guarantees that different vertices have different labels in the Stern–Brocot tree, we shall hereafter freely identify a vertex with its label without explicit mention.

Take an irrational number $\alpha\in(0,1)\setminus\mathbb{Q}$ arbitrarily. Then it should be either smaller or larger than $\frac{1}{2}$. Suppose that it is smaller than $\frac{1}{2}$. Then it should be either smaller or larger than the left (equivalently, by Proposition 1 (ii), smaller) child $\frac{1}{3}$ of $\frac{1}{2}$. Suppose that it is larger than $\frac{1}{3}$. Then it should be either smaller or larger than the right (equivalently, by Proposition 1 (ii), larger) child $\frac{2}{5}$ of $\frac{1}{3}$. By continuing in this way, we can associate an infinite sequence of L's (for "left") and R's (for "right") with each irrational number. Let us make this idea precise (see [2]).

Definition 2. For each irrational number $\alpha\in(0,1)\setminus\mathbb{Q}$, define a fraction $f_j(\alpha)$ from SBT_j and $\mathrm{SB}_j(\alpha)\in\{\mathtt{L},\mathtt{R}\}$ for $j\geq 1$, inductively, as follows:

- Set $f_1(\alpha)=\frac{1}{2}$.
- Suppose that we have defined $f_j(\alpha)$. Then set

$$\mathrm{SB}_j(\alpha)=\begin{cases}\mathtt{L} & \text{if } \alpha<f_j(\alpha),\\ \mathtt{R} & \text{if } \alpha>f_j(\alpha).\end{cases}$$

  Define $f_{j+1}(\alpha)$ to be the left (resp. right) child of $f_j(\alpha)$ in the Stern–Brocot tree if $\mathrm{SB}_j(\alpha)=\mathtt{L}$ (resp. R).

Then the *Stern–Brocot representation of* $\alpha\in(0,1)\setminus\mathbb{Q}$ is the infinite sequence $\mathrm{SB}(\alpha)\coloneqq\langle\mathrm{SB}_1(\alpha),\mathrm{SB}_2(\alpha),\mathrm{SB}_3(\alpha),\ldots\rangle\in\{\mathtt{L},\mathtt{R}\}^\infty$.

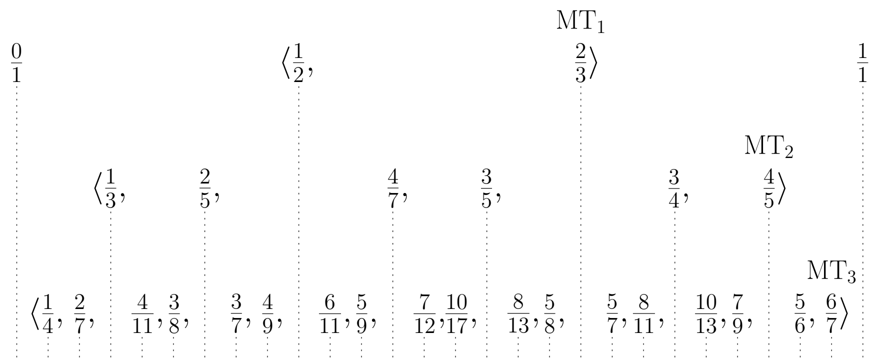

Figure 2: MT_i for the first few $i \geq 1$.

The following can readily be observed.

Proposition 2. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, we have*

(i) $\lim_{j\to\infty} f_j(\alpha) = \alpha$;

(ii) $\mathrm{SB}(\alpha) = \langle \mathtt{L}\rangle^{a_1-1} \frown \langle \mathtt{R}\rangle^{a_2} \frown \langle \mathtt{L}\rangle^{a_3} \frown \langle \mathtt{R}\rangle^{a_4} \frown \ldots$, *where* $[0; a_1, a_2, \ldots]$ *is the CF-representation of* α.

Write $\mathcal{SB}$ for the set $\{\mathrm{SB}(\alpha) \mid \alpha \in (0,1) \setminus \mathbb{Q}\}$. Then, from the second equation of the above proposition, the next corollary can be inferred.

Corollary 1. *The following relation holds:*

$$\mathcal{SB} = \{\langle x_j\rangle_{j=1}^{\infty} \in \{\mathtt{L},\mathtt{R}\}^{\infty} \mid \text{For no } j_0, \text{we have } \langle x_j\rangle_{j=j_0}^{\infty} = \langle \mathtt{L}\rangle^{\infty} \text{ or } = \langle \mathtt{R}\rangle^{\infty}\}.$$

2.3. The Mallows Tree and the Mallows Representation

Recall from Definition 1 that the ordered set SBT_{j+1} is obtained from the set $\left\{\frac{0}{1}, \frac{1}{1}\right\} \cup \mathrm{SBT}_1 \cup \mathrm{SBT}_2 \cup \cdots \cup \mathrm{SBT}_j$ by inserting one fraction between each pair of adjacent fractions. By inserting not one but two fractions, Mallows [5] introduced ordered sets MT_i ($i \geq 1$) of $2 \cdot 3^{i-1}$ fractions as follows. Let $\mathrm{MT}_1 = \left\{\frac{1}{2}, \frac{2}{3}\right\}$. Suppose that we have defined $\mathrm{MT}_1, \mathrm{MT}_2, \ldots, \mathrm{MT}_i$, and let $\frac{n_1}{m_1} < \frac{n_2}{m_2} < \cdots < \frac{n_{3^i+1}}{m_{3^i+1}}$ be the elements of $\left\{\frac{0}{1}, \frac{1}{1}\right\} \cup \mathrm{MT}_1 \cup \mathrm{MT}_2 \cup \cdots \cup \mathrm{MT}_i$. Then MT_{i+1} is the ordered set $\left\{\frac{n'_1}{m'_1}, \frac{n'_2}{m'_2}, \ldots, \frac{n'_{2\cdot 3^i}}{m'_{2\cdot 3^i}}\right\}$, where

$$\left\langle \frac{n'_{2k-1}}{m'_{2k-1}}, \frac{n'_{2k}}{m'_{2k}} \right\rangle = \begin{cases} \left\langle \frac{n_k+n_{k+1}}{m_k+m_{k+1}}, \frac{n_k+2n_{k+1}}{m_k+2m_{k+1}} \right\rangle & \text{if } n_k \text{ is even,} \\ \left\langle \frac{2n_k+n_{k+1}}{2m_k+m_{k+1}}, \frac{n_k+n_{k+1}}{m_k+m_{k+1}} \right\rangle & \text{if } n_k \text{ is odd} \end{cases}$$

for each $k \in \{1, 2, \ldots, 3^i\}$ (see Figure 2).

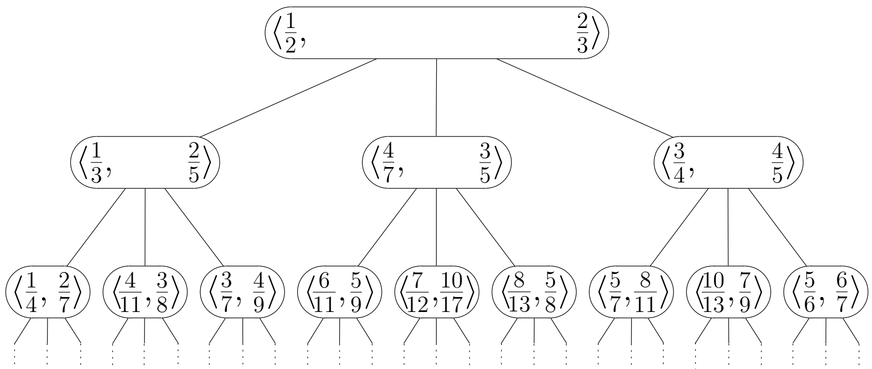

Figure 3: Top levels of the Mallows tree.

By identifying the ordered set $\mathrm{MT}_i = \left\{\frac{n_1}{m_1}, \frac{n_2}{m_2}, \dots, \frac{n_{2\cdot 3^{i-1}}}{m_{2\cdot 3^{i-1}}}\right\}$ with the ordered set $\left\{\langle\frac{n_1}{m_1}, \frac{n_2}{m_2}\rangle, \langle\frac{n_3}{m_3}, \frac{n_4}{m_4}\rangle, \dots, \langle\frac{n_{2\cdot 3^{i-1}-1}}{m_{2\cdot 3^{i-1}-1}}, \frac{n_{2\cdot 3^{i-1}}}{m_{2\cdot 3^{i-1}}}\rangle\right\}$ of 3^{i-1} pairs of fractions, we make the following definition.

Definition 3. The *Mallows tree* (MT) is the labeled ternary tree such that the labels of its ith level ($i \geq 1$), in left-to-right order, is MT_i (see Figure 3).

Remark 2. Define ordered sets $\widehat{\mathrm{MT}}_i$ ($i \geq 1$) of $2 \cdot 3^{i-1}$ fractions first by setting $\widehat{\mathrm{MT}}_1 = \left\{\frac{1}{1}, \frac{2}{1}\right\}$ and then by applying the above construction to the ordered set $\left\{\frac{0}{1}, \frac{1}{0}\right\} \cup \widehat{\mathrm{MT}}_1 \cup \widehat{\mathrm{MT}}_2 \cup \cdots \cup \widehat{\mathrm{MT}}_i$. Then it can readily be verified that MT_i is the left one-third of $\widehat{\mathrm{MT}}_{i+1}$ for each $i \geq 1$. Moreover, it can be proved that each positive fraction appears precisely once in an $\widehat{\mathrm{MT}}_i$ ($i \geq 1$), which can be seen as an extension of Part (i) of the next proposition.

Let us present two known properties (see, [3, 5]).

Proposition 3. (i) *Each fraction* $\frac{n}{m}$ *from* $(0, 1) \cap \mathbb{Q}$ *appears precisely once in an* MT_i ($i \geq 1$).

(ii) *On each* MT_i ($i \geq 1$)*, the left-to-right order accords with the increasing order.*

Since it follows from Part (i) of the above proposition that different vertices have different labels in the Mallows tree, as we did for the Stern–Brocot tree, we shall hereafter freely identify a vertex with its label without explicit mention.

In [3], we explained how to obtain MT_i ($i \geq 1$) from the Stern–Brocot tree. For later use, let us recall it here. On the set $\left\{\frac{n}{m} \in (0, 1) \cap \mathbb{Q} \;\middle|\; n \text{ is odd}\right\}$, define two functions φ_{SBT} and ψ_{SBT} by setting

$$\varphi_{\mathrm{SBT}}\left(\frac{n}{m}\right) = \begin{cases} \langle\frac{n'}{m'}, \frac{n}{m}\rangle & \text{if the left child } \frac{n'}{m'} \text{ of } \frac{n}{m} \text{ has even numerator,} \\ \langle\frac{n}{m}, \frac{n''}{m''}\rangle & \text{if the right child } \frac{n''}{m''} \text{ of } \frac{n}{m} \text{ has even numerator} \end{cases}$$

and

$$\psi_{\text{SBT}}\big(\tfrac{n}{m}\big) = \begin{cases} \langle \frac{n'}{m'}, \frac{n'''}{m'''}, \frac{n''''}{m''''} \rangle & \text{if the left child } \frac{n'}{m'} \text{ of } \frac{n}{m} \text{ has odd numerator} \\ & \quad \text{and } \frac{n'''}{m'''} \text{ and } \frac{n''''}{m''''} \text{ are the left and right children} \\ & \quad \text{of the right child of } \frac{n}{m}, \text{ respectively;} \\ \langle \frac{n'''}{m'''}, \frac{n''''}{m''''}, \frac{n''}{m''} \rangle & \text{if the right child } \frac{n''}{m''} \text{ of } \frac{n}{m} \text{ has odd numerator} \\ & \quad \text{and } \frac{n'''}{m'''} \text{ and } \frac{n''''}{m''''} \text{ are the left and right children} \\ & \quad \text{of the left child of } \frac{n}{m}, \text{ respectively.} \end{cases}$$

(The parent and child relations referred to here is, of course, the ones in the Stern–Brocot tree.) Extend these two functions to the finite sequences of positive fractions with odd numerators by setting

$$\varphi_{\text{SBT}}\big(\langle \tfrac{n_1}{m_1}, \tfrac{n_2}{m_2}, \dots, \tfrac{n_\ell}{m_\ell} \rangle\big) = \varphi_{\text{SBT}}\big(\tfrac{n_1}{m_1}\big)^\frown \varphi_{\text{SBT}}\big(\tfrac{n_2}{m_2}\big)^\frown \cdots ^\frown \varphi_{\text{SBT}}\big(\tfrac{n_\ell}{m_\ell}\big) \quad \text{and}$$
$$\psi_{\text{SBT}}\big(\langle \tfrac{n_1}{m_1}, \tfrac{n_2}{m_2}, \dots, \tfrac{n_\ell}{m_\ell} \rangle\big) = \psi_{\text{SBT}}\big(\tfrac{n_1}{m_1}\big)^\frown \psi_{\text{SBT}}\big(\tfrac{n_2}{m_2}\big)^\frown \cdots ^\frown \psi_{\text{SBT}}\big(\tfrac{n_\ell}{m_\ell}\big).$$

(That these functions are well-defined is explained in the aforementioned article.) These functions were used in [3] to prove the following result.

Theorem 1 ([3])**.** *The equation* $\mathrm{MT}_i = \varphi_{\text{SBT}}\big(\psi_{\text{SBT}}^{i-1}\big(\frac{1}{2}\big)\big)$ *holds for any* $i \geq 1$.

Recall that, by using the Stern–Brocot tree, each irrational number from $(0,1) \setminus \mathbb{Q}$ can be represented as an infinite sequence over $\{\mathtt{L}, \mathtt{R}\}$ (see Definition 2). In the same spirit, we introduce an irrational number representation system by using the Mallows tree.

Definition 4. For each irrational number $\alpha \in (0,1) \setminus \mathbb{Q}$, define a pair of fractions $pf_i(\alpha)$ from MT_i and $\mathrm{M}_i(\alpha) \in \{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$ for $i \geq 1$, inductively, as follows:

- Set $pf_1(\alpha) = \langle \frac{1}{2}, \frac{2}{3} \rangle$.
- Suppose that we have defined $pf_i(\alpha)$, and write $pf_i^L(\alpha)$ and $pf_i^R(\alpha)$ for the left and right elements of it, respectively. Then set

$$\mathrm{M}_i(\alpha) = \begin{cases} \mathtt{L} & \text{if } \alpha < pf_i^L(\alpha), \\ \mathtt{M} & \text{if } pf_i^L(\alpha) < \alpha < pf_i^R(\alpha), \\ \mathtt{R} & \text{if } \alpha > pf_i^R(\alpha). \end{cases}$$

  Define $pf_{i+1}(\alpha)$ to be the left (resp. middle, right) child of $pf_i(\alpha)$ in the Mallows tree if $\mathrm{M}_i(\alpha) = \mathtt{L}$ (resp. $\mathtt{M}, \mathtt{R}$).

Then the *Mallows representation of* $\alpha \in (0,1) \setminus \mathbb{Q}$ is the infinite sequence $\mathrm{M}(\alpha) \coloneqq \langle \mathrm{M}_1(\alpha), \mathrm{M}_2(\alpha), \mathrm{M}_3(\alpha), \dots \rangle \in \{\mathtt{L}, \mathtt{M}, \mathtt{R}\}^\infty$.

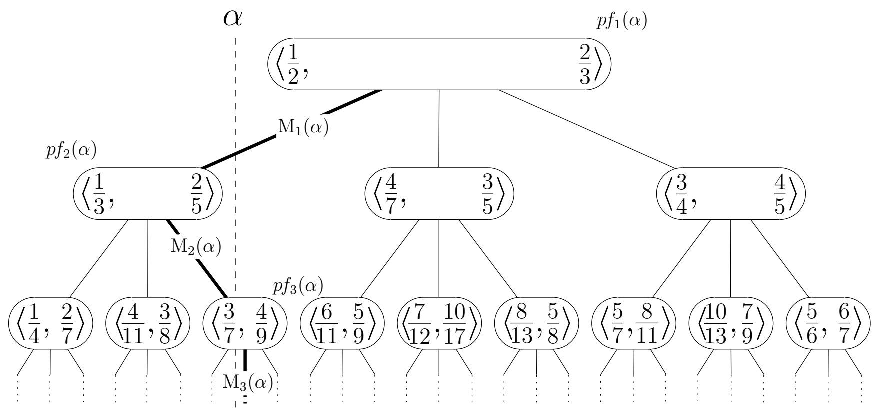

Figure 4: Illustration of the Mallows representation for an $\alpha \in (0,1) \setminus \mathbb{Q}$, which is between $\frac{3}{7}$ and $\frac{4}{9}$. We have $\mathrm{M}_1(\alpha) = \mathtt{L}, \mathrm{M}_2(\alpha) = \mathtt{R}$, and $\mathrm{M}_3(\alpha) = \mathtt{M}$.

See Figure 4 for the illustration of this representation. Note that in defining $\mathrm{M}_i(\alpha)$, we implicitly used the fact that $pf_i^L(\alpha) < pf_i^R(\alpha)$, which is true by Proposition 3 (ii).

Example 1. Proposition 3 of [3] states that a fraction from $(0,1) \cap \mathbb{Q}$ appears in the left half of MT_i for some $i \geq 1$ if and only if it is smaller than $2 - \sqrt{2}$. Using this and Proposition 3 (ii), one can show by induction that for any $i \geq 1$, the pair $pf_i(2-\sqrt{2})$ is the middle one in MT_i and $\mathrm{M}_i(2-\sqrt{2}) = \mathtt{M}$. Hence, we have $\mathrm{M}(2-\sqrt{2}) = \langle \mathtt{M}, \mathtt{M}, \mathtt{M}, \ldots \rangle$.

Write $\mathcal{M}$ for the set $\{\mathrm{M}(\alpha) \mid \alpha \in (0,1) \setminus \mathbb{Q}\}$. In the same spirit as Corollary 1, we shall next characterize this set.

Proposition 4. *The following relation holds:*

$$\mathcal{M} = \{\langle x_i \rangle_{i=1}^{\infty} \in \{\mathtt{L}, \mathtt{M}, \mathtt{R}\}^{\infty} \mid \text{For no } i_0, \text{we have } \langle x_i \rangle_{i=i_0}^{\infty} = \langle \mathtt{L} \rangle^{\infty} \text{ or } = \langle \mathtt{R} \rangle^{\infty}\}.$$

Proof. Let us first prove that $\mathcal{M}$ is contained in the right-hand side of the equation. To do so, take an $\alpha \in (0,1) \setminus \mathbb{Q}$ arbitrarily. Define intervals $I_i(\alpha)$ $(i \geq 1)$ inductively as follows:

$$I_1(\alpha) = \begin{cases} [0, \frac{1}{2}] & \text{if } \alpha < \frac{1}{2}, \\ [\frac{1}{2}, \frac{2}{3}] & \text{if } \frac{1}{2} < \alpha < \frac{2}{3}, \\ [\frac{2}{3}, 1] & \text{if } \frac{2}{3} < \alpha < 1, \end{cases}$$

$$I_{i+1}(\alpha) = \begin{cases} I_i(\alpha) \cap [0, pf_{i+1}^L(\alpha)] & \text{if } \alpha < pf_{i+1}^L(\alpha), \\ I_i(\alpha) \cap [pf_{i+1}^L(\alpha), pf_{i+1}^R(\alpha)] & \text{if } pf_{i+1}^L(\alpha) < \alpha < pf_{i+1}^R(\alpha), \\ I_i(\alpha) \cap [pf_{i+1}^R(\alpha), 1] & \text{if } \alpha > pf_{i+1}^R(\alpha). \end{cases}$$

Then it can be verified that the proper inclusions $I_1(\alpha) \supsetneq I_2(\alpha) \supsetneq I_3(\alpha) \supsetneq \cdots$ hold and also that the length of the interval $I_i(\alpha)$ converges to 0 as $i \to \infty$. Since $\alpha \in I_i(\alpha)$ for any $i \geq 1$, which can be proved by induction, we see that the intersection $\bigcap_{i=1}^{\infty} I_i(\alpha)$ is equal to the singleton set $\{\alpha\}$.

