ODD-INDEXED FIBONACCI NUMBERS VIA PATTERN-AVOIDING PERMUTATIONS

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Abstract

In this paper, we consider several combinatorial problems whose enumeration leads to the odd-indexed Fibonacci numbers, including certain types of Dyck paths, block fountains, directed column-convex polyominoes, and set partitions with no crossings and no nestings. Our goal is to provide bijective maps to pattern-avoiding permutations and derive generating functions that track certain positional statistics at the permutation level.

1. Introduction

The Fibonacci sequence, defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$, is one of the most common sequences in mathematics. It appears directly or indirectly (e.g., by means of the golden ratio) in various areas of mathematics as well as other fields like computer science, physics, and biology. In this paper, we are interested in the bisecting subsequence $(F_{2n-1})_{n \in \mathbb{N}}$. If we let $a_n = F_{2n-1}$, we then have $a_1 = 1$, $a_2 = 2$, and

$$a_n = 3a_{n-1} - a_{n-2} \text{ for } n \geq 3. \tag{1}$$

This sequence, starting with $1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, \ldots$, is listed as sequence A001519 in the OEIS [9] and has many interesting combinatorial interpretations on its own. We will focus on the following four:

▷ Dyck paths of height at most 3;

▷ Block fountains of coins;

$\triangleright$ Set partitions with no crossings and no nestings;

$\triangleright$ Directed column-convex polyominoes.

Our goal here is to bijectively connect each of these combinatorial families to a corresponding class of pattern-avoiding permutations. It is known that there are nine symmetric classes of permutations, classically avoiding one pattern of length 3 and one of length 4, that are all enumerated by the odd-indexed Fibonacci numbers; see Atkinson [1] or West [11].

Let us briefly review some terminology. A *permutation on* $[n] = \{1, \ldots, n\}$ is a one-to-one function $\sigma : [n] \to [n]$ that can be written as $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$, where $\sigma_i = \sigma(i)$ is the image of i under σ.[1] We let $\mathcal{S}_n$ denote the set of permutations on $[n]$ and let $\mathrm{id}_n = 12 \cdots n$.

Let $\tau \in \mathcal{S}_k$. A permutation σ is said to *contain the pattern* τ if it has a subsequence $\sigma(i_1), \ldots, \sigma(i_k)$ whose elements are in the same relative order as those in τ. For a set P of patterns, we let $\mathcal{S}_n(P)$ denote the set of permutations on $[n]$ that avoid (do not contain) any pattern $\tau \in P$. For example, the permutation $\sigma = 3\,4\,1\,2\,5\,6$ belongs to $\mathcal{S}_n(132, 321)$, but it has multiple occurrences of the other patterns of length 3. For instance, σ contains two occurrences of the pattern $\tau = 231$, namely $(3, 4, 1)$ and $(3, 4, 2)$.

A permutation σ is said to contain the *vincular pattern* $\underline{21}43$ if it contains a 2143 pattern at positions $i_1 < i_2 < i_3 < i_4$ with the additional condition that $i_2 - i_1 = 1$. That is, in one-line notation, the values $\sigma(i_1)$ and $\sigma(i_2)$ must be adjacent. For example, the permutation $\sigma = 3\,4\,1\,6\,2\,5$ contains two 2143 patterns, $(3, 1, 6, 5)$ and $(4, 1, 6, 5)$, but only the latter is a vincular $\underline{21}43$ pattern.

For more on pattern-avoiding permutations, we refer to the book by Kitaev [5].

The results of the paper are presented in four sections. Section 2 covers the class $\mathcal{S}_n(321, 4123)$ and its connection to Dyck paths of semilength n and height at most 3. In this section, we also review some terminology and recall a useful characterization of 321-avoiding permutations. Section 3 focuses on $\mathcal{S}_n(321, \underline{21}43)$ and their connection to block n-fountains. Section 4 examines $\mathcal{S}_n(321, 3412)$, also known as Boolean permutations, where using suitable arc diagrams on n nodes, we provide a simple bijection between Boolean permutations on $[n]$ and noncrossing, nonnesting set partitions of $[n]$. Finally, Section 5 considers $\mathcal{S}_n(231, 3124)$ and their connection to directed column-convex polyominoes of area n.

Each section is similarly structured. We let g_n be the cardinality of the corresponding set of permutations of size n and show that the sequence satisfies the recurrence relation given in Equation (1). We also consider subsets tracking either the position of the 1 or the position of the n in each permutation and derive corresponding enumerative triangles together with their bivariate generating functions. Finally, we discuss the combinatorial family associated with the permutations and

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[1]This way of writing permutations is referred to as *one-line notation*.

provide an explicit bijection. All of our bijective maps are constructive and reveal some information about the structure of the permutations under consideration.

2. Permutations Avoiding 321 and 4123

Before we dive into the content of this section, let us review some notation and discuss some properties of 321-avoiding permutations. Given two permutations π and σ of sizes m and n, respectively, their *direct sum* $\pi \oplus \sigma$ is the permutation of size $m + n$ consisting of π followed by a shifted copy of σ. Similarly, their *skew sum* $\pi \ominus \sigma$ is the permutation consisting of σ preceded by a shifted copy of π. For example, $3\,1\,2 \oplus 2\,1 = 3\,1\,2\,5\,4$ and $3\,1\,2 \ominus 2\,1 = 5\,3\,4\,2\,1$.

For a permutation σ, the value $\sigma(j)$ is called a *left-to-right maximum* of σ if $\sigma(j) > \sigma(i)$ for every $i < j$. It is straightforward to verify that a 321-avoiding permutation is uniquely determined by the positions and values of its left-to-right maxima. Indeed, suppose

$$1 = i_1 < i_2 < \cdots < i_k \leq n \text{ with } k \leq n, \text{ and } 1 \leq v_1 < v_2 < \cdots < v_k = n.$$

Let $A = \{\ell_1, \ldots, \ell_{n-k}\}$ be the complement of $\{i_1, \ldots, i_k\}$ in $\{1, \ldots, n\}$ and let $B = \{u_1, \ldots, u_{n-k}\}$ be the complement of $\{v_1, \ldots, v_k\}$, where $\ell_1 < \cdots < \ell_{n-k}$ and $u_1 < \cdots < u_{n-k}$. Let σ be the permutation on $[n]$ defined by

$$\sigma(i_j) = v_j \text{ for } j \in \{1, \ldots, k\},$$
$$\sigma(\ell_j) = u_j \text{ for } \ell_j \in A, u_j \in B, \text{ and } j \in \{1, \ldots, n-k\}.$$

By construction, σ consists of two increasing sequences and is therefore 321-avoiding.

We proceed with a simple proof of the following known statement.

Proposition 1. *For $n \in \mathbb{N}$, we have $|\mathcal{S}_n(321, 4123)| = F_{2n-1}$.*

Proof. Let $g_n = |\mathcal{S}_n(321, 4123)|$. Clearly, $g_1 = 1$ and $g_2 = 2$. For $n \geq 3$, every permutation $\sigma \in \mathcal{S}_n(321, 4123)$ must have its largest element in one of its last three positions.

Every σ having n at either the last or second-to-last position, can be obtained from a unique element of $\mathcal{S}_{n-1}(321, 4123)$ by inserting n into the corresponding position. Thus, there are $g_{n-1} + g_{n-1}$ such permutations. However, inserting n into the third-to-last position of a permutation of size $n - 1$ does not work if the permutation ends with a descent (it would create a forbidden 321 pattern), so we need to remove those. Now, since a permutation in $\mathcal{S}_{n-1}(321, 4123)$ ending with a descent must have its largest element in the second-to-last position, we conclude that there are g_{n-2} such permutations. Therefore, there is a total of $g_{n-1} - g_{n-2}$ elements of $\mathcal{S}_n(321, 4123)$ having n in the third-to-last position.