Assume for the sake of contradiction that there exists an i_0 such that $\mathrm{M}_i(\alpha) = \mathtt{L}$ (resp. $\mathtt{R}$) for all $i \geq i_0$. Then, by the definition of $\mathrm{M}_i(\alpha)$, we should have $\alpha < pf_i^L(\alpha)$ (resp. $\alpha > pf_i^R(\alpha)$) for all $i \geq i_0$. Write $[\frac{n_L}{m_L}, \frac{n_R}{m_R}]$ for $I_{i_0+1}(\alpha)$. Then induction shows that $\frac{n_L}{m_L} \in I_i(\alpha)$ (resp. $\frac{n_R}{m_R} \in I_i(\alpha)$) for all $i > i_0$. Consequently, $\frac{n_L}{m_L}$ (resp. $\frac{n_R}{m_R}$) $\in \bigcap_{i=1}^{\infty} I_i(\alpha) = \{\alpha\}$, contradicting the assumption that α is irrational.

To prove the opposite inclusion, take an arbitrary infinite sequence $\langle x_1, x_2, x_3, \ldots \rangle$ over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$ that does not eventually accord with $\langle \mathtt{L} \rangle^\infty$ or $\langle \mathtt{R} \rangle^\infty$. Set $pf_1 = \langle \frac{1}{2}, \frac{2}{3} \rangle$, and define pf_{i+1} to be the left (resp. middle, right) child of pf_i in the Mallows tree if $x_i = \mathtt{L}$ (resp. $x_i = \mathtt{M}, \mathtt{R}$). Write pf_i^L and pf_i^R for the left and right elements of pf_i, respectively. Also, define intervals I_i $(i \geq 1)$ inductively as follows:

$$I_1 = \begin{cases} [0, \frac{1}{2}] & \text{if } x_1 = \mathtt{L}, \\ [\frac{1}{2}, \frac{2}{3}] & \text{if } x_1 = \mathtt{M}, \\ [\frac{2}{3}, 1] & \text{if } x_1 = \mathtt{R}, \end{cases}$$

$$I_{i+1} = \begin{cases} I_i \cap [0, pf_{i+1}^L] & \text{if } x_{i+1} = \mathtt{L}, \\ I_i \cap [pf_{i+1}^L, pf_{i+1}^R] & \text{if } x_{i+1} = \mathtt{M}, \\ I_i \cap [pf_{i+1}^R, 1] & \text{if } x_{i+1} = \mathtt{R}. \end{cases}$$

As before, we can show that the intersection $\bigcap_{i=1}^{\infty} I_i$ is a singleton set, say $\{\alpha\}$, for a real number $\alpha \in [0, 1]$.

We first prove that α is not rational. To do so, assume for the sake of contradiction that α is rational. Then, as it follows from the assumption $\langle x_1, x_2, x_3, \ldots \rangle \neq \langle \mathtt{L} \rangle^\infty, \langle \mathtt{R} \rangle^\infty$ that $\alpha \neq 0, 1$, the rational number α should appear in MT_{i_0} for some $i_0 \geq 1$ by Proposition 3 (i). In view of the definition of MT_{i_0+1}, it is evident that either $\alpha < pf_{i_0+1}^L$ or $\alpha > pf_{i_0+1}^R$ should hold. As one can inductively show (by using the way that MT_{i+1} was constructed from $\left\{\frac{0}{1}, \frac{1}{1}\right\} \cup \mathrm{MT}_1 \cup \mathrm{MT}_2 \cup \cdots \cup \mathrm{MT}_i$ and the fact that pf_{i+1} is a child of pf_i) that if $\alpha < pf_{i_0+1}^L$ (resp. $\alpha > pf_{i_0+1}^R$) then $\alpha < pf_{i+1}^L$ (resp. $\alpha > pf_{i+1}^R$) for all $i \geq i_0$, it follows that we should have either $\alpha < pf_{i+1}^L$ for all $i \geq i_0$ or $\alpha > pf_{i+1}^R$ for all $i \geq i_0$. However, if $\alpha < pf_{i+1}^L$ (resp. $\alpha > pf_{i+1}^R$) for all $i \geq i_0$ then, because α does not belong to $[pf_{i+1}^L, pf_{i+1}^R]$ or $[pf_{i+1}^R, 1]$ (resp. $[0, pf_{i+1}^L]$ or $[pf_{i+1}^L, pf_{i+1}^R]$), it follows from the inductive definition of I_{i+1} and the relation $\alpha \in \bigcap_{i=1}^{\infty} I_i \subset I_{i+1}$ that x_{i+1} should be $\mathtt{L}$ (resp. $\mathtt{R}$) for all $i \geq i_0$, contrary to the assumption on the infinite sequence $\langle x_1, x_2, x_3, \ldots \rangle$.

Having proved that the real number α is not rational, it now makes sense to talk about the intervals $I_i(\alpha)$ $(i \geq 1)$ defined in the first paragraph of this proof. Using the equation $\bigcap_{i=1}^{\infty} I_i(\alpha) = \{\alpha\} = \bigcap_{i=1}^{\infty} I_i$, one can show by simultaneous induction on i that $I_i = I_i(\alpha), pf_i = pf_i(\alpha)$, and $x_i = \mathrm{M}_i(\alpha)$ for all $i \geq 1$. Therefore, the arbitrarily

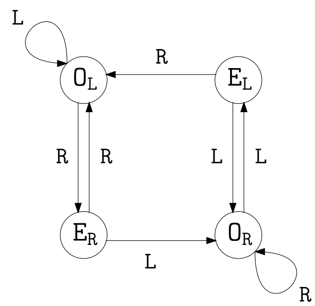

Figure 5: Diagram depicting the relation from Q_j to Q_{j+1} with arrows labeled by x_j.

chosen infinite sequence $\langle x_1, x_2, x_3, \ldots\rangle$ is equal to $\langle \mathrm{M}_1(\alpha), \mathrm{M}_2(\alpha), \mathrm{M}_3(\alpha), \ldots\rangle = \mathrm{M}(\alpha) \in \mathcal{M}$, completing the proof of the opposite inclusion. □

3. Linking Two Representations

In this section, we shall link the Stern–Brocot and Mallows representations. Instead of directly linking the Stern–Brocot representations $\mathcal{SB}$ and the Mallows representations $\mathcal{M}$, we link them via the ensuing intermediate set. Let $\mathcal{I}$ be the set of all infinite sequences $\langle x_1, x_2, x_3, \ldots\rangle$ over the quarternary set $\{\mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{R}, \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{L}\}$ that satisfy $x_1 = \mathtt{O}_\mathtt{L}$ and do not contain an occurrence of any of the following:

$$\begin{array}{llllll} \langle \mathtt{O}_\mathtt{L}, \mathtt{O}_\mathtt{R}\rangle, & \langle \mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{L}\rangle, & \langle \mathtt{E}_\mathtt{R}, \mathtt{E}_\mathtt{L}\rangle, & \langle \mathtt{E}_\mathtt{R}, \mathtt{E}_\mathtt{R}\rangle, & \langle \mathtt{O}_\mathtt{R}, \mathtt{O}_\mathtt{L}\rangle, & \langle \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{R}\rangle, \\ \langle \mathtt{E}_\mathtt{L}, \mathtt{E}_\mathtt{L}\rangle, & \langle \mathtt{E}_\mathtt{L}, \mathtt{E}_\mathtt{R}\rangle, & \langle \mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{R}\rangle^\infty, & \langle \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{L}\rangle^\infty, & \langle \mathtt{O}_\mathtt{L}\rangle^\infty, & \langle \mathtt{O}_\mathtt{R}\rangle^\infty. \end{array}$$

Let us first link $\mathcal{SB}$ to $\mathcal{I}$. For $\langle x_1, x_2, x_3, \ldots\rangle \in \mathcal{SB}$, define $Q_j(\langle x_1, x_2, x_3, \ldots\rangle) \in \{\mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{R}, \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{L}\}$ $(j \geq 1)$ inductively as follows:

$$Q_1(\langle x_1, x_2, x_3, \ldots\rangle) = \mathtt{O}_\mathtt{L},$$

$$Q_{j+1}(\langle x_1, x_2, x_3, \ldots\rangle) = \begin{cases} \mathtt{O}_\mathtt{L} & \text{if } (Q_j(\langle x_1, x_2, x_3, \ldots\rangle), x_j) \in \{(\mathtt{O}_\mathtt{L}, \mathtt{L}), (\mathtt{E}_\mathtt{R}, \mathtt{R}), (\mathtt{E}_\mathtt{L}, \mathtt{R})\}, \\ \mathtt{E}_\mathtt{R} & \text{if } (Q_j(\langle x_1, x_2, x_3, \ldots\rangle), x_j) = (\mathtt{O}_\mathtt{L}, \mathtt{R}), \\ \mathtt{O}_\mathtt{R} & \text{if } (Q_j(\langle x_1, x_2, x_3, \ldots\rangle), x_j) \in \{(\mathtt{E}_\mathtt{R}, \mathtt{L}), (\mathtt{O}_\mathtt{R}, \mathtt{R}), (\mathtt{E}_\mathtt{L}, \mathtt{L})\}, \\ \mathtt{E}_\mathtt{L} & \text{if } (Q_j(\langle x_1, x_2, x_3, \ldots\rangle), x_j) = (\mathtt{O}_\mathtt{R}, \mathtt{L}). \end{cases}$$

(The inductive definition of $Q_{j+1}(\langle x_1, x_2, x_3, \ldots\rangle)$ is depicted in Figure 5.) Write $Q(\mathrm{SB}(\alpha))$ for the infinite sequence $\langle Q_1(\mathrm{SB}(\alpha)), Q_2(\mathrm{SB}(\alpha)), Q_3(\mathrm{SB}(\alpha)), \ldots\rangle$ over the quarternary set $\{\mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{R}, \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{L}\}$. Then we have the following, which can be verified readily.

Proposition 5. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, the infinite sequence $Q(\mathrm{SB}(\alpha))$ belongs to $\mathcal{I}$. Moreover, $Q\colon \mathcal{SB} \to \mathcal{I}$ is bijective.*

Before presenting the next link, let us explain the reason for using the four letters $\mathtt{O_L}, \mathtt{E_R}, \mathtt{O_R}, \mathtt{E_L}$.

Proposition 6. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, the following two statements are true for all $j \geq 1$:*

(i) *The numerator of $f_j(\alpha)$ is odd (resp. even) if and only if $Q_j(\mathrm{SB}(\alpha))$ is $\mathtt{O_L}$ or $\mathtt{O_R}$ (resp. $\mathtt{E_R}$ or $\mathtt{E_L}$).*

(ii) *The fraction $f_j(\alpha)$ is the left (resp. right) element of $\langle \frac{n_L}{m_L}, \frac{n_R}{m_R} \rangle$ if and only if $Q_j(\mathrm{SB}(\alpha))$ is $\mathtt{O_L}$ or $\mathtt{E_L}$ (resp. $\mathtt{O_R}$ or $\mathtt{E_R}$), where $\langle \frac{n_L}{m_L}, \frac{n_R}{m_R} \rangle$ is the label from the Mallows tree such that either of the two fractions is $f_j(\alpha)$.*

Proof. Take an arbitrary $\alpha \in (0,1) \setminus \mathbb{Q}$. By induction on j, we shall prove the correctness of the two statements for this α.

For $j = 1$, two statements are evidently correct. Assume that we have verified the correctness of the two statements for some $j \geq 1$. To complete the proof of the induction step, we do case analysis according as the value of $Q_{j+1}(\mathrm{SB}(\alpha)) \in \{\mathtt{O_L}, \mathtt{E_R}, \mathtt{O_R}, \mathtt{E_L}\}$. Before doing so, let us note that the parent and child relations used in the remainder of this proof is the ones in the Stern–Brocot tree.

`Case 1:` $Q_{j+1}(\mathrm{SB}(\alpha)) = \mathtt{O_L}$. By the definition of Q, there are the following three possibilities: $(Q_j(\mathrm{SB}(\alpha)), \mathrm{SB}_j(\alpha)) = (\mathtt{O_L}, \mathtt{L}), (\mathtt{E_R}, \mathtt{R})$ or $(\mathtt{E_L}, \mathtt{R})$. Let us thus divide the argument further into these subcases. Write $\langle \frac{n_L}{m_L}, \frac{n_R}{m_R} \rangle$ (resp. $\langle \frac{n'_L}{m'_L}, \frac{n'_R}{m'_R} \rangle$) for the labels from the Mallows tree such that either of the two fractions is $f_j(\alpha)$ (resp. $f_{j+1}(\alpha)$).

If $Q_j(\mathrm{SB}(\alpha)) = \mathtt{O_L}$ and $\mathrm{SB}_j(\alpha) = \mathtt{L}$ then $f_{j+1}(\alpha)$ is the left child of $f_j(\alpha)$. Also, by the induction hypothesis, the numerator of $f_j(\alpha)$ is odd and $f_j(\alpha) = \frac{n_L}{m_L}$. Since $\varphi_{\mathrm{SBT}}(\frac{n_L}{m_L}) = \langle \frac{n_L}{m_L}, \frac{n_R}{m_R} \rangle$ by Theorem 1, the right child of $f_j(\alpha) = \frac{n_L}{m_L}$ should have even numerator. As precisely one child of $f_j(\alpha) = \frac{n_L}{m_L}$ has odd numerator by Proposition 2 of [3], we conclude that $f_{j+1}(\alpha)$ has odd numerator, which shows that Statement (i) is true in this subcase. By the mediant construction, the right child of $f_{j+1}(\alpha)$ should have even numerator. Since $\varphi_{\mathrm{SBT}}(f_{j+1}(\alpha)) = \langle \frac{n'_L}{m'_L}, \frac{n'_R}{m'_R} \rangle$ by Theorem 1, it thus follows from the definition of φ_{SBT} that $f_{j+1}(\alpha)$ is the left element $\frac{n'_L}{m'_L}$ of the pair $\langle \frac{n'_L}{m'_L}, \frac{n'_R}{m'_R} \rangle$. Hence, Statement (ii) is also true in this subcase.

If $Q_j(\mathrm{SB}(\alpha)) = \mathtt{E_R}$ (resp. $\mathtt{E_L}$) and $\mathrm{SB}_j(\alpha) = \mathtt{R}$ then $f_{j+1}(\alpha)$ is the right child of $f_j(\alpha)$. Also, by the induction hypothesis, the numerator of $f_j(\alpha)$ is even and $f_j(\alpha) = \frac{n_R}{m_R}$ (resp. $\frac{n_L}{m_L}$). As both children of $f_j(\alpha)$ have odd numerator by Proposition 2 of [3], Statement (i) is true in these subcases. By the mediant construction, the right child of $f_{j+1}(\alpha)$ should have even numerator. Since $\varphi_{\mathrm{SBT}}(f_{j+1}(\alpha)) = \langle \frac{n'_L}{m'_L}, \frac{n'_R}{m'_R} \rangle$ by Theorem 1, it thus follows from the definition of φ_{SBT} that $f_{j+1}(\alpha)$ is the left element $\frac{n'_L}{m'_L}$ of the pair $\langle \frac{n'_L}{m'_L}, \frac{n'_R}{m'_R} \rangle$. Therefore, Statement (ii) is also true in these subcases.

`Case 2:` $Q_{j+1}(\mathrm{SB}(\alpha)) = \mathtt{E_R}$. In this case, by the definition of Q, we have $Q_j(\mathrm{SB}(\alpha)) = \mathtt{O_L}$ and $\mathrm{SB}_j(\alpha) = \mathtt{R}$, which implies that $f_{j+1}(\alpha)$ is the right child of $f_j(\alpha)$. Also, by the induction hypothesis, the numerator of $f_j(\alpha)$ is odd and $f_j(\alpha) = \frac{n_L}{m_L}$. Since $\varphi_{\mathrm{SBT}}(f_j(\alpha)) = \langle \frac{n_L}{m_L}, \frac{n_R}{m_R} \rangle$ by Theorem 1, we conclude that $f_{j+1}(\alpha)$ is the right element $\frac{n_R}{m_R}$ of the pair $\langle \frac{n_R}{m_R}, \frac{n_L}{m_L} \rangle$ and has even numerator by the definition of φ_{SBT}. Therefore, Statements (i) and (ii) are both correct in this case.
`Cases 3 and 4:` $Q_{j+1}(\mathrm{SB}(\alpha)) = \mathtt{O_R}$ or $\mathtt{E_L}$. The proofs closely parallel that for Cases 1 and 2, respectively. □

To link $\mathcal{I}$ to $\mathcal{M}$, let us introduce one notation here. Observe that for any $\langle x_1, x_2, x_3, \ldots \rangle \in \mathcal{I}$, if x_j is equal to $\mathtt{O_L}$ or $\mathtt{O_R}$ then at least one of x_{j+1} and x_{j+2} is equal to $\mathtt{O_L}$ or $\mathtt{O_R}$. Consequently, the set $\{j \geq 1 \mid x_j = \mathtt{O_L} \text{ or } \mathtt{O_R}\}$ contains infinitely many elements. Let us write $1 = o_{\langle x_1,x_2,x_3,\ldots\rangle}(1) < o_{\langle x_1,x_2,x_3,\ldots\rangle}(2) < o_{\langle x_1,x_2,x_3,\ldots\rangle}(3) < \cdots$ for the enumeration of the infinite set $\{j \geq 1 \mid x_j = \mathtt{O_L} \text{ or } \mathtt{O_R}\}$. Whenever it is clear which sequence $\langle x_1, x_2, x_3, \ldots \rangle \in \mathcal{I}$ is referred to, we shall henceforth suppress the explicit dependence on it in the notation of the enumeration.

The ensuing proposition follows readily from the observation made in the last paragraph.

Proposition 7. *For any* $\langle x_1, x_2, x_3, \ldots \rangle \in \mathcal{I}$ *and* $i \geq 1$, *we have* $o(i+1) > o(i) + 1$ *if and only if* $o(i+1) = o(i) + 2$.

For $\langle x_1, x_2, x_3, \ldots \rangle \in \mathcal{I}$ and a positive integer i, define $T_i(\langle x_1, x_2, x_3, \ldots \rangle) \in \{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$ as follows:

- If $o(i+1) = o(i) + 1$, then set

$$T_i(\langle x_1, x_2, x_3, \ldots \rangle) = \begin{cases} \mathtt{L} & \text{when } x_{o(i)} = \mathtt{O_L}, \\ \mathtt{R} & \text{when } x_{o(i)} = \mathtt{O_R}. \end{cases}$$

- If $o(i+1) > o(i) + 1$ (equivalently, by Proposition 7, $o(i+1) = o(i) + 2$), then set

$$T_i(\langle x_1, x_2, x_3, \ldots \rangle) = \begin{cases} \mathtt{L} & \text{when } x_{o(i)} = x_{o(i+1)} = \mathtt{O_R}, \\ \mathtt{R} & \text{when } x_{o(i)} = x_{o(i+1)} = \mathtt{O_L}, \\ \mathtt{M} & \text{when } x_{o(i)} \neq x_{o(i+1)}. \end{cases}$$

Let us write $T(\langle x_1, x_2, x_3, \ldots \rangle)$ for the infinite sequence $\langle T_1(\langle x_1, x_2, x_3, \ldots \rangle), T_2(\langle x_1, x_2, x_3, \ldots \rangle), T_3(\langle x_1, x_2, x_3, \ldots \rangle), \ldots \rangle$ over the ternary set $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$. Then the following can be proved.

Proposition 8. *For any* $\langle x_1, x_2, x_3, \ldots \rangle \in \mathcal{I}$, *the infinite sequence* $T(\langle x_1, x_2, x_3, \ldots \rangle)$ *belongs to* $\mathcal{M}$. *Moreover,* $T\colon \mathcal{I} \to \mathcal{M}$ *is bijective.*

To obtain the main theorem of this section, the next property is necessary.

Lemma 1. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$ and $i \geq 1$, we have $\varphi_{\text{SBT}}(f_{o(i)}(\alpha)) = pf_i(\alpha)$.*

Proof. Take an arbitrary $\alpha \in (0,1) \setminus \mathbb{Q}$. We shall prove by induction that the stated equation holds for any $i \geq 1$.

The correctness of the equation for $i = 1$ is plain. Assume that we have verified the correctness of the equation for some $i \geq 1$. Since proofs for the two cases, $Q_{o(i)}(\text{SB}(\alpha)) = \texttt{O}_\texttt{L}$ and $Q_{o(i)}(\text{SB}(\alpha)) = \texttt{O}_\texttt{R}$, run parallel to each other, we shall consider the former case only. Let us further divide the argument into the following cases.