In conclusion, $g_1 = 1$, $g_2 = 2$, and $g_n = 3g_{n-1} - g_{n-2}$ for $n \geq 3$. Consequently, $g_n = F_{2n-1}$. □

We now consider the set $\mathcal{S}_n^{k \mapsto 1}(321, 4123)$ of permutations in $\mathcal{S}_n(321, 4123)$ such that $\sigma(k) = 1$. In one-line notation this means that entry 1 is at position k. Note that for every $n > 1$, the set $\mathcal{S}_n^{n \mapsto 1}(321, 4123)$ consists of the single permutation $\sigma = \mathrm{id}_{n-1} \ominus 1$.

Proposition 2. *For $1 \leq k < n$, we have $|\mathcal{S}_n^{k \mapsto 1}(321, 4123)| = k \cdot |\mathcal{S}_{n-k}(321, 4123)|$. The function $g(x, t) = \sum\limits_{n \geq 1} \sum\limits_{k=1}^{n} |\mathcal{S}_n^{k \mapsto 1}(321, 4123)| \, t^k x^n$ satisfies*

$$g(x, t) = \frac{tx}{1 - tx} + \frac{tx}{(1 - tx)^2} \cdot \frac{x - x^2}{1 - 3x + x^2}.$$

Proof. Every permutation $\sigma \in \mathcal{S}_n(321, 4123)$ with $\sigma(1) = 1$ must be of the form $\sigma = 1 \oplus \sigma'$ with $\sigma' \in \mathcal{S}_{n-1}(321, 4123)$. Thus the statement is clear for $k = 1$.

Suppose now that 1 is at position k with $1 < k < n$. If there is an entry $a > k+1$ to the left of 1, then there can be at most $k - 2$ elements from the set $\{2, \ldots, k+1\}$ to the left of 1. So, at least two elements from that set, say $2 \leq b < c \leq k + 1$, would have to be to the right of 1. But then $(a, 1, c, b)$ would make a 321 pattern and $(a, 1, b, c)$ would make a 4123 pattern. Since all $k-1$ entries to the left of 1 must form an increasing sequence, and we now have that the sequence must be made of elements from the set $\{2, \ldots, k + 1\}$, there are $\binom{k}{k-1} = k$ ways to choose such a sequence. The remaining $n - k$ entries, one from $\{2, \ldots, k+1\}$ and $n - k - 1$ from the set $\{k + 2, \ldots, n\}$, can be placed to the right of 1 in any order as long as their reduced permutation avoids the patterns 321 and 4123. There are $|\mathcal{S}_{n-k}(321, 4123)|$ such arrangements, and so $|\mathcal{S}_n^{k \mapsto 1}(321, 4123)| = k \cdot |\mathcal{S}_{n-k}(321, 4123)|$.

To derive $g(x, t)$, we isolate the $k = n$ case and use the above formula to get

$$g(x, t) = \sum_{n=1}^{\infty} t^n x^n + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} k g_{n-k} t^k x^n,$$

where $g_{n-k} = |\mathcal{S}_{n-k}(321, 4123)| = F_{2(n-k)-1}$. Then,

$$g(x, t) = \frac{tx}{1 - tx} + \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} k g_{n-k} (tx)^k x^{n-k}$$

$$= \frac{tx}{1 - tx} + tx \sum_{k=1}^{\infty} k (tx)^{k-1} \sum_{n=k+1}^{\infty} g_{n-k} x^{n-k}$$

$$= \frac{tx}{1 - tx} + \frac{tx}{(1 - tx)^2} \sum_{m=1}^{\infty} g_m x^m$$

$$= \frac{tx}{1 - tx} + \frac{tx}{(1 - tx)^2} \cdot \frac{x - x^2}{1 - 3x + x^2},$$

using the fact that $\sum\limits_{m=1}^{\infty} g_m x^m = \dfrac{x - x^2}{1 - 3x + x^2}$. $\qquad\qquad\qquad\qquad\qquad$ $\square$

A few terms of $|\mathcal{S}_n^{k \mapsto 1}(321, 4123)|$ are listed in Table 1.