`Case 1:` $o(i+1) = o(i) + 1$. In this case, we have $Q_{o(i+1)}(\text{SB}(\alpha)) = \texttt{O}_\texttt{L}$ and $\text{SB}_{o(i)}(\alpha) = \texttt{L}$. The latter implies $\alpha < f_{o(i)}(\alpha)$. Also, it follows from the induction hypothesis and Proposition 6 that $f_{o(i)}(\alpha)$ is the left element of $pf_i(\alpha)$. These two facts imply $\text{M}_i(\alpha) = \texttt{L}$, which in turn shows that $pf_{i+1}(\alpha)$ is the left child of $pf_i(\alpha)$ in the Mallows tree.

Both $f_{o(i)}(\alpha)$ and $f_{o(i+1)}(\alpha)$ have odd numerators by Proposition 6. Also, it follows from the equation $\text{SB}_{o(i)}(\alpha) = \texttt{L}$ that $f_{o(i+1)}(\alpha)$ is the left child of $f_{o(i)}(\alpha)$ in the Stern–Brocot tree. It can then be inferred from Theorem 1 that $\varphi_{\text{SBT}}(f_{o(i+1)}(\alpha))$ is the left child of $\varphi_{\text{SBT}}(f_{o(i)}(\alpha))$ in the Mallows tree.

The induction hypothesis and what we have shown above combine to prove the correctness of the equation for $i + 1$ in this case.

`Case 2:` $o(i+1) > o(i) + 1$ and $Q_{o(i+1)}(\text{SB}(\alpha)) = \texttt{O}_\texttt{L}$. In this case, we have $Q_{o(i)+1}(\text{SB}(\alpha)) = \texttt{E}_\texttt{R}$, $\text{SB}_{o(i)}(\alpha) = \texttt{R}$, $Q_{o(i)+2}(\text{SB}(\alpha)) = \texttt{O}_\texttt{L}$, and $\text{SB}_{o(i)+1}(\alpha) = \texttt{R}$. It follows that $\alpha > f_{o(i)+1}(\alpha) > f_{o(i)}(\alpha)$. The induction hypothesis and Proposition 6 imply $pf_i(\alpha) = \langle f_{o(i)}(\alpha), f_{o(i)+1}(\alpha)\rangle$. These prove $\text{M}_i(\alpha) = \texttt{R}$, which in turn proves that $pf_{i+1}(\alpha)$ is the right child of $pf_i(\alpha)$ in the Mallows tree.

Since it follows from the equation $\text{SB}_{o(i)+1}(\alpha) = \texttt{R}$ that $f_{o(i)+2}(\alpha)$ is the right child of $f_{o(i)+1}(\alpha)$ in the Stern–Brocot tree, it can be inferred from Theorem 1 that $\varphi_{\text{SBT}}(f_{o(i)+2}(\alpha))$ is the right child of $\varphi_{\text{SBT}}(f_{o(i)}(\alpha))$ in the Mallows tree.

By combining what we have shown above with the induction hypothesis, we get the desired equation for $i + 1$ in this case.

`Case 3:` $o(i+1) > o(i) + 1$ and $Q_{o(i+1)}(\text{SB}(\alpha)) = \texttt{O}_\texttt{R}$. In this case, we have $Q_{o(i)+1}(\text{SB}(\alpha)) = \texttt{E}_\texttt{R}$, $\text{SB}_{o(i)}(\alpha) = \texttt{R}$, $Q_{o(i)+2}(\text{SB}(\alpha)) = \texttt{O}_\texttt{R}$, and $\text{SB}_{o(i)+1}(\alpha) = \texttt{L}$. It follows that $f_{o(i)}(\alpha) < \alpha < f_{o(i)+1}(\alpha)$. Also, it can be seen from the induction hypothesis and Proposition 6 that $pf_i(\alpha) = \langle f_{o(i)}(\alpha), f_{o(i)+1}(\alpha)\rangle$. These imply $\text{M}_i(\alpha) = \texttt{M}$, which in turn implies that $pf_{i+1}(\alpha)$ is the middle child of $pf_i(\alpha)$ in the Mallows tree.

From the values of $\text{SB}_{o(i)+1}(\alpha)$ and $\text{SB}_{o(i)}(\alpha)$, it is immediate that in the Stern–Brocot tree, $f_{o(i)+2}(\alpha)$ is the left child of $f_{o(i)+1}(\alpha)$, which is the right child of $f_{o(i)}(\alpha)$. It can then be inferred from Proposition 6 and Theorem 1 that $\varphi_{\text{SBT}}(f_{o(i)+2}(\alpha))$ is the middle child of $\varphi_{\text{SBT}}(f_{o(i)}(\alpha))$ in the Mallows tree.

The combination of what we have shown above and the induction hypothesis proves the equation for $i+1$ in this case. □

Having finished preparations, we can now prove that the composition of two bijective maps $Q\colon \mathcal{SB} \to \mathcal{I}$ and $T\colon \mathcal{I} \to \mathcal{M}$ links two representations as follows.

Theorem 2. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, we have $T \circ Q(\mathrm{SB}(\alpha)) = \mathrm{M}(\alpha)$.*

Proof. Take an $\alpha \in (0,1) \setminus \mathbb{Q}$ arbitrarily. To prove that $T_i(Q(\mathrm{SB}(\alpha))) = \mathrm{M}_i(\alpha)$ holds for any $i \geq 1$, we do case analysis.

`Case 1:` $Q_{o(i)}(\mathrm{SB}(\alpha)) = \mathtt{O}_{\mathtt{L}}$. From Proposition 6 and Lemma 1, it follows that $f_{o(i)}(\alpha)$ is the left element of $pf_i(\alpha)$ and has odd numerator. From the definition of φ_{SBT} and Lemma 1, it also follows that the right element $pf_i^R(\alpha)$ of $pf_i(\alpha)$ has even numerator and is the right child of $f_{o(i)}(\alpha)$ in the Stern–Brocot tree.

If $\alpha < pf_i^L(\alpha) = f_{o(i)}(\alpha)$ then $\mathrm{SB}_{o(i)}(\alpha) = \mathtt{L}$; hence, $Q_{o(i)+1}(\mathrm{SB}(\alpha)) = \mathtt{O}_{\mathtt{L}}$. Consequently, $o(i+1) = o(i)+1$. Thus, $T_i(Q(\mathrm{SB}(\alpha))) = \mathtt{L} = \mathrm{M}_i(\alpha)$.

If $f_{o(i)} = pf_i^L(\alpha) < \alpha < pf_i^R(\alpha)$ (resp. $\alpha > pf_i^R(\alpha)$) then $\mathrm{SB}_{o(i)}(\alpha) = \mathtt{R}$ and $\mathrm{SB}_{o(i)+1}(\alpha) = \mathtt{L}$ (resp. $\mathtt{R}$). Hence, $Q_{o(i)+1}(\mathrm{SB}(\alpha)) = \mathtt{E}_{\mathtt{R}}$ and $Q_{o(i)+2}(\mathrm{SB}(\alpha)) = \mathtt{O}_{\mathtt{R}}$ (resp. $\mathtt{O}_{\mathtt{L}}$). Consequently, $o(i+1) = o(i)+2$. Thus, $T_i(Q(\mathrm{SB}(\alpha))) = \mathtt{M}$ (resp. $\mathtt{R}$) $= \mathrm{M}_i(\alpha)$.

`Case 2:` $Q_{o(i)}(\mathrm{SB}(\alpha)) = \mathtt{O}_{\mathtt{R}}$. From Proposition 6 and Lemma 1, it follows that $f_{o(i)}(\alpha)$ is the right element of $pf_i(\alpha)$ and has odd numerator. From the definition of φ_{SBT} and Lemma 1, it also follows that the left element $pf_i^L(\alpha)$ of $pf_i(\alpha)$ has even numerator and is the left child of $f_{o(i)}(\alpha)$ in the Stern–Brocot tree.

If $\alpha < pf_i^L(\alpha)$ (resp. $pf_i^L(\alpha) < \alpha < pf_i^R(\alpha) = f_{o(i)}(\alpha)$) then $\mathrm{SB}_{o(i)}(\alpha) = \mathtt{L}$ and $\mathrm{SB}_{o(i)+1}(\alpha) = \mathtt{L}$ (resp. $\mathtt{R}$). Hence, $Q_{o(i)+1}(\mathrm{SB}(\alpha)) = \mathtt{E}_{\mathtt{L}}$ and $Q_{o(i)+2}(\mathrm{SB}(\alpha)) = \mathtt{O}_{\mathtt{R}}$ (resp. $\mathtt{O}_{\mathtt{L}}$). Consequently, $o(i+1) = o(i)+2$. Thus, $T_i(Q(\mathrm{SB}(\alpha))) = \mathtt{L}$ (resp. $\mathtt{M}$) $= \mathrm{M}_i(\alpha)$.

If $\alpha > pf_i^R(\alpha) = f_{o(i)}(\alpha)$ then $\mathrm{SB}_{o(i)}(\alpha) = \mathtt{R}$ and hence $Q_{o(i)+1}(\mathrm{SB}(\alpha)) = \mathtt{O}_{\mathtt{R}}$. Consequently, $o(i+1) = o(i)+1$. Thus, $T_i(Q(\mathrm{SB}(\alpha))) = \mathtt{R} = \mathrm{M}_i(\alpha)$. □

4. Combinatorial Results

In this section, we shall present various combinatorial results on Mallows representations of irrational numbers. Specifically, in the first three subsections, we shall study the relationship between the Mallows and Stern–Brocot representations in terms of eventual periodicity, positive Poisson stability, and transitivity, respectively. Then, in the fourth subsection, we characterize those irrational numbers whose Mallows representations do not contain an occurrence of $\mathtt{L}, \mathtt{M}, \mathtt{R}$, respectively. In the last subsection, we study certain questions concerning the periodicity of the sequence of parities of the numerators of $f_1(\alpha), f_2(\alpha), f_3(\alpha), \ldots$.

The following set of facts can readily be proved by induction but is worth stating here explicitly because it will be used repeatedly in the ensuing subsections.

Lemma 2. *Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and $\ell \geq 1$.*

(i) *For any positive integers $j_1 < j_2$, the following two statements are equivalent:*

- *Both of the two equations $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j_1}^{j_1+\ell-1} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=j_2}^{j_2+\ell-1}$ and $Q_{j_1}(\mathrm{SB}(\alpha)) = Q_{j_2}(\mathrm{SB}(\alpha))$ hold.*
- *The equation $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_1}^{j_1+\ell} = \langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_2}^{j_2+\ell}$ holds.*

(ii) *For any positive integers $i_1 < i_2$, the following two statements are equivalent:*

- *Both of the two equations $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_1}^{i_1+\ell-1} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_2}^{i_2+\ell-1}$ and $Q_{o(i_1)}(\mathrm{SB}(\alpha)) = Q_{o(i_2)}(\mathrm{SB}(\alpha))$ hold.*
- *The equation $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_1)}^{o(i_1+\ell)} = \langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_2)}^{o(i_2)+o(i_1+\ell)-o(i_1)}$ holds.*

Moreover, if any (hence, both) of the above two statements is correct, then $o(i_2+k) - o(i_2) = o(i_1+k) - o(i_1)$ for all $k \in \{0,1,\ldots,\ell\}$.

4.1. Eventual Periodicity

Theorem 3. *For any $\alpha \in (0,1)\setminus\mathbb{Q}$, its Mallows representation is eventually periodic if and only if its Stern–Brocot representation is eventually periodic.*

Proof. Take an $\alpha \in (0,1) \setminus \mathbb{Q}$ arbitrarily.

Assume first that $\mathrm{M}(\alpha)$ is eventually periodic. Then, by Theorem 2, there exists an $i_0 \geq 1$ such that $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ is periodic with period p. Clearly, by the pigeonhole principle, there exist $i_1 < i_2 \in \{i_0, i_0+p, i_0+2p\}$ such that $Q_{o(i_1)}(\mathrm{SB}(\alpha)) = Q_{o(i_2)}(\mathrm{SB}(\alpha)) \in \{\mathtt{O_L}, \mathtt{O_R}\}$. Then for any $\ell \geq 1$, since we have $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_1}^{i_1+\ell-1} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_2}^{i_2+\ell-1}$ by the definition of p, an application of Lemma 2 proves $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_1)}^{o(i_1+\ell)} = \langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_2)}^{o(i_2)+o(i_1+\ell)-o(i_1)}$, which in turn implies $\langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_1)}^{o(i_1+\ell)-1} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_2)}^{o(i_2)+o(i_1+\ell)-o(i_1)-1}$ by the same lemma. Since this holds for any $\ell \geq 1$, we conclude that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_1)}^{\infty} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_2)}^{\infty}$, which clearly implies that $\mathrm{SB}(\alpha)$ is eventually periodic.

Conversely, suppose that $\mathrm{SB}(\alpha)$ is eventually periodic. Then there exists a $j_0 \geq 1$ such that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j_0}^{\infty}$ is periodic with period p. By the pigeonhole principle, there exist $j_1 < j_2 \in \{j_0, j_0+p, j_0+2p, j_0+3p, j_0+4p\}$ such that $Q_{j_1}(\mathrm{SB}(\alpha)) = Q_{j_2}(\mathrm{SB}(\alpha))$. This equation and the periodicity of $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j_0}^{\infty}$ combine to imply, by Lemma 2, that $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_1}^{j_1+2\ell+1} = \langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_2}^{j_2+2\ell+1}$ holds for any $\ell \geq 1$. Let i_1 be the smallest integer satisfying $o(i_1) \geq j_1$. Then the inequality $o(i_1+\ell) \leq o(i_1) + 2\ell$ holds for any $\ell \geq 1$, which can be verified by induction. Observe also that if $Q_{j_1}(\mathrm{SB}(\alpha)) \notin \{\mathtt{O_L}, \mathtt{O_R}\}$ (equivalently, $o(i_1) \neq j_1$) then $Q_{j_1+1}(\mathrm{SB}(\alpha)) \in \{\mathtt{O_L}, \mathtt{O_R}\}$

(equivalently, $o(i_1) = j_1 + 1$), which can be readily seen from the definition of Q. From these, we can derive the inequality $o(i_1 + \ell) \le j_1 + 2\ell + 1$, which, along with the above equation, shows that $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_1)}^{o(i_1+\ell)} = \langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_2+o(i_1)-j_1}^{j_2+o(i_1+\ell)-j_1}$. As the integer $j_2 + o(i_1) - j_1$ can be written as $o(i_2)$ for some $i_2 > i_1$ because $Q_{j_2+o(i_1)-j_1}(\mathrm{SB}(\alpha)) = Q_{o(i_1)}(\mathrm{SB}(\alpha)) \in \{\mathtt{O_L}, \mathtt{O_R}\}$, we can apply Lemma 2 to obtain $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_1}^{i_1+\ell-1} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_2}^{i_2+\ell-1}$. Since this equation holds for any $\ell \ge 1$, we conclude $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_1}^{\infty} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_2}^{\infty}$, from which it can readily be seen that $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=1}^{\infty}$ is eventually periodic. An application of Theorem 2 then finishes the proof. □

From Proposition 2, it can readily be inferred that $\mathrm{SB}(\alpha)$ is eventually periodic if and only if the elements of α are eventually periodic. Also, it is known [4] that the elements of α are eventually periodic if and only if α is quadratic. From these and the above theorem, we can deduce the following corollary.

Corollary 2. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, its Mallows representation is eventually periodic if and only if it is quadratic.*

4.2. Positive Poisson Stability

Let $f\colon X \to X$ be a continuous map on a metric space X. We say that a point of X is *positively Poisson stable* [6] if its positive orbit and its ω-limit set intersect. Thus, in the shift dynamical system over a non-empty set Σ, a point $\langle x_1, x_2, x_3, \ldots\rangle \in \Sigma^\infty$ is positively Poisson stable if and only if there exists an $n_0 \ge 1$ such that for each $\ell \ge 1$, one can find a positive integer $p(\ell)$ satisfying $\langle x_n\rangle_{n=n_0}^{n_0+\ell-1} = \langle x_n\rangle_{n=n_0+p(\ell)}^{n_0+p(\ell)+\ell-1}$ (equivalently, there exists an $n_0 \ge 1$ and a function $p\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that this equation holds for any $\ell \ge 1$).

By viewing the Mallows and Stern–Brocot representations as points of the shift dynamical systems over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}^\infty$ and $\{\mathtt{L}, \mathtt{R}\}^\infty$, respectively, let us study the relationship between their positive Poisson stability.

Theorem 4. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, its Mallows representation is positively Poisson stable if and only if its Stern–Brocot representation is positively Poisson stable.*

Proof. Take an $\alpha \in (0,1) \setminus \mathbb{Q}$ arbitrarily.

To prove the "only if" part of the statement, suppose that $\mathrm{M}(\alpha)$ is positively Poisson stable. Then, by Theorem 2, there exists an $i_0 \ge 1$ and a function $p\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that the equation

$$\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{i_0+\ell-1} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0+p(\ell)}^{i_0+p(\ell)+\ell-1} \tag{1}$$

holds for any $\ell \ge 1$.

We first claim that there exists an $i_1 \ge i_0$ such that for each $\ell \ge 1$, one can find an $i_2 > i_1$ satisfying both $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_1}^{i_1+\ell-1} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_2}^{i_2+\ell-1}$ and

$Q_{o(i_1)}(\mathrm{SB}(\alpha)) = Q_{o(i_2)}(\mathrm{SB}(\alpha))$. If i_0 stands as i_1, then there is nothing to do. If on the other hand i_0 does not stand as i_1, then there exists an $\ell' \geq 1$ such that for each $i' > i_0$, we have either $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{i_0+\ell'-1} \neq \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i'}^{i'+\ell'-1}$ or $Q_{o(i_0)}(\mathrm{SB}(\alpha)) \neq Q_{o(i')}(\mathrm{SB}(\alpha))$. By substituting $i_0 + p(\ell')$ into i', we get $Q_{o(i_0)}(\mathrm{SB}(\alpha)) \neq Q_{o(i_0+p(\ell'))}(\mathrm{SB}(\alpha))$. We shall show that $i_0 + p(\ell')$ stands as i_1. To do so, take an $\ell \geq 1$ arbitrarily. By substituting $p(\ell') + \max\{\ell, \ell'\}$ into ℓ in Equation (1) and then by taking the length $\max\{\ell, \ell'\}$ closing subsequences, we obtain

$$\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0+p(\ell')}^{i_0+p(\ell')+\max\{\ell,\ell'\}-1} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_2}^{i_2+\max\{\ell,\ell'\}-1}, \tag{2}$$

where $i_2 \coloneqq i_0+p(p(\ell')+\max\{\ell,\ell'\})+p(\ell')$. From this equation and Equation (1) with ℓ' substituted for ℓ, we can deduce $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{i_0+\ell'-1} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_2}^{i_2+\ell'-1}$. This equation and the definition of ℓ' combine to guarantee $Q_{o(i_0)}(\mathrm{SB}(\alpha)) \neq Q_{o(i_2)}(\mathrm{SB}(\alpha))$. Since we also have $Q_{o(i_0)}(\mathrm{SB}(\alpha)) \neq Q_{o(i_0+p(\ell'))}(\mathrm{SB}(\alpha))$ as has been mentioned already, we conclude that $Q_{o(i_0+p(\ell'))}(\mathrm{SB}(\alpha)) = Q_{o(i_2)}(\mathrm{SB}(\alpha))$. Equation (2) and this equation show that $i_0 + p(\ell')$ has the desired property of i_1.

Using the established claim, we shall prove the positive Poisson stability of $\mathrm{SB}(\alpha)$ as follows. Take an i_1 as in the established claim. Then, by Lemma 2, for each $\ell \geq 1$, one can find an $i_2 > i_1$ satisfying $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_1)}^{o(i_1+\ell)} = \langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_2)}^{o(i_2)+o(i_1+\ell)-o(i_1)}$. By applying the same lemma to this equation, we get the equation $\langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_1)}^{o(i_1+\ell)-1}$ $= \langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_2)}^{o(i_2)+o(i_1+\ell)-o(i_1)-1}$ between two length $o(i_1+\ell) - o(i_1)$ ($\geq \ell$) sequences. As ℓ was chosen arbitrarily, this argument implies that $\mathrm{SB}(\alpha)$ is positively Poisson stable.

Having proved the "only if" part of the statement, we then substantiate the "if" part. To do so, suppose that $\mathrm{SB}(\alpha)$ is positively Poisson stable. Then there exists a $j_0 \geq 1$ and a function $p\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that the equation $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j_0}^{j_0+\ell-1}$ $= \langle \mathrm{SB}_j(\alpha)\rangle_{j=j_0+p(\ell)}^{j_0+p(\ell)+\ell-1}$ holds for any $\ell \geq 1$.

Let us first substantiate the claim that there exists a $j_1 \geq j_0$ such that for each $\ell \geq 1$, one can find a $j_2 > j_1$ satisfying both $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j_1}^{j_1+\ell-1} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=j_2}^{j_2+\ell-1}$ and $Q_{j_1}(\mathrm{SB}(\alpha)) = Q_{j_2}(\mathrm{SB}(\alpha))$. If j_0 already stands as j_1, then there is nothing to do. If on the other hand j_0 does not stand as j_1, then there exists an $\ell(1) \geq 1$ such that for any $j' > j_0$, we have either $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j_0}^{j_0+\ell(1)-1} \neq \langle \mathrm{SB}_j(\alpha)\rangle_{j=j'}^{j'+\ell(1)-1}$ or $Q_{j_0}(\mathrm{SB}(\alpha)) \neq Q_{j'}(\mathrm{SB}(\alpha))$. If $j(1) \coloneqq j_0 + p(\ell(1))$ stands as j_1, then there is nothing to do. If on the other hand $j(1)$ does not stand as j_1, then there exists an $\ell(2) \geq 1$ such that for any $j' > j(1)$, we have either $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j(1)}^{j(1)+\ell(2)-1} \neq \langle \mathrm{SB}_j(\alpha)\rangle_{j=j'}^{j'+\ell(2)-1}$ or $Q_{j(1)}(\mathrm{SB}(\alpha)) \neq Q_{j'}(\mathrm{SB}(\alpha))$. If $j(2) \coloneqq j(1) + p(j(1) + \max\{\ell(1), \ell(2)\} - j_0)$ stands as j_1, then there is nothing to do. If on the other hand $j(2)$ does not stand as j_1, then there exists an $\ell(3) \geq 1$ such that for any $j' > j(2)$, we have either $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j(2)}^{j(2)+\ell(3)-1} \neq \langle \mathrm{SB}_j(\alpha)\rangle_{j=j'}^{j'+\ell(3)-1}$ or $Q_{j(2)}(\mathrm{SB}(\alpha)) \neq Q_{j'}(\mathrm{SB}(\alpha))$. Set $j(3) \coloneqq j(2)+p(j(2)+\max\{\ell(1), \ell(2), \ell(3)\}-j_0)$. Then, it follows from the definition of the function $p\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that the equation

$$\langle \mathrm{SB}_j(\alpha)\rangle_{j=j(m)}^{j(m)+\max\{\ell(n)\mid 1\le n\le m+1\}-1} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=j(m+1)}^{j(m+1)+\max\{\ell(n)\mid 1\le n\le m+1\}-1}$$

holds for each $m \in \{0,1,2\}$, where we set $j(0) \coloneqq j_0$ for notational convenience. Combining these equations, we obtain

$$\langle \mathrm{SB}_j(\alpha)\rangle_{j=j(m_1)}^{j(m_1)+\ell(m_1+1)-1} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=j(m_2)}^{j(m_2)+\ell(m_1+1)-1} \tag{3}$$

for any $0 \le m_1 < m_2 \le 3$. The definitions of $\ell(1), \ell(2), \ell(3)$, and Equation (3), imply that four elements $Q_{j(0)}(\mathrm{SB}(\alpha)), Q_{j(1)}(\mathrm{SB}(\alpha)), Q_{j(2)}(\mathrm{SB}(\alpha)), Q_{j(3)}(\mathrm{SB}(\alpha))$ of $\{\mathtt{O_L}, \mathtt{E_R}, \mathtt{O_R}, \mathtt{E_L}\}$ are pairwise distinct. We shall show that $j(3)$ stands as j_1. To do so, take an ℓ arbitrarily. Then, by the definition of the function $p\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, we have

$$\langle \mathrm{SB}_j(\alpha)\rangle_{j(3)}^{j(3)+\max\{\ell,\ell(1),\ell(2),\ell(3)\}-1} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=j_2}^{j_2+\max\{\ell,\ell(1),\ell(2),\ell(3)\}-1}, \tag{4}$$

where $j_2 \coloneqq j(3) + p(j(3) + \max\{\ell, \ell(1), \ell(2), \ell(3)\} - j(0))$. From Equations (3) and (4), we deduce that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j(m)}^{j(m)+\ell(m+1)-1} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=j_2}^{j_2+\ell(m+1)-1}$ for each $m \in \{0,1,2\}$, which, along with the definitions of $\ell(1), \ell(2), \ell(3)$, implies that $Q_{j_2}(\mathrm{SB}(\alpha))$ is distinct from $Q_{j(0)}(\mathrm{SB}(\alpha)), Q_{j(1)}(\mathrm{SB}(\alpha)), Q_{j(2)}(\mathrm{SB}(\alpha))$. Because four elements $Q_{j(0)}(\mathrm{SB}(\alpha)), Q_{j(1)}(\mathrm{SB}(\alpha)), Q_{j(2)}(\mathrm{SB}(\alpha)), Q_{j(3)}(\mathrm{SB}(\alpha))$ of $\{\mathtt{O_L}, \mathtt{E_R}, \mathtt{O_R}, \mathtt{E_L}\}$ are pairwise distinct (as we mentioned already), it follows that $Q_{j(3)}(\mathrm{SB}(\alpha)) = Q_{j_2}(\mathrm{SB}(\alpha))$. From Equation (4) and this equation, we conclude that $j(3)$ indeed stands as j_1.

Using the substantiated claim, we shall establish positive Poisson stability of $\mathrm{M}(\alpha)$ as follows. Let j_1 be as in the substantiated claim, and let i_1 be the smallest integer satisfying $o(i_1) \ge j_1$. Take an $\ell \ge 1$ arbitrarily. Then there exists a $j_2 > j_1$ such that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j_1}^{o(i_1+\ell)-1} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=j_2}^{j_2+o(i_1+\ell)-j_1-1}$ and $Q_{j_1}(\mathrm{SB}(\alpha)) = Q_{j_2}(\mathrm{SB}(\alpha))$. By applying Lemma 2, we obtain

$$\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_1}^{o(i_1+\ell)} = \langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_2}^{j_2+o(i_1+\ell)-j_1}. \tag{5}$$

It follows in particular that $Q_{j_2+o(i_1)-j_1}(\mathrm{SB}(\alpha)) = Q_{o(i_1)}(\mathrm{SB}(\alpha)) \in \{\mathtt{O_L}, \mathtt{O_R}\}$, which implies that $j_2 + o(i_1) - j_1$ is of the form $o(i_2)$ for some $i_2 > i_1$. Being the length $o(i_1+\ell) - o(i_1) + 1$ closing subsequences of the left- and right-hand sides of Equation (5), two sequences $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_1)}^{o(i_1+\ell)}$ and $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_2)}^{o(i_2)+o(i_1+\ell)-o(i_1)}$ are identical, from which we obtain $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_1}^{i_1+\ell-1} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_2}^{i_2+\ell-1}$ by Lemma 2. As $\ell \ge 1$ was chosen arbitrarily, we conclude that $T(Q(\mathrm{SB}(\alpha)))$ is positively Poisson stable. An application of Theorem 2 finishes the proof. □

One corollary is in order before going to the next subsection.

Corollary 3. *For any $\alpha \in (0,1)\setminus\mathbb{Q}$, its Mallows representation is positively Poisson stable if and only if its CF-representation is positively Poisson stable.*

Here, we are identifying an infinite continued fraction $[0; a_1, a_2, \dots]$ with a point $\langle a_1, a_2, a_3, \dots\rangle$ of the shift dynamical system over $\mathbb{Z}_{>0}$.

Proof. Suppose first that $\mathrm{M}(\alpha)$ is positively Poisson stable. Then $\mathrm{SB}(\alpha)$ is positively Poisson stable by the above theorem. From this and Proposition 2, positive Poisson stability of the CF-representation of α can readily be inferred.

Conversely, suppose that the CF-representation $[0; a_1, a_2, \dots]$ of α is positively Poisson stable. Hence, there exists a $k_0 \geq 2$ and a function $p\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that the equation $\langle a_k\rangle_{k=k_0}^{k_0+\ell-1} = \langle a_k\rangle_{k=k_0+p(\ell)}^{k_0+p(\ell)+\ell-1}$ holds for any $\ell \geq 1$. Define a function $p'\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ by

$$p'(\ell) = \begin{cases} p(\ell) & \text{if } p(\ell) \text{ is even,} \\ p(p(\ell)+\ell) & \text{if } p(\ell) \text{ is odd and } p(p(\ell)+\ell) \text{ is even,} \\ p(p(\ell)+\ell)+p(\ell) & \text{if both } p(\ell) \text{ and } p(p(\ell)+\ell) \text{ are odd.} \end{cases}$$

Then it can readily be seen that $p'(\ell)$ is even for any $\ell \geq 1$.

We claim that the equation $\langle a_k\rangle_{k=k_0}^{k_0+\ell-1} = \langle a_k\rangle_{k=k_0+p'(\ell)}^{k_0+p'(\ell)+\ell-1}$ holds for any $\ell \geq 1$. To substantiate this claim, take an $\ell \geq 1$ arbitrarily. By the definition of the function $p\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, we have

$$\langle a_k\rangle_{k=k_0}^{k_0+\ell-1} = \langle a_k\rangle_{k=k_0+p(\ell)}^{k_0+p(\ell)+\ell-1} \text{ and } \langle a_k\rangle_{k=k_0}^{k_0+p(\ell)+\ell-1} = \langle a_k\rangle_{k=k_0+p(p(\ell)+\ell)}^{k_0+p(p(\ell)+\ell)+p(\ell)+\ell-1}. \quad (6)$$

Hence, if either $p(\ell)$ or $p(p(\ell)+\ell)$ is even, then the claimed equation is correct. If both $p(\ell)$ and $p(p(\ell)+\ell)$ are odd then, as the length ℓ closing subsequence of the left- (and hence, right-) hand side of the right equation of (6) is equal to $\langle a_k\rangle_{k=k_0}^{k_0+\ell-1}$ by the left equation of (6), it follows that the claimed equation is correct also in this case.

Having substantiated the claim, we can prove the positive Poisson stability of $\mathrm{M}(\alpha)$ as follows. Take an arbitrary $\ell \geq 1$. Then, because $p'(\ell)$ is even, it can be inferred from Proposition 2 and the substantiated claim that

$$\langle \mathrm{SB}_j(\alpha)\rangle_{j=\sum_{k=1}^{k_0-1} a_k}^{\sum_{k=1}^{k_0+\ell-1} a_k} = \langle \mathrm{SB}_j(\alpha)\rangle_{j=\sum_{k=1}^{k_0+p'(\ell)-1} a_k}^{\sum_{k=1}^{k_0+p'(\ell)+\ell-1} a_k}.$$

Since the length of the above sequences is $a_{k_0} + a_{k_0+1} + \cdots + a_{k_0+\ell-1} + 1\,(\geq \ell)$ and since ℓ was chosen arbitrarily, we conclude that $\mathrm{SB}(\alpha)$ is positively Poisson stable. Thus, by the above theorem, $\mathrm{M}(\alpha)$ is positively Poisson stable. □

4.3. Transitivity

Let $f\colon X \to X$ be a continuous map on a metric space X. We say that a point of X is *transitive* if its orbit is dense in X. Thus, in the shift dynamical system over a non-empty set Σ, a point $\langle x_1, x_2, x_3, \dots\rangle \in \Sigma^\infty$ is transitive if and only if every non-empty finite sequence over Σ appears in it.

In this subsection, by viewing the Mallows and Stern–Brocot representations as points of the shift dynamical systems over $\{\mathtt{L},\mathtt{M},\mathtt{R}\}^\infty$ and $\{\mathtt{L},\mathtt{R}\}^\infty$, respectively, we shall study the relationship between their transitivity. In doing so, we need the ensuing finite version $\dot{Q}$ of Q. For an $X \in \{\mathtt{O_L},\mathtt{E_R},\mathtt{O_R},\mathtt{E_L}\}$ and a finite sequence $\langle x_1, x_2, \dots, x_\ell\rangle$ ($\ell \geq 1$) over $\{\mathtt{L},\mathtt{R}\}$, define $\dot{Q}^X_j(\langle x_1, x_2, \dots, x_\ell\rangle)$ ($j \in \{1, 2, \dots, \ell+1\}$) inductively as follows:

$$\dot{Q}^X_1(\langle x_1, x_2, \dots, x_\ell\rangle) = X,$$

$$\dot{Q}^X_{j+1}(\langle x_1, x_2, \dots, x_\ell\rangle) = \begin{cases} \mathtt{O_L} & \text{if } (\dot{Q}^X_j(\langle x_1, x_2, \dots, x_\ell\rangle), x_j) \in \{(\mathtt{O_L},\mathtt{L}), (\mathtt{E_R},\mathtt{R}), (\mathtt{E_L},\mathtt{R})\},\\ \mathtt{E_R} & \text{if } (\dot{Q}^X_j(\langle x_1, x_2, \dots, x_\ell\rangle), x_j) = (\mathtt{O_L},\mathtt{R}),\\ \mathtt{O_R} & \text{if } (\dot{Q}^X_j(\langle x_1, x_2, \dots, x_\ell\rangle), x_j) \in \{(\mathtt{E_R},\mathtt{L}), (\mathtt{O_R},\mathtt{R}), (\mathtt{E_L},\mathtt{L})\},\\ \mathtt{E_L} & \text{if } (\dot{Q}^X_j(\langle x_1, x_2, \dots, x_\ell\rangle), x_j) = (\mathtt{O_R},\mathtt{L}). \end{cases}$$

We set $\dot{Q}^X(\langle x_1, x_2, \dots, x_\ell\rangle) \coloneqq \langle \dot{Q}^X_j(\langle x_1, x_2, \dots, x_\ell\rangle)\rangle_{j=1}^{\ell+1}$. The following finite version $\dot{T}$ of T is also necessary. Let $\langle x_1, x_2, \dots, x_\ell\rangle$ ($\ell \geq 1$) be a sequence over $\{\mathtt{O_L},\mathtt{E_R},\mathtt{O_R},\mathtt{E_L}\}$ such that the set $\{j \in \{1, 2, \dots, \ell\} \mid x_j = \mathtt{O_L} \text{ or } x_j = \mathtt{O_R}\}$ contains at least two, say n, elements. Then define $\dot{T}_i(\langle x_1, x_2, \dots, x_\ell\rangle)$ ($i \in \{1, 2, \dots, n-1\}$) as in the definition of T_i with $o(\cdot)$ replaced by $\dot{o}(\cdot)$, where $\dot{o}(1) < \dot{o}(2) < \cdots < \dot{o}(n)$ is the enumeration of the set $\{j \in \{1, 2, \dots, \ell\} \mid x_j = \mathtt{O_L} \text{ or } x_j = \mathtt{O_R}\}$, and set $\dot{T}(\langle x_1, x_2, \dots, x_\ell\rangle) \coloneqq \langle \dot{T}_i(\langle x_1, x_2, \dots, x_\ell\rangle)\rangle_{i=1}^{n-1}$.

The following property can readily be proved by induction.

Lemma 3. *Let X be either $\mathtt{O_L}$ or $\mathtt{O_R}$, and let $\langle x_1, x_2, \dots, x_\ell\rangle$ ($\ell \geq 1$) be a finite sequence over $\{\mathtt{L},\mathtt{R}\}$ such that the last letter of $\dot{Q}^X(\langle x_1, x_2, \dots, x_\ell\rangle)$ is either $\mathtt{O_L}$ or $\mathtt{O_R}$. Then for any $\alpha \in (0,1) \setminus \mathbb{Q}$ and $i_0 \geq 1$, the sequence $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^\infty$ starts with $\dot{T}(\dot{Q}^X(\langle x_1, x_2, \dots, x_\ell\rangle))$ and $Q_{o(i_0)}(\mathrm{SB}(\alpha)) = X$ if and only if we have $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_0)}^{o(i_0)+\ell} = \dot{Q}^X(\langle x_1, x_2, \dots, x_\ell\rangle)$.*

Having finished preparations, we can go to the main result of this subsection.

Theorem 5. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, its Mallows representation is transitive if and only if its Stern–Brocot representation is transitive.*

Proof. Take an $\alpha \in (0,1) \setminus \mathbb{Q}$ arbitrarily.

To prove the "only if" part of the statement, suppose that $\mathrm{M}(\alpha)$ is transitive. Take a non-empty finite sequence w over $\{\mathtt{L},\mathtt{R}\}$ arbitrarily. Then construct finite sequences w', w'' over $\{\mathtt{L},\mathtt{R}\}$ so that the following conditions are satisfied:

(i) the last letter of $\dot{Q}^{\mathtt{O_L}}(w^\frown w')$ is either $\mathtt{O_L}$ or $\mathtt{O_R}$;

(ii) the last letter of $\dot{Q}^{\mathtt{O_R}}(w'')$ is $\mathtt{O_L}$;

(iii) $\dot{T}(\dot{Q}^{\mathtt{O_R}}(w''))$ starts with $\dot{T}(\dot{Q}^{\mathtt{O_L}}(w^\frown w'))$.

By the transitivity of $\mathrm{M}(\alpha)$ and Theorem 2, there exists an $i_0 \geq 1$ such that the infinite sequence $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ starts with $\dot{T}(\dot{Q}^{\mathtt{O_R}}(w''))^\frown\dot{T}(\dot{Q}^{\mathtt{O_L}}(w^\frown w'))$. Let us note here that the sequence $\dot{T}(\dot{Q}^{\mathtt{O_R}}(w''))^\frown\dot{T}(\dot{Q}^{\mathtt{O_L}}(w^\frown w'))$ is identical to $\dot{T}(\dot{Q}^{\mathtt{O_R}}(w''^\frown w^\frown w'))$, which can be seen from the definitions of $\dot{Q}^{\mathtt{O_L}}$, $\dot{Q}^{\mathtt{O_R}}$, $\dot{T}$, and Condition (ii).

If $Q_{o(i_0)}(\mathrm{SB}(\alpha)) = \mathtt{O_L}$ then, because the infinite sequence $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ starts with $\dot{T}(\dot{Q}^{\mathtt{O_L}}(w^\frown w'))$ by the definition of i_0 and Condition (iii), one can apply Lemma 3 to obtain $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_0)}^{o(i_0)+\mathrm{lh}(w)} = \dot{Q}^{\mathtt{O_L}}(w)$, from which it can readily be inferred that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_0)}^{o(i_0)+\mathrm{lh}(w)-1} = w$.

If $Q_{o(i_0)}(\mathrm{SB}(\alpha)) = \mathtt{O_R}$ then, because the infinite sequence $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ starts with $\dot{T}(\dot{Q}^{\mathtt{O_R}}(w''^\frown w^\frown w'))$ by the definition of i_0 and the equation $\dot{T}(\dot{Q}^{\mathtt{O_R}}(w''))$ $^\frown\dot{T}(\dot{Q}^{\mathtt{O_L}}(w^\frown w')) = \dot{T}(\dot{Q}^{\mathtt{O_R}}(w''^\frown w^\frown w'))$, one can apply Lemma 3 to see that $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_0)}^{o(i_0)+\mathrm{lh}(w'')+\mathrm{lh}(w)} = \dot{Q}^{\mathtt{O_R}}(w''^\frown w)$, from which it can readily be inferred that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_0)+\mathrm{lh}(w'')}^{o(i_0)+\mathrm{lh}(w'')+\mathrm{lh}(w)-1} = w$.

Having proved the "only if" part of the statement, we then substantiate the "if" part. To do so, suppose that $\mathrm{SB}(\alpha)$ is transitive. Take a non-empty finite sequence w over $\{\mathtt{L},\mathtt{M},\mathtt{R}\}$ arbitrarily. Then construct a finite sequence u over $\{\mathtt{L},\mathtt{R}\}$ such that the last letter of $\dot{Q}^{\mathtt{O_L}}(u)$ is either $\mathtt{O_L}$ or $\mathtt{O_R}$ and the sequence $\dot{T}(\dot{Q}^{\mathtt{O_L}}(u))$ starts with w. Set

$$\upsilon(\mathtt{O_L}) = \langle \mathtt{L},\mathtt{L},\mathtt{L}\rangle, \quad \upsilon(\mathtt{E_R}) = \langle \mathtt{L},\mathtt{L},\mathtt{R}\rangle, \quad \upsilon(\mathtt{O_R}) = \langle \mathtt{L},\mathtt{R},\mathtt{L}\rangle, \quad \upsilon(\mathtt{E_L}) = \langle \mathtt{R},\mathtt{L},\mathtt{L}\rangle,$$

and consider the sequence $u^\frown u'^\frown u^\frown u''^\frown u^\frown u'''^\frown u$, where

$$\begin{aligned} u' &= \upsilon(\dot{Q}^{\mathtt{O_R}}_{\mathrm{lh}(u)}(u)),\\ u'' &= \upsilon(\dot{Q}^{\mathtt{E_L}}_{2\,\mathrm{lh}(u)+3}(u^\frown u'^\frown u)),\\ u''' &= \upsilon(\dot{Q}^{\mathtt{E_R}}_{3\,\mathrm{lh}(u)+6}(u^\frown u'^\frown u^\frown u''^\frown u)). \end{aligned}$$

By the transitivity of $\mathrm{SB}(\alpha)$, there exists a positive integer j_0 such that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=j_0}^{\infty}$ starts with $u^\frown u'^\frown u^\frown u''^\frown u^\frown u'''^\frown u$.