$n \setminus k$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	2	2	1					
4	5	4	3	1				
5	13	10	6	4	1			
6	34	26	15	8	5	1		
7	89	68	39	20	10	6	1	
8	233	178	102	52	25	12	7	1

Table 1: Triangle for $|\mathcal{S}_n^{k \mapsto 1}(321, 4123)|$ and $|\mathcal{S}_n^{k \mapsto 1}(231, 3124)|$. See [9, A105292].

2.1. Dyck Paths of Height at Most 3

A *Dyck path of semilength n* is a lattice path from $(0,0)$ to $(2n, 0)$ using up-steps $(1, 1)$, down-steps $(1, -1)$, and never going below the x-axis. An up-step immediately followed by a down-step is called a *peak*. The *height* of a Dyck path D, denoted $\mathrm{height}(D)$, is the height of its highest peak.

There are several bijections between $\mathcal{S}_n(321)$ and the set of Dyck paths of semilength n. The one that best fits our needs is a slightly modified version of a map given by Krattenthaler [6]; let us call it φ_K. Suppose $\sigma \in \mathcal{S}_n(321)$ has left-to-right maxima $\sigma(i_1), \ldots, \sigma(i_k)$, where $1 = i_1 < \cdots < i_k$. Reading the permutation σ from left to right, we construct the Dyck path $D_\sigma = \varphi_K(\sigma)$ as follows. Start with $\sigma(i_1)$ up-steps, followed by $i_2 - i_1$ down-steps. For each $j \in \{2, \ldots, k\}$, we then draw $\sigma(i_j) - \sigma(i_{j-1})$ up-steps, followed by $i_{j+1} - i_j$ down-steps (with the convention $i_{k+1} = n + 1$). Observe that the height of the jth peak of D_σ is

$$\sigma(i_1) - (i_2 - i_1) + (\sigma(i_2) - \sigma(i_1)) - (i_3 - i_2) + \cdots + (\sigma(i_j) - \sigma(i_{j-1})) = \sigma(i_j) - i_j + 1.$$

In particular, we have

$$\mathrm{height}(D_\sigma) > 3 \;\; \text{if and only if} \;\; \sigma(i_j) > i_j + 2 \;\; \text{for some } j \in \{1, \ldots, k\}. \qquad (2)$$

For example, the permutation $\sigma = 2\,4\,5\,1\,3$ has left-to-right maxima 2, 4, 5 at positions $i_1 = 1$, $i_2 = 2$, and $i_3 = 3$, so it corresponds to the path shown in Figure 1.

Proposition 3. *There is a bijection between* $\mathcal{S}_n(321, 4123)$ *and the set of Dyck paths of semilength n and height at most 3.*
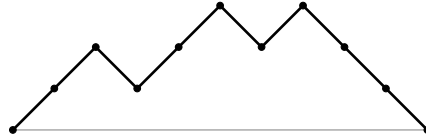
Figure 1: Dyck path corresponding to $\sigma = 2\,4\,5\,1\,3$. It has one peak of height 2 and two peaks of height 3.

Proof. This is a direct consequence of the above bijection φ_K. In fact, if $D_\sigma = \varphi_K(\sigma)$ has height$(D_\sigma) > 3$, then by Equation (2) we have $\sigma(i_j) > i_j + 2$ for some left-to-right maximum value $\sigma(i_j)$. Now, since σ has $i_j - 1$ entries to the left of $\sigma(i_j)$ and since there are $i_j + 2$ values smaller than $\sigma(i_j)$, at least three of these values must appear in increasing order to the right of $\sigma(i_j)$, creating a 4123 pattern. In conclusion, if σ is a $(321, 4123)$-avoiding permutation, then height$(D_\sigma) \leq 3$.