If $Q_{j_0}(\mathrm{SB}(\alpha)) = \mathtt{O_L}$ then $j_0 = o(i_0)$ for some $i_0 \geq 1$. As $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_0)}^{o(i_0)+\mathrm{lh}(u)}$ $= \dot{Q}^{\mathtt{O_L}}(u)$, Lemma 3 then implies that the infinite sequence $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ starts with $\dot{T}(\dot{Q}^{\mathtt{O_L}}(u))$ and hence with w.

If $Q_{j_0}(\mathrm{SB}(\alpha)) = \mathtt{O_R}$ then $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0}^{j_0+2\,\mathrm{lh}(u)+3} = \dot{Q}^{\mathtt{O_R}}(u^\frown u'^\frown u)$. By the definition of u', we have $\dot{Q}^{\mathtt{O_R}}_{\mathrm{lh}(u)+3}(u^\frown u'^\frown u) = \mathtt{O_L}$, which implies that the length $\mathrm{lh}(u)+1$ closing subsequence of $\dot{Q}^{\mathtt{O_R}}(u^\frown u'^\frown u)$ is equal to $\dot{Q}^{\mathtt{O_L}}(u)$. Therefore, we have $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0+\mathrm{lh}(u)+3}^{j_0+2\,\mathrm{lh}(u)+3} = \dot{Q}^{\mathtt{O_L}}(u)$. Since $Q_{j_0+\mathrm{lh}(u)+3}(\mathrm{SB}(\alpha)) = \mathtt{O_L}$, the integer $j_0+\mathrm{lh}(u)+3$ is of the form $o(i_0)$ for some $i_0 \geq 1$. An application of Lemma 3 then implies that the infinite sequence $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ starts with $\dot{T}(\dot{Q}^{\mathtt{O_L}}(u))$ and hence with w.

If $Q_{j_0}(\mathrm{SB}(\alpha)) = \mathtt{E_L}$ (resp. $\mathtt{E_R}$) then, by arguing as above, we can show that the length $\mathrm{lh}(u) + 1$ closing subsequence of $\dot{Q}^{\mathtt{E_L}}(u^\frown u'^\frown u^\frown u''^\frown u)$ (resp. $\dot{Q}^{\mathtt{E_R}}(u^\frown u'^\frown u^\frown u''^\frown u^\frown u'''^\frown u)$) is equal to $\dot{Q}^{\mathtt{O_L}}(u)$. Thus, the sequence $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0+2\,\mathrm{lh}(u)+6}^{j_0+3\,\mathrm{lh}(u)+6}$ (resp. $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0+3\,\mathrm{lh}(u)+9}^{j_0+4\,\mathrm{lh}(u)+9}$) is equal to $\dot{Q}^{\mathtt{O_L}}(u)$. Since $Q_{j_0+2\,\mathrm{lh}(u)+6}(\mathrm{SB}(\alpha))$ (resp. $Q_{j_0+3\,\mathrm{lh}(w)+9}(\mathrm{SB}(\alpha))) = \mathtt{O_L}$, the integer $j_0 + 2\,\mathrm{lh}(u) + 6$ (resp. $j_0 + 3\,\mathrm{lh}(u) + 9$) is of the form $o(i_0)$ for some $i_0 \geq 1$. Then, by Lemma 3, the infinite sequence $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ starts with $\dot{T}(\dot{Q}^{\mathtt{O_L}}(u))$ and hence with w. □

Before going to the next subsection, let us present one corollary. In the following, as we did in Corollary 3, we identify an infinite continued fraction $[0; a_1, a_2, \dots]$ with a point $\langle a_1, a_2, a_3, \dots\rangle$ of the shift dynamical system over $\mathbb{Z}_{>0}$.

Corollary 4. *For any $\alpha \in (0, 1) \setminus \mathbb{Q}$, its Mallows representation is transitive if and only if its CF-representation is transitive.*

Proof. The proof for the "only if" part of the statement being analogous to that in Corollary 3, we shall prove the "if" part of the statement only. Suppose that the CF-representation $[0; a_1, a_2, \dots]$ of α is transitive. Take a non-empty finite sequence w over $\{\mathtt{L}, \mathtt{R}\}$ arbitrarily, and write $\langle\mathtt{L}\rangle^\frown w^\frown\langle\mathtt{L}\rangle$ as $\langle\mathtt{L}\rangle^{e_1\frown}\langle\mathtt{R}\rangle^{e_2\frown}\langle\mathtt{L}\rangle^{e_3\frown}\langle\mathtt{R}\rangle^{e_4\frown}\cdots{}^\frown\langle\mathtt{L}\rangle^{e_{2\ell-1}}$ for an $\ell \geq 1$ and positive integers $e_1, e_2, \dots, e_{2\ell-1}$. By the transitivity of the CF-representation of α, there exists a $k_0 \geq 2$ such that $\langle a_k\rangle_{k=k_0}^{k_0+4\ell-3} = \langle e_1, e_2, \dots, e_{2\ell-1}\rangle^2$. If k_0 is odd, then it follows from the equation $\langle a_k\rangle_{k=k_0}^{k_0+2\ell-2} = \langle e_1, e_2, \dots, e_{2\ell-1}\rangle$ and Proposition 2 that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=\sum_{k=1}^{k_0-1} a_k}^{-1+\sum_{k=1}^{k_0+2\ell-2} a_k} = \langle\mathtt{L}\rangle^\frown w^\frown\langle\mathtt{L}\rangle$. If k_0 is odd, then it follows from the equation $\langle a_k\rangle_{k=k_0+2\ell-1}^{k_0+4\ell-3} = \langle e_1, e_2, \dots, e_{2\ell-1}\rangle$ and Proposition 2 that $\langle \mathrm{SB}_j(\alpha)\rangle_{j=\sum_{k=1}^{k_0+2\ell-2} a_k}^{-1+\sum_{k=1}^{k_0+4\ell-3} a_k} = \langle\mathtt{L}\rangle^\frown w^\frown\langle\mathtt{L}\rangle$. In either case, therefore, $\mathrm{SB}(\alpha)$ contains an occurrence of the arbitrarily taken sequence w. Since this implies that $\mathrm{SB}(\alpha)$ is transitive, an application of the above theorem proves that $\mathrm{M}(\alpha)$ is transitive, completing the proof of the "if" part of the statement. □

4.4. Not Containing a Particular Letter

In this subsection, we shall characterize those $\alpha \in (0, 1) \setminus \mathbb{Q}$ whose Mallows representations do not contain an occurrence of a particular letter. To do so, let us introduce one notation here. For any $\alpha \in (0, 1) \setminus \mathbb{Q}$ which satisfies the condition that both $\mathtt{O_L}$ and $\mathtt{O_R}$ appear infinitely many times in $Q(\mathrm{SB}(\alpha))$, we write $e_1, e_2, e_3, \dots$ for the positive integers satisfying

$$\langle Q_{o(1)}(\mathrm{SB}(\alpha)), Q_{o(2)}(\mathrm{SB}(\alpha)), Q_{o(3)}(\mathrm{SB}(\alpha)), \dots\rangle = \langle\mathtt{O_L}\rangle^{e_1\frown}\langle\mathtt{O_R}\rangle^{e_2\frown}\langle\mathtt{O_L}\rangle^{e_3\frown}\langle\mathtt{O_R}\rangle^{e_4\frown}\cdots. \tag{7}$$

Here, the explicit dependence on α is suppressed in the notation because, whenever we use this notation later, it will be clear which α is referred to.

Proposition 9. *For any $\alpha = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$, its Mallows representation does not contain an occurrence of* $\mathtt{M}$ *if and only if a_{2k} is even for all $k \geq 1$.*

Proof. From the definition of T, it can be inferred that for any $\alpha \in (0,1) \setminus \mathbb{Q}$, its Mallows representation does not contain an occurrence of $\mathtt{M}$ if and only if $\langle Q_{o(1)}(\mathrm{SB}(\alpha)), Q_{o(2)}(\mathrm{SB}(\alpha)), Q_{o(3)}(\mathrm{SB}(\alpha)), \dots \rangle = \langle \mathtt{O}_\mathtt{L} \rangle^\infty$. Suppose that an $\alpha \in (0,1) \setminus \mathbb{Q}$ satisfies the latter condition. Then it is immediate from the definition of Q that for any $i \geq 1$, if $o(i+1) = o(i)+1$ then $\mathrm{SB}_{o(i)}(\alpha) = \mathtt{L}$. Also, if $o(i+1) > o(i)+1$ then $o(i+1) = o(i)+2$ by Proposition 7 and $\mathrm{SB}_{o(i)}(\alpha) = \mathrm{SB}_{o(i)+1}(\alpha) = \mathtt{R}$. Being not eventually constant, $\mathrm{SB}(\alpha)$ should thus be of the form

$$\langle \mathtt{L} \rangle^{e'_1} {}^\frown \langle \mathtt{R}, \mathtt{R} \rangle^\frown \langle \mathtt{L} \rangle^{e'_2} {}^\frown \langle \mathtt{R}, \mathtt{R} \rangle^\frown \langle \mathtt{L} \rangle^{e'_3} {}^\frown \langle \mathtt{R}, \mathtt{R} \rangle^\frown \cdots$$

for non-negative integers $e'_1, e'_2, e'_3, \dots$, infinitely many of which are positive. Conversely, a calculation shows that if the Stern–Brocot representation of an $\alpha \in (0,1) \setminus \mathbb{Q}$ is of the above form then $\langle Q_{o(1)}(\mathrm{SB}(\alpha)), Q_{o(2)}(\mathrm{SB}(\alpha)), Q_{o(3)}(\mathrm{SB}(\alpha)), \dots \rangle = \langle \mathtt{O}_\mathtt{L} \rangle^\infty$. Since the above form can equivalently be written as

$$\langle \mathtt{L} \rangle^{e''_1} {}^\frown \langle \mathtt{R} \rangle^{2e''_2} {}^\frown \langle \mathtt{L} \rangle^{e''_3} {}^\frown \langle \mathtt{R} \rangle^{2e''_4} {}^\frown \langle \mathtt{L} \rangle^{e''_5} {}^\frown \langle \mathtt{R} \rangle^{2e''_6} {}^\frown \cdots$$

for a non-negative integer e''_1 and positive integers $e''_2, e''_3, e''_4, \dots$, an application of Proposition 2 finishes the proof. □

Remark 3. Even if only one k does not satisfy $a_{2k} \in 2\mathbb{Z}_{>0}$, the Mallows representation can contain infinitely many occurrences of $\mathtt{M}$. For example, the Mallows representation of $2 - \sqrt{2} = [0; 1, 1, 2, 2, 2, 2, \dots]$ is $\langle \mathtt{M} \rangle^\infty$ by Example 1.

Proposition 10. *For any $\alpha = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$, its Mallows representation does not contain an occurrence of* $\mathtt{R}$ *if and only if*

$$\langle a_1, a_2, a_3, \dots \rangle = \langle e'_1, 1, 2e'_2 \rangle^\frown w_1^\frown \langle 2e'_3 \rangle^\frown w_2^\frown \langle 2e'_4 \rangle^\frown w_3^\frown \cdots,$$

where $e'_1, e'_2, e'_3, \dots$ are positive integers and w_k ($k \geq 1$) is either equal to $\langle 2 \rangle$ or of the form $\langle 1, e''_k, 1 \rangle$ for some $e''_k \geq 1$.

Proof. Let $\alpha = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$.

Suppose that $\mathrm{M}(\alpha)$ does not contain an occurrence of $\mathtt{R}$. Then, because $\mathrm{SB}(\alpha)$ is not eventually constant, if $Q_{o(i)}(\mathrm{SB}(\alpha)) = \mathtt{O}_\mathtt{L}$ then there should be an $i' > i$ such that $Q_{o(i')}(\mathrm{SB}(\alpha)) = \mathtt{O}_\mathtt{R}$. Likewise, if $Q_{o(i)}(\mathrm{SB}(\alpha)) = \mathtt{O}_\mathtt{R}$ then there should be an $i' > i$ such that $Q_{o(i')}(\mathrm{SB}(\alpha)) = \mathtt{O}_\mathtt{L}$. It follows that both $\mathtt{O}_\mathtt{L}$ and $\mathtt{O}_\mathtt{R}$ appear infinitely many times in $Q(\mathrm{SB}(\alpha))$. Let $e_1, e_2, e_3, \dots$ be positive integers as in Equation (7). Then, it follows from the assumption on $\mathrm{M}(\alpha) = T(Q(\mathrm{SB}(\alpha)))$ that $\mathrm{SB}(\alpha)$ should be the ensuing one:

$$\langle \mathtt{L} \rangle^{e_1 - 1} {}^\frown \langle \mathtt{R} \rangle^\frown \langle \mathtt{L} \rangle^{2e_2} {}^\frown \langle \mathtt{R} \rangle^\frown \langle \mathtt{L} \rangle^{e_3 - 1} {}^\frown \langle \mathtt{R} \rangle^\frown \langle \mathtt{L} \rangle^{2e_4} {}^\frown \langle \mathtt{R} \rangle^\frown \langle \mathtt{L} \rangle^{e_5 - 1} {}^\frown \langle \mathtt{R} \rangle^\frown \langle \mathtt{L} \rangle^{2e_6} {}^\frown \langle \mathtt{R} \rangle^\frown \cdots.$$

We then have $\langle a_1, a_2, a_3, \ldots\rangle = \langle e_1, 1, 2e_2\rangle^\frown w_1^\frown \langle 2e_4\rangle^\frown w_2^\frown \langle 2e_6\rangle^\frown w_3^\frown \cdots$ by Proposition 2, where w_k is equal to $\langle 2\rangle$ or $\langle 1, e_{2k+1} - 1, 1\rangle$ according as $e_{2k+1} = 1$ or $e_{2k+1} > 1$.

Conversely, suppose that the elements of α are as stated. Then

$$\mathrm{SB}(\alpha) = \langle \mathtt{L}\rangle^{e'_1-1\frown}\langle \mathtt{R}\rangle^\frown\langle \mathtt{L}\rangle^{2e'_2\frown} u_1^\frown \langle \mathtt{L}\rangle^{2e'_3\frown} u_2^\frown \langle \mathtt{R}\rangle^{2e'_4\frown} u_3^\frown \cdots$$

by Proposition 2, where $u_j = \langle \mathtt{R}\rangle^2$ or $\langle \mathtt{R}\rangle^\frown\langle \mathtt{L}\rangle^{e''_j\frown}\langle \mathtt{R}\rangle$. A calculation then reveals that $\mathrm{M}(\alpha) = T(Q(\mathrm{SB}(\alpha)))$ does not contain an occurrence of $\mathtt{R}$. □

Proposition 11. *For any $\alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q}$, its Mallows representation does not contain an occurrence of* L *if and only if*

$$\langle a_1, a_2, a_3, \ldots\rangle = \langle 1, 2e'_1 - 1\rangle^\frown w_1^\frown \langle 2e'_2\rangle^\frown w_2^\frown \langle 2e'_3\rangle^\frown w_3^\frown \cdots,$$

where $e'_1, e'_2, e'_3, \ldots$ are positive integers and w_k ($k \geq 1$) is either equal to $\langle 2\rangle$ or of the form $\langle 1, e''_k, 1\rangle$ for some $e''_k \geq 1$.

Proof. Arguing as in the proof of the foregoing proposition, we can show that for any $\alpha \in (0,1) \setminus \mathbb{Q}$, if $\mathrm{M}(\alpha)$ does not contain an occurrence of L then both $\mathtt{O}_\mathtt{L}$ and $\mathtt{O}_\mathtt{R}$ appear infinitely many times in $Q(\mathrm{SB}(\alpha))$.

Let $\alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q}$. If $\mathrm{M}(\alpha) = T(Q(\mathrm{SB}(\alpha)))$ does not contain an occurrence of L, then it can be verified that $\mathrm{SB}(\alpha)$ is equal to

$$\langle \mathtt{R}\rangle^{2e_1-1\frown}\langle \mathtt{L}\rangle^\frown\langle \mathtt{R}\rangle^{e_2-1\frown}\langle \mathtt{L}\rangle^\frown\langle \mathtt{R}\rangle^{2e_3\frown}\langle \mathtt{L}\rangle^\frown\langle \mathtt{R}\rangle^{e_4-1\frown}\langle \mathtt{L}\rangle^\frown\langle \mathtt{R}\rangle^{2e_5\frown}\langle \mathtt{L}\rangle^\frown\langle \mathtt{R}\rangle^{e_6-1\frown}\langle \mathtt{L}\rangle^\frown\cdots,$$

where positive integers $e_1, e_2, e_3, \ldots$ are as in Equation (7). Along with Proposition 2, this implies $\langle a_1, a_2, a_3, \ldots\rangle = \langle 1, 2e_1 - 1\rangle^\frown w_1^\frown \langle 2e_3\rangle^\frown w_2^\frown \langle 2e_5\rangle^\frown w_3^\frown \cdots$, where w_k is equal to $\langle 2\rangle$ or $\langle 1, e_{2k} - 1, 1\rangle$ according as $e_{2k} = 1$ or $e_{2k} > 1$.

Conversely, suppose that the elements of α are as stated. Then

$$\mathrm{SB}(\alpha) = \langle \mathtt{R}\rangle^{2e'_1-1\frown} u_1^\frown \langle \mathtt{R}\rangle^{2e'_2\frown} u_2^\frown \langle \mathtt{R}\rangle^{2e'_3\frown} u_3^\frown \cdots$$

by Proposition 2, where $u_j = \langle \mathtt{L}\rangle^2$ or $\langle \mathtt{L}\rangle^\frown\langle \mathtt{R}\rangle^{e''_j\frown}\langle \mathtt{L}\rangle$. A calculation then reveals that $\mathrm{M}(\alpha) = T(Q(\mathrm{SB}(\alpha)))$ does not contain an occurrence of L. □

4.5. Parities of Numerators

In this subsection, we are concerned with periodicity of the sequence of parities of the numerators of $f_1(\alpha), f_2(\alpha), f_3(\alpha), \ldots$. Among various questions concerning periodicity, what we shall study is the characterization (in terms of Mallows representations) of those $\alpha \in (0,1) \setminus \mathbb{Q}$ for which the above parity sequence is eventually periodic with a certain period. To do so, we need the following lemma, which can readily be inferred from the definition of T.

Lemma 4. *Let $\alpha \in (0,1) \setminus \mathbb{Q}$ be such that both $\mathtt{O_L}$ and $\mathtt{O_R}$ appear infinitely many times in $Q(\mathrm{SB}(\alpha))$, and define positive integers $e_1, e_2, e_3, \ldots$ by Equation (7). Then for any $i \geq 1$, we have $T_i(Q(\mathrm{SB}(\alpha))) = \mathtt{M}$ if and only if $i = e_1 + e_2 + \cdots + e_k$ for some $k \geq 1$.*

Let us first study those $\alpha \in (0,1) \setminus \mathbb{Q}$ for which the parity sequence is eventually periodic with period 1 (in other words, eventually constant).

Proposition 12. *Let $\alpha \in (0,1) \setminus \mathbb{Q}$. Then the sequence of parities of the numerators of $f_1(\alpha), f_2(\alpha), f_3(\alpha), \ldots$ is not eventually constant.*

Proof. In view of Proposition 6, it is sufficient to show that there is no $j \geq 1$ such that $\langle Q_j(\mathrm{SB}(\alpha)), Q_{j+1}(\mathrm{SB}(\alpha)), Q_{j+2}(\mathrm{SB}(\alpha)), \ldots\rangle$ consists of $\mathtt{O_L}$'s and $\mathtt{O_R}$'s only (resp. $\mathtt{E_L}$'s and $\mathtt{E_R}$'s only).

Take a $j \geq 1$ arbitrarily. If $Q_j(\mathrm{SB}(\alpha)) = \mathtt{E_L}$ or $Q_j(\mathrm{SB}(\alpha)) = \mathtt{E_R}$ (equivalently, by Proposition 6, the numerator of $f_j(\alpha)$ is even), it is immediate from the definition of Q that $Q_{j+1}(\mathrm{SB}(\alpha))$ is neither $\mathtt{E_L}$ nor $\mathtt{E_R}$. If $Q_j(\mathrm{SB}(\alpha)) = \mathtt{O_L}$ (resp. $\mathtt{O_R}$), then the only way that we have $Q_{j+1}(\mathrm{SB}(\alpha)) \in \{\mathtt{O_L}, \mathtt{O_R}\}$ is that $\mathrm{SB}_j(\alpha) = \mathtt{L}$ (resp. $\mathtt{R}$). And in that case, we have $Q_{j+1}(\mathrm{SB}(\alpha)) = Q_j(\mathrm{SB}(\alpha))$. A similar argument shows that if $Q_{j+2}(\mathrm{SB}(\alpha)) \in \{\mathtt{O_L}, \mathtt{O_R}\}$ then we have $\mathrm{SB}_{j+1}(\alpha) = \mathtt{L}$ (resp. $\mathtt{R}$) and $Q_{j+2}(\mathrm{SB}(\alpha)) = Q_{j+1}(\mathrm{SB}(\alpha))$. Continuing in this way, we can prove that $Q_j(\mathrm{SB}(\alpha)), Q_{j+1}(\mathrm{SB}(\alpha)), \ldots, Q_{j+\ell}(\mathrm{SB}(\alpha)) \in \{\mathtt{O_L}, \mathtt{O_R}\}$ implies $\mathrm{SB}_j(\alpha) = \mathrm{SB}_{j+1}(\alpha) = \cdots = \mathrm{SB}_{j+\ell-1}(\alpha)$ for any $\ell \geq 1$. Since $\mathrm{SB}(\alpha)$ is not eventually constant, this guarantees that $Q_{j+\ell'}(\mathrm{SB}(\alpha)) \notin \{\mathtt{O_L}, \mathtt{O_R}\}$ for some ℓ'. □

We then characterize those $\alpha \in (0,1) \setminus \mathbb{Q}$ for which the parity sequence is eventually periodic with period 2 (in other words, eventually alternating).

Proposition 13. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, the following two statements are equivalent:*

(i) *The sequence of parities of the numerators of $f_1(\alpha), f_2(\alpha), f_3(\alpha), \ldots$ is eventually alternating.*

(ii) $\mathtt{M}$ *appears infinitely many times in $\mathrm{M}(\alpha)$, and there exists a $k \geq 1$ such that*

$$\langle \mathrm{M}_i(\alpha)\rangle_{i=m(2k)}^{\infty} = \langle \mathtt{M}\rangle^\frown \langle \mathtt{R}\rangle^{e_1'\frown} \langle \mathtt{M}\rangle^\frown \langle \mathtt{L}\rangle^{e_2'\frown} \langle \mathtt{M}\rangle^\frown \langle \mathtt{R}\rangle^{e_3'\frown} \langle \mathtt{M}\rangle^\frown \langle \mathtt{L}\rangle^{e_4'\frown} \cdots,$$

where $e_1', e_2', e_3', \ldots$ are non-negative integers and $m(1) < m(2) < m(3) < \cdots$ is the enumeration of the infinite set $\{i \geq 1 \mid \mathrm{M}_i(\alpha) = \mathtt{M}\}$.

Proof. Take an $\alpha \in (0,1) \setminus \mathbb{Q}$ arbitrarily.

To show that Statement (i) implies Statement (ii), assume that α is as in Statement (i). Observe that for any $j_0 \geq 1$, if $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0}^{\infty}$ starts with the length $\ell + 1$ alternating sequence of $\mathtt{O_L}$'s and $\mathtt{E_R}$'s (resp. $\mathtt{O_R}$'s and $\mathtt{E_L}$'s), then $\mathrm{SB}_{j_0}(\alpha) = \mathrm{SB}_{j_0+1}(\alpha) = \cdots = \mathrm{SB}_{j_0+\ell-1}(\alpha) = \mathtt{R}$ (resp. $\mathtt{L}$). The sequence $\mathrm{SB}(\alpha)$

being not eventually constant, it follows from this and Proposition 6 that both $\mathtt{O}_\mathtt{L}$ and $\mathtt{O}_\mathtt{R}$ appear infinitely many times in $\mathrm{M}(\alpha)$. Let $e_1, e_2, e_3, \ldots$ be as in Equation (7). Then it can be inferred from the assumption and Proposition 6 that there exists a $k \geq 1$ such that

$$\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(e_1+e_2+\cdots+e_{2k})}^{\infty} = \langle \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{L}\rangle^\frown \langle \mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{R}\rangle^{e_{2k+1}\frown} \langle \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{L}\rangle^{e_{2k+2}\frown} \langle \mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{R}\rangle^{e_{2k+3}\frown} \cdots .$$

A calculation then shows that $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=e_1+e_2+\cdots+e_{2k}}^{\infty} = \langle \mathtt{M}\rangle^\frown \langle \mathtt{R}\rangle^{e_{2k+1}-1\frown} \langle \mathtt{M}\rangle^\frown \langle \mathtt{L}\rangle^{e_{2k+2}-1\frown} \langle \mathtt{M}\rangle^\frown \langle \mathtt{R}\rangle^{e_{2k+3}-1\frown} \cdots$. Since $e_1+e_2+\cdots+e_{2k} = m(2k)$ by Lemma 4, we conclude that Statement (ii) is true.

Having proved one implication, it remains to show that Statement (ii) implies Statement (i). If α is as in Statement (ii) then it can be seen from the definition of T that both $\mathtt{O}_\mathtt{L}$ and $\mathtt{O}_\mathtt{R}$ appear infinitely many times in $Q(\mathrm{SB}(\alpha))$. Let $e_1, e_2, e_3, \ldots$ be as in Equation (7). Then, by Lemma 4, we have $m(2k) = e_1 + e_2 + \cdots + e_{2k}$, which, together with Equation (7), implies $Q_{o(m(2k))}(\mathrm{SB}(\alpha)) = \mathtt{O}_\mathtt{R}$. From this equation and the equation from Statement (ii), it can be inferred that

$$\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(m(2k))}^{\infty} = \langle \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{L}\rangle^\frown \langle \mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{R}\rangle^{e'_1+1\frown} \langle \mathtt{O}_\mathtt{R}, \mathtt{E}_\mathtt{L}\rangle^{e'_2+1\frown} \langle \mathtt{O}_\mathtt{L}, \mathtt{E}_\mathtt{R}\rangle^{e'_3+1\frown} \cdots .$$

This and Proposition 6 combine to show that the parities of numerators of $f_{o(m(2k))}(\alpha), f_{o(m(2k))+1}(\alpha), f_{o(m(2k))+2}(\alpha), \ldots$ are alternating. □

Lastly, we characterize those $\alpha \in (0,1) \setminus \mathbb{Q}$ for which the parity sequence is eventually periodic with period 3.

Proposition 14. *For any $\alpha \in (0,1) \setminus \mathbb{Q}$, the following two statements are equivalent:*

(i) *The sequence of parities of the numerators of $f_1(\alpha), f_2(\alpha), f_3(\alpha), \ldots$ is eventually periodic with period 3.*

(ii) *At least one of the ensuing two statements is satisfied:*

- $\mathrm{M}(\alpha)$ *eventually accords with* $\langle \mathtt{L}, \mathtt{R}\rangle^\infty$.
- $\mathtt{M}$ *appears infinitely many times in* $\mathrm{M}(\alpha)$ *and, if we remove all occurrences of* $\mathtt{M}$ *from* $\mathrm{M}(\alpha)$*, then the resulting sequence eventually accords with* $\langle \mathtt{L}, \mathtt{R}\rangle^\infty$*. Moreover, there exists a* $k \geq 1$ *such that for all* $k' \geq k$*, the* $2k'$th *(resp.* $(2k'+1)$st*) occurrence of* $\mathtt{M}$ *is preceded by* $\mathtt{R}$ *(resp.* $\mathtt{L}$*) and is succeeded by* $\mathtt{L}$ *(resp.* $\mathtt{R}$*).*

Proof. Take an arbitrary $\alpha \in (0,1) \setminus \mathbb{Q}$.

We first prove that Statement (i) implies Statement (ii). In the Stern–Brocot tree, if a fraction has even numerator then both of its children have odd numerator [3, Proposition 2]. Consequently, if α is as in Statement (i), then there exists a $j_0 \geq 1$

such that the numerators of $f_{j_0}(\alpha), f_{j_0+1}(\alpha), f_{j_0+2}(\alpha), \ldots$ are odd, odd, even, odd, odd, even, ... There are then the following three cases:

`Case 1`: $\mathtt{O_R}$ appears only finitely many times in $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0}^{\infty}$. Let i_0 be such that $\mathtt{O_R}$ does not appear in $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_0)}^{\infty}$. Then, from the definition of Q and Proposition 6, it can be shown that $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_0)}^{\infty}$ is equal to either $\langle \mathtt{O_L}, \mathtt{O_L}, \mathtt{E_R}\rangle^{\infty}$ or $\langle \mathtt{O_L}, \mathtt{E_R}, \mathtt{O_L}\rangle^{\infty}$. A calculation then shows that, in the former (resp. latter) case, $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty} = \langle \mathtt{L}, \mathtt{R}\rangle^{\infty}$ (resp. $\langle \mathtt{R}\rangle ^\frown \langle \mathtt{L}, \mathtt{R}\rangle^{\infty}$). An application of Theorem 2 proves that, in either case, the former condition of Statement (ii) is satisfied.

`Case 2`: $\mathtt{O_L}$ appears only finitely many times in $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0}^{\infty}$. By the same kind of reasoning to the above case, one can show that the former condition of Statement (ii) is satisfied.

`Case 3`: Both $\mathtt{O_R}$ and $\mathtt{O_L}$ appear infinitely many times in $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0}^{\infty}$. Assume without loss of generality that $Q_{j_0}(\mathrm{SB}(\alpha)) = \mathtt{O_L}$. Then, from the definition of Q and the assumption on the numerators of $f_{j_0}(\alpha), f_{j_0+1}(\alpha), f_{j_0+2}(\alpha), \ldots$, it follows that we have

$$\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=j_0}^{\infty} = \langle \mathtt{O_L}, \mathtt{O_L}, \mathtt{E_R}\rangle^{e_1'} {}^\frown \langle \mathtt{O_R}, \mathtt{O_R}, \mathtt{E_L}\rangle^{e_2'} {}^\frown \langle \mathtt{O_L}, \mathtt{O_L}, \mathtt{E_R}\rangle^{e_3'} {}^\frown \langle \mathtt{O_R}, \mathtt{O_R}, \mathtt{E_L}\rangle^{e_4'} {}^\frown \cdots \tag{8}$$

for some positive integers $e_1', e_2', e_3', \ldots$. Then

$$\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty} = \langle \mathtt{L}, \mathtt{R}\rangle^{e_1'-1} {}^\frown \langle \mathtt{L}, \mathtt{M}\rangle ^\frown \langle \mathtt{R}, \mathtt{L}\rangle^{e_2'-1} {}^\frown \langle \mathtt{R}, \mathtt{M}\rangle \\ {}^\frown \langle \mathtt{L}, \mathtt{R}\rangle^{e_3'-1} {}^\frown \langle \mathtt{L}, \mathtt{M}\rangle ^\frown \langle \mathtt{R}, \mathtt{L}\rangle^{e_4'-1} {}^\frown \langle \mathtt{R}, \mathtt{M}\rangle ^\frown \cdots,$$

where $i_0 \geq 1$ is such that $o(i_0) = j_0$. Since $Q_{o(i_0+2e_1'-1)}(\mathrm{SB}(\alpha)) = \mathtt{O_L}$ by Equation (8) and since $Q_{o(e_1+e_2+\cdots+e_{2k})}(\mathrm{SB}(\alpha)) = \mathtt{O_R}$ for any $k \geq 1$, where positive integers $e_1, e_2, e_3, \ldots$ are as in Equation (7), it follows from the equation $T_{i_0+2e_1'-1}(Q(\mathrm{SB}(\alpha))) = \mathtt{M}$ and Lemma 4 that $i_0 + 2e_1' - 1$ is of the form $e_1 + e_2 + \cdots + e_{2k+1}$ for some $k \geq 0$. Thus, by the same lemma, the first occurrence $T_{i_0+2e_1'-1}(Q(\mathrm{SB}(\alpha)))$ of $\mathtt{M}$ in $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ is the $(2k+1)$st one. That the latter condition of Statement (ii) is satisfied can now be verified readily.

Having proved that Statement (i) implies Statement (ii), we now prove the converse implication. If $\langle \mathrm{M}_i(\alpha)\rangle_{i=i_0}^{\infty} = \langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_0}^{\infty}$ is identical to $\langle \mathtt{L}, \mathtt{R}\rangle^{\infty}$ and if $Q_{o(i_0)}(\mathrm{SB}(\alpha)) = \mathtt{O_L}$ (resp. $\mathtt{O_R}$), then we have $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_0)}^{\infty} = \langle \mathtt{O_L}, \mathtt{O_L}, \mathtt{E_R}\rangle^{\infty}$ (resp. $\langle \mathtt{O_R}, \mathtt{E_L}, \mathtt{O_R}\rangle^{\infty}$). An application of Proposition 6 then proves that the sequence of parities of the numerators of $f_{o(i_0)}(\alpha), f_{o(i_0)+1}(\alpha), f_{o(i_0)+2}(\alpha), \ldots$ is periodic with period 3.

If $\mathrm{M}(\alpha)$ satisfies the latter condition of Statement (ii), then it can be seen from the definition of T that both $\mathtt{O_L}$ and $\mathtt{O_R}$ appear infinitely many times in $Q(\mathrm{SB}(\alpha))$. Let $e_1, e_2, e_3, \ldots$ be as in Equation (7). Then, for a sufficiently large positive integer m, we should have

$$\langle \mathrm{M}_i(\alpha)\rangle_{i=i_0}^{\infty} = \langle \mathtt{R}, \mathtt{M}\rangle ^\frown \langle \mathtt{L}, \mathtt{R}\rangle^{e_1'} {}^\frown \langle \mathtt{L}, \mathtt{M}\rangle ^\frown \langle \mathtt{R}, \mathtt{L}\rangle^{e_2'} {}^\frown \langle \mathtt{R}, \mathtt{M}\rangle ^\frown \langle \mathtt{L}, \mathtt{R}\rangle^{e_3'} {}^\frown \langle \mathtt{L}, \mathtt{M}\rangle ^\frown \langle \mathtt{R}, \mathtt{L}\rangle^{e_4'} {}^\frown \langle \mathtt{R}, \mathtt{M}\rangle ^\frown \cdots$$

for some non-negative integers $e'_1, e'_2, e'_3, \ldots$, where $i_0 = -1 + e_1 + e_2 + \cdots + e_{2mk}$. Since $Q_{o(i_0+1)}(\mathrm{SB}(\alpha)) = Q_{o(e_1+e_2+\cdots+e_{2mk})}(\mathrm{SB}(\alpha)) = \mathtt{O_R}$ by Equation (7), it follows from the above equation and Theorem 2 that

$$\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_0+1)}^{\infty} = \langle \mathtt{O_R}, \mathtt{E_L}, \mathtt{O_L}\rangle^\frown \langle \mathtt{O_L}, \mathtt{E_R}, \mathtt{O_L}\rangle^{e'_1\frown} \langle \mathtt{O_L}, \mathtt{E_R}, \mathtt{O_R}\rangle^\frown \langle \mathtt{O_R}, \mathtt{E_L}, \mathtt{O_R}\rangle^{e'_2} \\ {}^\frown \langle \mathtt{O_R}, \mathtt{E_L}, \mathtt{O_L}\rangle^\frown \langle \mathtt{O_L}, \mathtt{E_R}, \mathtt{O_L}\rangle^{e'_3\frown} \langle \mathtt{O_L}, \mathtt{E_R}, \mathtt{O_R}\rangle^\frown \cdots .$$

An application of Proposition 6 then proves that the sequence of numerators of $f_{o(i_0+1)}(\alpha), f_{o(i_0+1)+1}(\alpha), f_{o(i_0+1)+2}(\alpha), \ldots$ is periodic with period 3. □

5. Metric Theoretic Results

In this section, we shall present various metric theoretic results on Mallows representations. The first one below is about three combinatorial properties studied in the preceding section.

Proposition 15. (i) *Almost no $\alpha \in (0,1) \setminus \mathbb{Q}$ has an eventually periodic Mallows representation.*

(ii) *Almost every $\alpha \in (0,1) \setminus \mathbb{Q}$ has a positively Poisson stable Mallows representation.*

(iii) *Almost every $\alpha \in (0,1) \setminus \mathbb{Q}$ has a transitive Mallows representation.*

Proof. Since there are only countably many quadratic irrationals, Part (i) is an immediate consequence of Corollary 2.

Part (ii) being a consequence of Part (iii) (because transitivity implies positive Poisson stability), we then prove the third part. To do so, observe that, for each non-empty finite sequence w over $\mathbb{Z}_{>0}$, the set $\{\alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q} \mid$ infinitely many k satisfy $\langle a_k, a_{k+1}, \ldots, a_{k+\mathrm{lh}(w)-1}\rangle = w\}$ has full measure, which follows directly from Birkhoff's ergodic theorem. Since there are only countably many finite sequences over $\mathbb{Z}_{>0}$, the set $\{\alpha \in (0,1) \setminus \mathbb{Q} \mid$ the CF-representation of α is transitive$\} = \bigcap_w \{\alpha = [0; a_1, a_2, \ldots] \in (0,1) \setminus \mathbb{Q} \mid$ infinitely many k satisfy $\langle a_k, a_{k+1}, \ldots, a_{k+\mathrm{lh}(w)-1}\rangle = w\}$ has full measure too. An application of Corollary 4 then finishes the proof. □

Before proving new results, let us present some immediate corollaries. The first one is concerned with the complexity of Mallows representations. Recall that the *factor complexity* of an infinite sequence is a function $\rho\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $\rho(n)$ is the total number of length n subsequences of the given infinite sequence. Since every length n subsequence over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$ appears as a subsequence in $\mathrm{M}(\alpha)$ if $\mathrm{M}(\alpha)$ is transitive, Part (iii) of the above proposition implies the following corollary.

Corollary 5. *For almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, the factor complexity of its Mallows representation is $\rho(n) = 3^n$.*

We say that an infinite sequence contains *relatively dense occurrences* of a finite sequence w if there exists an $\ell > 0$ such that in every length ℓ interval of the given infinite sequence, one can find an occurrence of w.

Corollary 6. *For almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, its Mallows representation does not contain relatively dense occurrences of some finite sequence over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$.*

Proof. If $\mathrm{M}(\alpha)$ is transitive then for any $\ell > 0$, both $\langle \mathtt{L} \rangle^\ell$ and $\langle \mathtt{R} \rangle^\ell$ appear as subsequences of $\mathrm{M}(\alpha)$, which implies that $\mathrm{M}(\alpha)$ cannot contain relatively dense occurrences of any finite sequence over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$. This fact and Part (iii) of the above proposition prove the assertion. □

From Proposition 15 (iii), we know that for almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, its Mallows representation contains infinitely many occurrences of w for each non-empty finite sequence w over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$. However, we do not know the frequency that w appears in $\mathrm{M}(\alpha)$. Let us thus study this point.