Conversely, let D be a Dyck path and let $\sigma = \varphi_K^{-1}(D)$. If σ has a 4123 pattern, then there is a left-to-right maximum $\sigma(i_j)$ (i.e., $\sigma(i_j) > \sigma(i)$ for every $i < i_j$) and there are values $a < b < c < \sigma(i_j)$ such that $i_j < \sigma^{-1}(a) < \sigma^{-1}(b) < \sigma^{-1}(c)$. In other words, we must have $\sigma(i_j) > (i_j - 1) + 3 = i_j + 2$, which by Equation (2) implies height$(D) > 3$. Therefore, if height$(D) \leq 3$, then $\sigma = \varphi_K^{-1}(D)$ avoids the patterns 321 and 4123. $\qquad\square$

Remark 1. At the level of Dyck paths, the first column in Table 1 gives the number of Dyck paths of height at most 3 that start with an up-step followed by a down-step. For $k > 1$, the table gives the counting of Dyck paths of height at most 3 by the position of the first long descent (more than one consecutive down-steps).

3. Permutations Avoiding 321 and $\underline{21}43$

In this section, we show that the elements of the set $\mathcal{S}_n(321, \underline{21}43)$ are enumerated by the odd-indexed Fibonacci numbers. We also establish a bijection to block fountains.

Proposition 4. *For $n \in \mathbb{N}$, we have $|\mathcal{S}_n(321, \underline{21}43)| = F_{2n-1}$.*

Proof. Let $g_n = |\mathcal{S}_n(321, \underline{21}43)|$. Clearly, $g_1 = 1$, $g_2 = 2$. For $n \geq 3$, we decompose $\mathcal{S}_n(321, \underline{21}43)$ as a disjoint union of three sets, say $A_n^{1*} \cup A_n^{*n} \cup B_n$, where

$$\begin{aligned}
A_n^{1*} &= \{\sigma \in \mathcal{S}_n(321, \underline{21}43) : \sigma(1) = 1\}, \\
A_n^{*n} &= \{\sigma \in \mathcal{S}_n(321, \underline{21}43) : \sigma(1) \neq 1 \text{ and } \sigma(n) = n\}, \\
B_n &= \{\sigma \in \mathcal{S}_n(321, \underline{21}43) : \sigma(1) \neq 1 \text{ and } \sigma(n) \neq n\}.
\end{aligned}$$

Every element of $\mathcal{S}_n(321, \underline{21}43)$ that starts with 1 or ends with n is of the form $1 \oplus \sigma'$ or $\sigma' \oplus 1$, respectively, with $\sigma' \in \mathcal{S}_{n-1}(321, \underline{21}43)$. In each case, there are g_{n-1} such permutations. Moreover, every permutation in $\mathcal{S}_n(321, \underline{21}43)$ that starts with 1 and ends with n is of the form $1 \oplus \sigma'' \oplus 1$ with $\sigma'' \in \mathcal{S}_{n-2}(321, \underline{21}43)$, so there are g_{n-2} of those. Therefore, $|A_n^{1*}| = g_{n-1}$ and $|A_n^{*n}| = g_{n-1} - g_{n-2}$. Finally, there is a bijective map

$$\varphi : B_n \to \mathcal{S}_{n-1}(321, \underline{21}43)$$

defined as follows. Let $\sigma \in B_n$ and let $\sigma(i_1), \ldots, \sigma(i_m)$ be its left-to-right maxima listed in increasing order. Note that $i_1 = 1$, $\sigma(i_1) > 1$, $\sigma(i_m) = n$, and $i_m < n$. We let $\tau = \varphi(\sigma)$ be the unique permutation in $\mathcal{S}_{n-1}(321)$ with left-to-right maxima $\tau(i_1), \ldots, \tau(i_m)$, where $\tau(i_j) = \sigma(i_j) - 1$ for $j \in \{1, \ldots, m\}$. For example,

$$2\,5\,1\,6\,3\,4 = \mathbf{2\,5}\,1\,\mathbf{6}\,3\,4 \mapsto \mathbf{1\,4}\,2\,\mathbf{5}\,3.$$