Theorem 6. *Let w be a finite sequence over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$ that contains either $\mathtt{M}$ or both $\mathtt{L}$ and $\mathtt{R}$. Then, we have $\lim_{n\to\infty} \frac{|\{i \le n \mid \langle \mathrm{M}_i(\alpha), \mathrm{M}_{i+1}(\alpha), \ldots, \mathrm{M}_{i+\mathrm{lh}(w)-1}(\alpha)\rangle = w\}|}{n} = 0$ for almost every $\alpha \in (0,1) \setminus \mathbb{Q}$.*

Proof. Take an arbitrary finite sequence w over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$, and assume that the letter at position i_0 in w is $\mathtt{M}$. It is evident from the definition of T that for any $i \ge 1$, we have $T_i(Q(\mathrm{SB}(\alpha))) = \mathtt{M}$ if and only if $Q_{o(i)}(\mathrm{SB}(\alpha)) \neq Q_{o(i+1)}(\mathrm{SB}(\alpha))$. Also, if $Q_{o(i)}(\mathrm{SB}(\alpha)) \neq Q_{o(i+1)}(\mathrm{SB}(\alpha))$ then $\mathrm{SB}_{o(i)}(\alpha) \neq \mathrm{SB}_{o(i)+1}(\alpha)$. Moreover, we have $o(i) \le 2i$ for any $i \ge 1$, which can be verified by induction. Using these facts, Theorem 2, and Proposition 2, we obtain

$$\begin{aligned}
&\frac{|\{i \le n \mid \langle \mathrm{M}_i(\alpha), \mathrm{M}_{i+1}(\alpha), \ldots, \mathrm{M}_{i+\mathrm{lh}(w)-1}(\alpha)\rangle = w\}|}{n} \\
&\le \frac{|\{i \le n \mid \mathrm{M}_{i+i_0-1}(\alpha) = \mathtt{M}\}|}{n} \\
&= \frac{|\{i \le n \mid Q_{o(i+i_0-1)}(\mathrm{SB}(\alpha)) \neq Q_{o(i+i_0)}(\mathrm{SB}(\alpha))\}|}{n} \\
&\le \frac{|\{i \le n \mid \mathrm{SB}_{o(i+i_0-1)}(\alpha) \neq \mathrm{SB}_{o(i+i_0-1)+1}(\alpha)\}|}{n} \\
&\le \frac{|\{j \le 2(n+i_0-1) \mid \mathrm{SB}_j(\alpha) \neq \mathrm{SB}_{j+1}(\alpha)\}|}{n} \\
&= \frac{2(n+i_0-1)}{n} \cdot \frac{|\{j \le 2(n+i_0-1) \mid \mathrm{SB}_j(\alpha) \neq \mathrm{SB}_{j+1}(\alpha)\}|}{2(n+i_0-1)}
\end{aligned}$$

$$\leq \frac{2(n+i_0-1)}{n} \cdot \frac{m(n)+1}{m(n)} \cdot \frac{m(n)}{\sum_{k=1}^{m(n)} a_k}$$

for any $\alpha = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$ and a large n, where $m(n)$ is the largest integer $m \geq 1$ satisfying the inequality $\sum_{k=1}^{m} a_k \leq 2(n+i_0-1)$. Since $\lim_m \frac{\sum_{k=1}^{m} a_k}{m} = \infty$ for almost every $\alpha = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$, which is known in ergodic theory and since $m(n) \to \infty$ as $n \to \infty$, the above inequality proves the asserted equation for this w.

We then take up those finite sequences w that contain both L and R. If w contains the letter M then the correctness of the asserted equation follows from the above argument. If w does not contain the letter M, then there exists an i_0 such that the letters at positions i_0 and $i_0 + 1$ of w are different. A calculation shows that if $\langle T_{i+i_0-1}(Q(\mathrm{SB}(\alpha))), T_{i+i_0}(Q(\mathrm{SB}(\alpha)))\rangle = \langle \mathtt{L}, \mathtt{R}\rangle$ and if $Q_{o(i+i_0-1)}(\mathrm{SB}(\alpha)) = \mathtt{O}_\mathtt{L}$ (resp. $\mathtt{O}_\mathtt{R}$), then $\langle \mathrm{SB}_{o(i+i_0-1)}(\alpha), \mathrm{SB}_{o(i+i_0-1)+1}(\alpha), \mathrm{SB}_{o(i+i_0-1)+2}(\alpha)\rangle = \langle \mathtt{L}, \mathtt{R}, \mathtt{R}\rangle$ (resp. $\langle \mathtt{L}, \mathtt{L}, \mathtt{R}\rangle$). Also, if $\langle T_{i+i_0-1}(Q(\mathrm{SB}(\alpha))), T_{i+i_0}(Q(\mathrm{SB}(\alpha)))\rangle = \langle \mathtt{R}, \mathtt{L}\rangle$ and if $Q_{o(i+i_0-1)}(\mathrm{SB}(\alpha)) = \mathtt{O}_\mathtt{L}$ (resp. $\mathtt{O}_\mathtt{R}$), then $\langle \mathrm{SB}_{o(i+i_0-1)}(\alpha), \mathrm{SB}_{o(i+i_0-1)+1}(\alpha), \mathrm{SB}_{o(i+i_0-1)+2}(\alpha)\rangle = \langle \mathtt{R}, \mathtt{R}, \mathtt{L}\rangle$ (resp. $\langle \mathtt{R}, \mathtt{L}, \mathtt{L}\rangle$). Together with Theorem 2, these imply the following:

$$\begin{aligned}
&\frac{|\{i \leq n \mid \langle \mathrm{M}_i(\alpha), \mathrm{M}_{i+1}(\alpha), \dots, \mathrm{M}_{i+\mathrm{lh}(w)-1}(\alpha)\rangle = w\}|}{n} \\
&\leq \frac{|\{i \leq n \mid \langle \mathrm{M}_{i+i_0-1}(\alpha), \mathrm{M}_{i+i_0}(\alpha)\rangle = \langle \mathtt{L}, \mathtt{R}\rangle \text{ or } \langle \mathtt{R}, \mathtt{L}\rangle\}|}{n} \\
&\leq \frac{\left|\left\{ i \leq n \;\middle|\; \begin{array}{r} \langle \mathrm{SB}_{o(i+i_0-1)}(\alpha), \mathrm{SB}_{o(i+i_0-1)+1}(\alpha), \mathrm{SB}_{o(i+i_0-1)+2}(\alpha)\rangle \\ \in \{\langle \mathtt{L}, \mathtt{R}, \mathtt{R}\rangle, \langle \mathtt{L}, \mathtt{L}, \mathtt{R}\rangle, \langle \mathtt{R}, \mathtt{R}, \mathtt{L}\rangle, \langle \mathtt{R}, \mathtt{L}, \mathtt{L}\rangle\} \end{array} \right\}\right|}{n} \\
&\leq \frac{|\{j \leq 2(n+i_0-1) \mid \mathrm{SB}_j(\alpha) \neq \mathrm{SB}_{j+1}(\alpha) \text{ or } \mathrm{SB}_{j+1}(\alpha) \neq \mathrm{SB}_{j+2}(\alpha)\}|}{n} \\
&\leq \frac{2|\{j \leq 2(n+i_0-1) \mid \mathrm{SB}_j(\alpha) \neq \mathrm{SB}_{j+1}(\alpha)\}| + 1}{n}.
\end{aligned}$$

Since the last line is, up to a convergent term $\frac{1}{n}$, twice the fifth line of the inequality in the preceding case, an argument similar to the foregoing completes the proof. □

We next study matching fractions [7] of Mallows representations. For an infinite sequence $\langle x_1, x_2, x_3, \dots\rangle$, its ℓth *matching fraction*, $\ell \geq 1$, is defined as the limit $\lim_{n\to\infty} \frac{|\{i \leq n \mid x_i = x_{i+\ell}\}|}{n}$ (if it exists).

Theorem 7. *For almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, the ℓth matching fraction of $\mathrm{M}(\alpha)$ is 1 for all $\ell \geq 1$.*

Proof. It is sufficient to show that for each $\ell \geq 1$, the ℓth matching fraction of $\mathrm{M}(\alpha)$ is 1 for almost every $\alpha \in (0,1) \setminus \mathbb{Q}$. To do so, take an $\ell \geq 1$ and an $\alpha = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$ arbitrarily. In this proof, we shall use the following notation. For each positive integer $m \geq 1$, we write i_m for the integer i satisfying the inequality $o(i) \leq \sum_{k=1}^{m} a_k < o(i+1)$. Also, we let m_n denote the largest integer m satisfying $i_m \leq n$.

It can readily be seen that if $i_1 > 1$ then $a_1 \geq 2$. In that case, the length $a_1 - 1$ (hence, non-empty) sequence $\langle \mathrm{SB}_j(\alpha)\rangle_{j=1}^{a_1-1}$ is a constant sequence of $\mathtt{L}$'s by Proposition 2. This and the definition of Q imply that $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=1}^{a_1}$ is a constant sequence of $\mathtt{O_L}$'s, which, along with the definition of T, in turn implies that $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=1}^{i_1-1}$ is a constant sequence. From this, the validity, when $i_1 > 1$, of the inequality

$$|\{i \in \{1, 2, \ldots, i_1\} \mid T_i(Q(\mathrm{SB}(\alpha))) = T_{i+\ell}(Q(\mathrm{SB}(\alpha)))\}| \geq i_1 - \ell - 1 \tag{9}$$

follows at once. Let us note that this inequality is plainly valid even when $i_1 = 1$.

Similarly, for each $m \geq 2$, the length a_m sequence $\langle \mathrm{SB}_j(\alpha)\rangle_{j=\sum_{k=1}^{m-1} a_k}^{-1+\sum_{k=1}^{m} a_k}$ is a constant sequence by Proposition 2. It follows that if $i_m > i_{m-1} + 1$ then the length $o(i_m) - o(i_{m-1}+1)$ (hence, non-empty) subsequence $\langle \mathrm{SB}_j(\alpha)\rangle_{j=o(i_{m-1}+1)}^{o(i_m)-1}$ is also a constant sequence, which, together with the definition of Q, implies that $\langle Q_j(\mathrm{SB}(\alpha))\rangle_{j=o(i_{m-1}+1)}^{o(i_m)}$ is a constant sequence of $\mathtt{O_L}$'s (resp. alternating sequence of $\mathtt{O_R}$'s and $\mathtt{E_L}$'s, alternating sequence of $\mathtt{O_L}$'s and $\mathtt{E_R}$'s, constant sequence of $\mathtt{O_R}$'s) when $(\mathrm{SB}_{o(i_{m-1}+1)}(\alpha), Q_{o(i_{m-1}+1)}(\mathrm{SB}(\alpha))) = (\mathtt{L}, \mathtt{O_L})$ (resp. $(\mathtt{L}, \mathtt{O_R}), (\mathtt{R}, \mathtt{O_L}), (\mathtt{R}, \mathtt{O_R})$). A calculation then reveals that the sequence $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_{m-1}+1}^{i_m-1}$ is, in either case, a constant sequence, which in turn proves the validity, when $i_m > i_{m-1} + 1$, of the inequality

$$|\{i \in \{i_{m-1}+1, i_{m-1}+2, \ldots, i_m\} \mid T_i(Q(\mathrm{SB}(\alpha))) = T_{i+\ell}(Q(\mathrm{SB}(\alpha)))\}| \geq i_m - i_{m-1} - \ell - 1. \tag{10}$$

Plainly, this inequality is valid even when $i_m = i_{m-1} + 1$.

Take a large n. If $i_{m_n} = n$ then

$$\begin{aligned}
&\frac{|\{i \leq n \mid \mathrm{M}_i(\alpha) = \mathrm{M}_{i+\ell}(\alpha)\}|}{n} \\
&= \frac{1}{n}\Bigg(|\{i \in \{1, 2, \ldots, i_1\} \mid T_i(Q(\mathrm{SB}(\alpha))) = T_{i+\ell}(Q(\mathrm{SB}(\alpha)))\}| \\
&\qquad + \sum_{m=2}^{m_n} |\{i \in \{i_{m-1}+1, i_{m-1}+2, \ldots, i_m\} \mid T_i(Q(\mathrm{SB}(\alpha))) = T_{i+\ell}(Q(\mathrm{SB}(\alpha)))\}|\Bigg) \\
&\geq \frac{1}{n}\left(i_1 - \ell - 1 + \sum_{m=2}^{m_n} \big(i_m - i_{m-1} - \ell - 1\big)\right) \\
&= \frac{1}{n}\big(n - (\ell+1)m_n\big) \\
&= 1 - (\ell+1)\frac{m_n}{n}
\end{aligned} \tag{11}$$

by Theorem 2 and Inequalities (9) and (10).

If $i_{m_n} < n$ then, as an argument similar to the one above shows that if $i_{m_n+1} > i_{m_n} + 1$ then $\langle T_i(Q(\mathrm{SB}(\alpha)))\rangle_{i=i_{m_n}+1}^{n}$ is a constant sequence, the inequality

$$|\{i \in \{i_{m_n}+1, i_{m_n}+2, \dots, n\} \mid T_i(Q(\mathrm{SB}(\alpha))) = T_{i+\ell}(Q(\mathrm{SB}(\alpha)))\}| \geq n - i_{m_n} - \ell \quad (12)$$

holds. From Theorem 2 and Inequalities (9), (10), and (12), we get

$$\begin{aligned} &\frac{|\{i \leq n \mid \mathrm{M}_i(\alpha) = \mathrm{M}_{i+\ell}(\alpha)\}|}{n} \\ &= \frac{1}{n}\Bigg(|\{i \in \{1,2,\dots,i_1\} \mid T_i(Q(\mathrm{SB}(\alpha))) = T_{i+\ell}(Q(\mathrm{SB}(\alpha)))\}| \\ &\qquad + \sum_{m=2}^{m_n} |\{i \in \{i_{m-1}+1, i_{m-1}+2, \dots, i_m\} \mid T_i(Q(\mathrm{SB}(\alpha))) = T_{i+\ell}(Q(\mathrm{SB}(\alpha)))\}| \\ &\qquad\qquad + |\{i \in \{i_{m_n}+1, i_{m_n}+2, \dots, n\} \mid T_i(Q(\mathrm{SB}(\alpha))) = T_{i+\ell}(Q(\mathrm{SB}(\alpha)))\}|\Bigg) \\ &\geq \frac{1}{n}\left(i_1 - \ell - 1 + \sum_{m=2}^{m_n}\big(i_m - i_{m-1} - \ell - 1\big) + n - i_{m_n} - \ell\right) \\ &= \frac{1}{n}\big(n - (\ell+1)m_n - \ell\big) \\ &= 1 - \frac{\ell}{n} - (\ell+1)\frac{m_n}{n}. \end{aligned} \quad (13)$$

In view of Inequalities (11) and (13), to prove $\lim_{n\to\infty} \frac{|\{i \leq n \mid \mathrm{M}_i(\alpha) = \mathrm{M}_{i+\ell}(\alpha)\}|}{n} = 1$, it is sufficient to show that $\frac{m_n}{n} \to 0$ as $n \to \infty$. To do so, observe that if $o(n+1)$ were smaller than or equal to $\sum_{k=1}^{m_n} a_k$ then i_{m_n} should be at least $n+1$, contrary to the definition of m_n. Hence, $o(n+1) > \sum_{k=1}^{m_n} a_k$. As we have $2(n+1) \geq o(n+1)$, which can be verified by induction, it follows that $2(n+1) > \sum_{k=1}^{m_n} a_k$, and hence $\frac{m_n}{n} = \frac{n+1}{n} \cdot \frac{m_n}{n+1} < \frac{2(n+1)}{n} \cdot \frac{m_n}{\sum_{k=1}^{m_n} a_k}$. Since $m_n \to \infty$ as $n \to \infty$ and since $\lim_m \frac{\sum_{k=1}^{m} a_k}{m} = \infty$ for almost every $\alpha = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$, which is known in ergodic theory, it follows that $\frac{m_n}{n} \to 0$ as $n \to \infty$ for almost every $\alpha \in (0,1) \setminus \mathbb{Q}$, completing the proof. □

Recall from Proposition 15 that for each non-empty finite sequence w over $\{\mathtt{L}, \mathtt{M}, \mathtt{R}\}$, the measure of the set of those $\alpha \in (0,1) \setminus \mathbb{Q}$ whose Mallows representations contain w as a subsequence is 1. Then, what is the measure of the set of those $\alpha \in (0,1) \setminus \mathbb{Q}$ whose Mallows representations contain w as the *initial* subsequence (in other words, whose Mallows representations start with w)? In the following theorem, by letting w range over $\{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell$ for a fixed ℓ, we study the maximum and minimum of the measure.

Theorem 8. *Let ℓ be a positive integer. Then*

(i) $\displaystyle\max_{w\in\{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell} \mu(\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \mathrm{M}(\alpha) \text{ starts with } w\}) = \frac{1}{\ell+1},$

(ii) $\displaystyle\min_{w\in\{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell} \mu(\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \mathrm{M}(\alpha) \text{ starts with } w\})$

$$= \frac{4\sqrt{2}}{(1+\sqrt{2})^{2\ell+2}-(1-\sqrt{2})^{2\ell+2}}.$$

Here and throughout, μ stands for Lebesgue measure.

Proof. In this proof, we shall utilize the following set of notation. For a non-negative integer $i \geq 1$, we write $\frac{n_{(i,1)}}{m_{(i,1)}} < \frac{n_{(i,2)}}{m_{(i,2)}} < \cdots < \frac{n_{(i,3^i+1)}}{m_{(i,3^i+1)}}$ for the elements of $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup\mathrm{MT}_1\cup\mathrm{MT}_2\cup\cdots\cup\mathrm{MT}_i$. Also, we let $\mathrm{lm}(i,k)$ and $\mathrm{sm}(i,k)$ ($i\geq 1, k\in \{2,3,\ldots,3^i+1\}$) denote the larger and smaller of $m_{(i,k-1)}$ and $m_{(i,k)}$, respectively. Let $\nu\colon \{\mathtt{L},\mathtt{M},\mathtt{R}\}\to\{0,1,2\}$ be a function defined by $\nu(\mathtt{L})=0, \nu(\mathtt{M})=1$, and $\nu(\mathtt{R})=2$, and extend it to a function $\hat{\nu}$ over $\bigcup_{i=1}^\infty\{\mathtt{L},\mathtt{M},\mathtt{R}\}^i$ by setting $\hat{\nu}(\langle x_1,x_2,\ldots,x_i\rangle) = 1+3^{i-1}\nu(x_1)+3^{i-2}\nu(x_2)+\cdots+3^0\nu(x_i)$. Then it can be readily seen that, when restricted to $\{\mathtt{L},\mathtt{M},\mathtt{R}\}^i$, the function $\hat{\nu}$ is a bijection between $\{\mathtt{L},\mathtt{M},\mathtt{R}\}^i$ and $\{1,2,\ldots,3^i\}$. Also, it can be proved by induction on $i\geq 1$ that for any $\alpha\in(0,1)\setminus\mathbb{Q}$ and $w\in\{\mathtt{L},\mathtt{M},\mathtt{R}\}^i$, the Mallows representation of α starts with w if and only if α belongs to the open interval $\left(\frac{n_{(i,\hat{\nu}(w))}}{m_{(i,\hat{\nu}(w))}}, \frac{n_{(i,\hat{\nu}(w)+1)}}{m_{(i,\hat{\nu}(w)+1)}}\right)$.

Take a positive integer ℓ arbitrarily. By using Theorem 1, we can inductively prove that any adjacent fractions from $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup\mathrm{MT}_1\cup\mathrm{MT}_2\cup\cdots\cup\mathrm{MT}_\ell$ is also adjacent in $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup\mathrm{SBT}_1\cup\mathrm{SBT}_2\cup\cdots\cup\mathrm{SBT}_{\ell'}$ for some $\ell'\geq\ell$. It can also be proved inductively that if $\frac{n_L}{m_L}<\frac{n_R}{m_R}$ are adjacent fractions from $\left\{\frac{0}{1},\frac{1}{1}\right\}\cup\mathrm{SBT}_1\cup\mathrm{SBT}_2\cup\cdots\cup\mathrm{SBT}_{\ell'}$ then $n_Lm_R-n_Rm_L=-1$ and either m_L or m_R is at least $\ell'+1$. Therefore,

$$\begin{aligned}
&\max_{w\in\{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell} \mu(\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \mathrm{M}(\alpha) \text{ starts with } w\})\\
&= \max_{w\in\{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell}\left|\frac{n_{(\ell,\hat{\nu}(w)+1)}}{m_{(\ell,\hat{\nu}(w)+1)}}-\frac{n_{(\ell,\hat{\nu}(w))}}{m_{(\ell,\hat{\nu}(w))}}\right|\\
&= \max_{w\in\{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell}\frac{1}{m_{(\ell,\hat{\nu}(w)+1)}m_{(\ell,\hat{\nu}(w))}}\\
&= \max_{k\in\{1,2,\ldots,3^\ell\}}\frac{1}{m_{(\ell,\,k+1)}m_{(\ell,\,k)}}\\
&\leq \frac{1}{\ell+1}.
\end{aligned}$$

Since $\frac{n_{(\ell,1)}}{m_{(\ell,1)}} = \frac{0}{1}$ and $\frac{n_{(\ell,2)}}{m_{(\ell,2)}} = \frac{1}{\ell+1}$, the last equality is indeed attained. Hence, the proof of Equation (i) is complete.

To prove Equation (ii), we first validate the claim that for any $\ell \geq 1$, the following statement is true: for any $i \geq 1$ and $k \in \{1, 2, \dots, 3^{i-1}\}$, we have

$$\begin{aligned}
&\mathrm{lm}(i+\ell, 3^{\ell+1}(k-1) + \tfrac{3^{\ell+1}+3}{2}) \\
&\quad = \frac{(1+\sqrt{2})^\ell + (1-\sqrt{2})^\ell}{2} \mathrm{lm}(i, 3k) + \frac{(1+\sqrt{2})^\ell - (1-\sqrt{2})^\ell}{\sqrt{2}} \mathrm{sm}(i, 3k) \quad \text{and} \\
&\mathrm{sm}(i+\ell, 3^{\ell+1}(k-1) + \tfrac{3^{\ell+1}+3}{2}) \\
&\quad = \frac{(1+\sqrt{2})^\ell - (1-\sqrt{2})^\ell}{2\sqrt{2}} \mathrm{lm}(i, 3k) + \frac{(1+\sqrt{2})^\ell + (1-\sqrt{2})^\ell}{2} \mathrm{sm}(i, 3k).
\end{aligned}$$

The proof is by induction on ℓ. To validate the claim for $\ell = 1$, take an $i \geq 1$ and a $k \in \{1, 2, \dots, 3^{i-1}\}$ arbitrarily. If $n_{(i,3k-1)}$ is odd (resp. even) then it is immediate from the definition of MT_{i+1} that $m_{(i+1,9k-4)} = 2m_{(i,3k-1)} + m_{(i,3k)} \geq m_{(i,3k-1)} + m_{(i,3k)} = m_{(i+1,9k-3)}$ (resp. $m_{(i+1,9k-4)} = m_{(i,3k-1)} + m_{(i,3k)} \leq m_{(i,3k-1)} + 2m_{(i,3k)} = m_{(i+1,9k-3)}$). Also, it can be inferred from Theorem 1 and the mediant construction of the Stern–Brocot tree that $m_{(i,3k-1)} \leq m_{(i,3k)}$ (resp. $m_{(i,3k-1)} \geq m_{(i,3k)}$). Using these, one can verify the correctness of the claim for $\ell = 1$ by hand. To prove the induction step, assume that we have verified the correctness of the claim for some $\ell \geq 1$, and again take arbitrary integers $i \geq 1$ and $k \in \{1, 2, \dots, 3^{i-1}\}$. Then, by the already verified base case and the induction hypothesis (more precisely, by combining the resulting equations obtained by substituting $1, i, k$ and $\ell, i+1, 3k-1$ into ℓ, i, k in the statement, respectively), one can verify the correctness of the claim for $\ell + 1$.

We next substantiate the claim that for any $\ell \geq 1$, the ensuing statement is true: for any $i \geq 1$ and $k' \in \{1, 2, \dots, 3^{i-1}\}$, we have

$$\max_{k \in \{1,2,\dots,3^\ell\}} \mathrm{lm}(i+\ell, 3^{\ell+1}(k'-1) + 3k) = \mathrm{lm}(i+\ell, 3^{\ell+1}(k'-1) + \tfrac{3^{\ell+1}+3}{2})$$

and

$$\max_{k \in \{1,2,\dots,3^\ell\}} \mathrm{sm}(i+\ell, 3^{\ell+1}(k'-1) + 3k) = \mathrm{sm}(i+\ell, 3^{\ell+1}(k'-1) + \tfrac{3^{\ell+1}+3}{2}).$$

The proof is by induction on ℓ. To show the correctness of the claim for $\ell = 1$, take integers $i \geq 1$ and $k' \in \{1, 2, \dots, 3^{i-1}\}$ arbitrarily, and suppose that $n_{(i,3k'-2)}$ is odd. (The proof for the case $n_{(i,3k'-2)}$ being even is analogous and hence is not presented here.) A calculation shows that

$$\begin{aligned}
&m_{(i+1,9k'-7)} = 4m_{(i,3k'-2)} + m_{(i,3k'+1)}, \quad && m_{(i+1,9k'-6)} = 3m_{(i,3k'-2)} + m_{(i,3k'+1)}, \\
&m_{(i+1,9k'-4)} = 3m_{(i,3k'-2)} + 2m_{(i,3k'+1)}, \quad && m_{(i+1,9k'-3)} = 4m_{(i,3k'-2)} + 3m_{(i,3k'+1)},
\end{aligned}$$

$$m_{(i+1,9k'-1)} = 2m_{(i,3k'-2)} + 3m_{(i,3k'+1)}, \qquad m_{(i+1,9k')} = m_{(i,3k'-2)} + 2m_{(i,3k'+1)}, \tag{14}$$

from which the correctness of the claim for $\ell = 1$ can readily be seen. Assume then that we have verified the correctness of the claim for some $\ell \geq 1$, and take integers $i \geq 1$ and $k' \in \{1, 2, \ldots, 3^{i-1}\}$ arbitrarily. As before, we suppose that $n_{(i,3k'-2)}$ is odd. Then, by the validated first claim and the induction hypothesis (with $\ell, i+1, 3k'-j$ ($j \in \{0,1,2\}$) substituted for ℓ, i, k', respectively), we obtain

$$\begin{aligned}
&\max_{k\in\{1,2,\ldots,3^{\ell+1}\}} \operatorname{lm}(i+\ell+1, 3^{\ell+2}(k'-1)+3k)\\
&\quad= \max_{j'\in\{0,1,2\}} \max_{k\in\{1,2,\ldots,3^{\ell}\}} \operatorname{lm}(i+\ell+1, 3^{\ell+2}(k'-1)+3^{\ell+1}j'+3k)\\
&\quad= \max_{j\in\{0,1,2\}} \operatorname{lm}(i+1+\ell, 3^{\ell+1}(3k'-j-1)+\tfrac{3^{\ell+1}+3}{2})\\
&\quad= \max_{j\in\{0,1,2\}} \Bigg(\frac{(1+\sqrt{2})^{\ell}+(1-\sqrt{2})^{\ell}}{2} \operatorname{lm}(i+1, 9k'-3j)\\
&\qquad\qquad + \frac{(1+\sqrt{2})^{\ell}-(1-\sqrt{2})^{\ell}}{\sqrt{2}} \operatorname{sm}(i+1, 9k'-3j) \Bigg)\\
&\quad= \frac{(1+\sqrt{2})^{\ell}+(1-\sqrt{2})^{\ell}}{2} \operatorname{lm}(i+1, 9k'-3)\\
&\qquad + \frac{(1+\sqrt{2})^{\ell}-(1-\sqrt{2})^{\ell}}{\sqrt{2}} \operatorname{sm}(i+1, 9k'-3)\\
&\quad= \operatorname{lm}(i+1+\ell, 3^{\ell+1}(3k'-2)+\tfrac{3^{\ell+1}+3}{2})\\
&\quad= \operatorname{lm}(i+\ell+1, 3^{\ell+2}(k'-1)+\tfrac{3^{\ell+2}+3}{2}).
\end{aligned}$$

(The validity of the fourth equality sign can readily be seen from Equation (14).) The second equation of the claim for $\ell + 1$ can be proved in much the same way.

If $\ell \geq 2$, then it is evident from the definition of MT_{ℓ} that for any $k \in \{1, 2, \ldots, 3^{\ell-1}\}$, we have $m_{(\ell,3k-2)} = m_{(\ell-1,k)}$, $m_{(\ell,3k+1)} = m_{(\ell-1,k+1)}$, and $\langle m_{(\ell,3k-1)}, m_{(\ell,3k)} \rangle$ equals to either $\langle m_{(\ell-1,k)} + m_{(\ell-1,k+1)}, m_{(\ell-1,k)} + 2m_{(\ell-1,k+1)} \rangle$ or $\langle 2m_{(\ell-1,k)} + m_{(\ell-1,k+1)}, m_{(\ell-1,k)} + m_{(\ell-1,k+1)} \rangle$. In both cases, neither $\operatorname{lm}(\ell, 3k-1)$ nor $\operatorname{lm}(\ell, 3k+1)$ (resp. neither $\operatorname{sm}(\ell, 3k-1)$ nor $\operatorname{sm}(\ell, 3k+1)$) exceeds $\operatorname{lm}(\ell, 3k)$ (resp. $\operatorname{sm}(\ell, 3k)$). These facts and the substantiated second claim (with $\ell - 1, 1, 1$ substituted for ℓ, i, k', respectively) imply

$$\begin{aligned}
&\max_{k'\in\{1,2,\ldots,3^{\ell}\}} \operatorname{lm}(\ell, k'+1)\\
&\quad= \max_{k\in\{1,2,\ldots,3^{\ell-1}\}} \max\{\operatorname{lm}(\ell, 3k-1), \operatorname{lm}(\ell, 3k), \operatorname{lm}(\ell, 3k+1)\}\\
&\quad= \max_{k\in\{1,2,\ldots,3^{\ell-1}\}} \operatorname{lm}(\ell, 3k)\\
&\quad= \operatorname{lm}(\ell, \tfrac{3^{\ell}+3}{2}).
\end{aligned} \tag{15}$$

Likewise, one can show that

$$\max_{k' \in \{1,2,\ldots,3^\ell\}} \mathrm{sm}(\ell, k'+1) = \mathrm{sm}(\ell, \tfrac{3^\ell+3}{2}). \tag{16}$$

Note that Equations (15) and (16) are both valid even when $\ell = 1$.

Using Equations (15) and (16) and the validated first claim, we can complete the proof of Equation (ii) as follows:

$$\begin{aligned}
&\min_{w \in \{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell} \mu(\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \mathrm{M}(\alpha) \text{ starts with } w\}) \\
&= \min_{w \in \{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell} \left| \frac{n_{(\ell,\hat{\nu}(w)+1)}}{m_{(\ell,\hat{\nu}(w)+1)}} - \frac{n_{(\ell,\hat{\nu}(w))}}{m_{(\ell,\hat{\nu}(w)}} \right| \\
&= \min_{w \in \{\mathtt{L},\mathtt{M},\mathtt{R}\}^\ell} \frac{1}{m_{(\ell,\hat{\nu}(w)+1)} m_{(\ell,\hat{\nu}(w))}} \\
&= \min_{k' \in \{1,2,\ldots,3^\ell\}} \frac{1}{m_{(\ell,\, k'+1)} m_{(\ell,\, k')}} \\
&= \min_{k' \in \{1,2,\ldots,3^\ell\}} \frac{1}{\mathrm{lm}(\ell, k'+1)\, \mathrm{sm}(\ell, k'+1)} \\
&= \frac{1}{\mathrm{lm}(\ell, \frac{3^\ell+3}{2})\, \mathrm{sm}(\ell, \frac{3^\ell+3}{2})} \\
&= \frac{4\sqrt{2}}{\big((1+\sqrt{2})^{\ell+1} + (1-\sqrt{2})^{\ell+1}\big)\big((1+\sqrt{2})^{\ell+1} - (1-\sqrt{2})^{\ell+1}\big)} \\
&= \frac{4\sqrt{2}}{(1+\sqrt{2})^{2\ell+2} - (1-\sqrt{2})^{2\ell+2}}.
\end{aligned}$$

□

An implication of Equation (i) of this theorem is that for any integer $n \geq 1$ and irrational numbers $\alpha, \alpha' \in (0,1) \setminus \mathbb{Q}$, if $d(\mathrm{M}(\alpha), \mathrm{M}(\alpha')) \leq 2^{-n}$ then $|\alpha - \alpha'| < \frac{1}{n}$, where d is a metric on $\mathcal{M}$ defined by

$$d(\langle x_i \rangle_{i=1}^\infty, \langle x'_i \rangle_{i=1}^\infty) = \begin{cases} 2^{-\min\{i \geq 1 \,|\, x_i \neq x'_i\}} & \text{if } \langle x_i \rangle_{i=1}^\infty \neq \langle x'_i \rangle_{i=1}^\infty, \\ 0 & \text{if } \langle x_i \rangle_{i=1}^\infty = \langle x'_i \rangle_{i=1}^\infty. \end{cases}$$

This fact in turn implies that the map $\mathcal{M} \ni \mathrm{M}(\alpha) \mapsto \alpha \in (0,1) \setminus \mathbb{Q}$ is not only continuous but even uniformly continuous. On the other hand, the inverse $(0,1) \setminus \mathbb{Q} \ni \alpha \mapsto \mathrm{M}(\alpha) \in \mathcal{M}$ of the above map is continuous but not uniformly continuous, which can be seen from the fact that, for any arbitrarily close pair α, α' of positive irrational numbers satisfying $\alpha < \frac{1}{2} < \alpha'$, we have $\mathrm{M}_1(\alpha) \neq \mathrm{M}_1(\alpha')$ (see Figure 4), and hence $d(\mathrm{M}(\alpha), \mathrm{M}(\alpha')) = \frac{1}{2}$.

From Proposition 15, it follows that the set of those $\alpha \in (0,1) \setminus \mathbb{Q}$ whose Mallows representations satisfy $\mathrm{M}_i(\alpha) = \mathtt{M}$ for some i has full measure. Then what is the measure if we fix i? Concerning this question, we have the ensuing result.

Theorem 9. *The inequality* $\mu(\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \mathrm{M}_i(\alpha) = \mathtt{M}\}) < 3 - 2\sqrt{2}$ *holds for any* $i \geq 1$.

Proof. In this proof, we shall use the notation used in the proof of the last theorem.

Since we have $\mu(\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \mathrm{M}_1(\alpha) = \mathtt{M}\}) = \frac{1}{6} < 3 - 2\sqrt{2}$, the inequality for $i = 1$ is correct. To complete the proof, take an arbitrary positive integer $i \geq 2$. Then

$$
\begin{aligned}
&\{\alpha \in (0,1) \setminus \mathbb{Q} \mid \mathrm{M}_i(\alpha) = \mathtt{M}\} \\
&= \bigcup_{w \in \{\mathtt{L},\mathtt{M},\mathtt{R}\}^{i-1}} \{\alpha \in (0,1) \setminus \mathbb{Q} \mid \mathrm{M}(\alpha) \text{ starts with } w^\frown\langle\mathtt{M}\rangle\} \\
&= \bigcup_{w \in \{\mathtt{L},\mathtt{M},\mathtt{R}\}^{i-1}} \Big(\frac{n_{(i,\hat{\nu}(w^\frown\langle\mathtt{M}\rangle))}}{m_{(i,\hat{\nu}(w^\frown\langle\mathtt{M}\rangle))}}, \frac{n_{(i,\hat{\nu}(w^\frown\langle\mathtt{M}\rangle)+1)}}{m_{(i,\hat{\nu}(w^\frown\langle\mathtt{M}\rangle)+1)}}\Big) \\
&= \bigcup_{k=1}^{3^{i-1}} \Big(\frac{n_{(i,3k-1)}}{m_{(i,3k-1)}}, \frac{n_{(i,3k)}}{m_{(i,3k)}}\Big).
\end{aligned}
$$

It can be inferred from the definition of MT_i that the interval $\Big(\frac{n_{(i,3k-1)}}{m_{(i,3k-1)}}, \frac{n_{(i,3k)}}{m_{(i,3k)}}\Big)$ is equal to $\Big(\frac{n_{(i-1,k)}+n_{(i-1,k+1)}}{m_{(i-1,k)}+m_{(i-1,k+1)}}, \frac{n_{(i-1,k)}+2n_{(i-1,k+1)}}{m_{(i-1,k)}+2m_{(i-1,k+1)}}\Big)$ if $n_{(i-1,k)}$ is even and to $\Big(\frac{2n_{(i-1,k)}+n_{(i-1,k+1)}}{2m_{(i-1,k)}+m_{(i-1,k+1)}}, \frac{n_{(i-1,k)}+n_{(i-1,k+1)}}{m_{(i-1,k)}+m_{(i-1,k+1)}}\Big)$ if $n_{(i-1,k)}$ is odd. Also, as has been mentioned already in the proof of the last theorem, we have $n_{(i',k)}m_{(i',k+1)} - n_{(i',k+1)}m_{(i',k)} = -1$ for any $i' \geq 1$ and $k \in \{1, 2, \ldots, 3^{i'}\}$. Hence,

$$
\begin{aligned}
\mu\Big(\Big(\frac{n_{(i,3k-1)}}{m_{(i,3k-1)}}, \frac{n_{(i,3k)}}{m_{(i,3k)}}\Big)\Big) &= \frac{1}{(m_{(i-1,k)} + m_{(i-1,k+1)})(m_{(i-1,k)} + 2m_{(i-1,k+1)})} \\
&= \frac{1}{m_{(i-1,k)}m_{(i-1,k+1)}} \cdot \frac{1}{3 + \frac{m_{(i-1,k)}}{m_{(i-1,k+1)}} + 2\frac{m_{(i-1,k+1)}}{m_{(i-1,k)}}} \\
&\leq \frac{1}{m_{(i-1,k)}m_{(i-1,k+1)}} \cdot \frac{1}{\min_{x>0} 3 + \frac{1}{x} + 2x} \\
&= \frac{3 - 2\sqrt{2}}{m_{(i-1,k)}m_{(i-1,k+1)}}
\end{aligned}
$$

if $n_{(i-1,k)}$ is even and

$$
\begin{aligned}
\mu\Big(\Big(\frac{n_{(i,3k-1)}}{m_{(i,3k-1)}}, \frac{n_{(i,3k)}}{m_{(i,3k)}}\Big)\Big) &= \frac{1}{(2m_{(i-1,k)} + m_{(i-1,k+1)})(m_{(i-1,k)} + m_{(i-1,k+1)})} \\
&= \frac{1}{m_{(i-1,k)}m_{(i-1,k+1)}} \cdot \frac{1}{3 + 2\frac{m_{(i-1,k)}}{m_{(i-1,k+1)}} + \frac{m_{(i-1,k+1)}}{m_{(i-1,k)}}} \\
&\leq \frac{1}{m_{(i-1,k)}m_{(i-1,k+1)}} \cdot \frac{1}{\min_{x>0} 3 + \frac{2}{x} + x}
\end{aligned}
$$

$$= \frac{3-2\sqrt{2}}{m_{(i-1,k)}m_{(i-1,k+1)}}$$

if $n_{(i-1,k)}$ is odd. In either case, because the minimum of $3+\frac{1}{x}+2x$ (resp. $3+\frac{2}{x}+x$) is attained at $x=\frac{1}{\sqrt{2}}$ (resp. $\sqrt{2}$) and because $\frac{m_{(i-1,k)}}{m_{(i-1,k+1)}}$ is a rational number, the above equality is not attained. Therefore,

$$\begin{aligned}
&\mu(\{\alpha \in (0,1)\setminus\mathbb{Q} \mid \mathrm{M}_i(\alpha)=\mathtt{M}\}) \\
&= \mu\Big(\bigcup_{k=1}^{3^{i-1}}\Big(\frac{n_{(i,3k-1)}}{m_{(i,3k-1)}},\frac{n_{(i,3k)}}{m_{(i,3k)}}\Big)\Big) \\
&= \sum_{k=1}^{3^{i-1}}\mu\Big(\Big(\frac{n_{(i,3k-1)}}{m_{(i,3k-1)}},\frac{n_{(i,3k)}}{m_{(i,3k)}}\Big)\Big) \\
&< \sum_{k=1}^{3^{i-1}}\frac{3-2\sqrt{2}}{m_{(i-1,k)}m_{(i-1,k+1)}} \\
&= (3-2\sqrt{2})\sum_{k=1}^{3^{i-1}}\mu\Big(\Big(\frac{n_{(i-1,k)}}{m_{(i-1,k)}},\frac{n_{(i-1,k+1)}}{m_{(i-1,k+1)}}\Big)\Big) \\
&= (3-2\sqrt{2})\mu\Big(\bigcup_{k=1}^{3^{i-1}}\Big(\frac{n_{(i-1,k)}}{m_{(i-1,k)}},\frac{n_{(i-1,k+1)}}{m_{(i-1,k+1)}}\Big)\Big) \\
&= (3-2\sqrt{2})\mu\Big((0,1)\setminus\Big\{\frac{n_{(i-1,1)}}{m_{(i-1,1)}},\frac{n_{(i-1,2)}}{m_{(i-1,2)}},\dots,\frac{n_{(i-1,3^i+1)}}{m_{(i-1,3^i+1)}}\Big\}\Big) \\
&= 3-2\sqrt{2}. \qquad \square
\end{aligned}$$

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