Suppose τ contains a leftmost $\underline{21}43$ pattern, say (b, a, d, c) with $a < b < c < d$. Then b and d are left-to-right maxima of τ, entries b and a are adjacent, and $d - 1$ must be to the right of d. Therefore, σ must have $b + 1$ and $d + 1$ as left-to-right maxima at the same positions of b and d in τ, and d cannot be a left-to-right maximum of σ because $d - 1$ is not one for τ. This means that $\sigma^{-1}(d) > \sigma^{-1}(d + 1)$. Thus, if a' is the element adjacent to the right of $b + 1$ in σ, then $(b + 1, a', d + 1, d)$ is a $\underline{21}43$ pattern. In other words, if σ avoids $\underline{21}43$, so does $\varphi(\sigma)$. This map is clearly bijective, which implies $|B_n| = g_{n-1}$. In conclusion, $g_1 = 1$, $g_2 = 2$, and $g_n = 3g_{n-1} - g_{n-2}$ for $n \geq 3$. Consequently, $g_n = F_{2n-1}$. $\qquad \square$

For the application discussed at the end of this section, it is meaningful to count the elements of $\mathcal{S}_n(321, \underline{21}43)$ by the position of their largest entry. Let $\mathcal{S}_n^{k \mapsto n}(321, \underline{21}43)$ be the set of permutations in $\mathcal{S}_n(321, \underline{21}43)$ such that $\sigma(k) = n$.

Proposition 5. *Let* $a_{n,k} = |\mathcal{S}_n^{k \mapsto n}(321, \underline{21}43)|$. *We have*

$$a_{n,1} = 1 \ \text{ for } n \geq 1,$$
$$a_{n,n} = F_{2n-3} \ \text{ for } n > 1,$$
$$a_{n,k} = a_{n-1,k-1} + a_{n-1,k} \ \text{ for } 1 < k < n.$$

The generating function $g(x,t) = \sum\limits_{n \geq 1} \sum\limits_{k=1}^{n} a_{n,k}\, t^k x^n$ *satisfies*

$$g(x,t) = \frac{tx(1 - tx)(1 - 2tx)}{(1 - x - tx)(1 - 3tx + t^2 x^2)}.$$

Proof. The set $\mathcal{S}_n^{1 \mapsto n}(321, \underline{21}43)$ consists of the permutation $1 \ominus \mathrm{id}_{n-1}$, so $a_{n,1} = 1$. As discussed in the proof of Proposition 4, every element $\sigma \in \mathcal{S}_n(321, \underline{21}43)$ can be uniquely obtained from an element $\sigma' \in \mathcal{S}_{n-1}(321, \underline{21}43)$ as a direct sum, $1 \oplus \sigma'$

or $\sigma' \oplus 1$, or by using the inverse map φ^{-1}. In particular, every permutation in $\mathcal{S}_n^{n \mapsto n}(321, \underline{2}143)$ is of the form $\sigma' \oplus 1$, thus $a_{n,n} = |\mathcal{S}_{n-1}(321, \underline{2}143)| = F_{2n-3}$.

Now, for $1 < k < n$, every $\sigma \in \mathcal{S}_n^{k \mapsto n}(321, \underline{2}143)$ is either of the form $\sigma = 1 \oplus \sigma'$ with $\sigma' \in \mathcal{S}_{n-1}^{k-1 \mapsto n-1}(321, \underline{2}143)$ or $\sigma = \varphi^{-1}(\sigma')$ with $\sigma' \in \mathcal{S}_{n-1}^{k \mapsto n-1}(321, \underline{2}143)$. Note that φ and φ^{-1} preserve the position of the largest element. In conclusion, $a_{n,k} = a_{n-1,k-1} + a_{n-1,k}$.

The claimed expression for $g(x,t)$ follows from routine algebraic manipulations, together with the generating function for the odd-indexed Fibonacci numbers. $\qquad\square$

$n \setminus k$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	2	2					
4	1	3	4	5				
5	1	4	7	9	13			
6	1	5	11	16	22	34		
7	1	6	16	27	38	56	89	
8	1	7	22	43	65	94	145	233

Table 2: Triangle for $|\mathcal{S}_n^{k \mapsto n}(321, \underline{2}143)|$. Reverse of A121460 in [9].

3.1. Block Fountains of Coins

A *block n-fountain* of coins is an arrangement of coins in rows such that the bottom row consists of n coins forming a contiguous block, and each higher row consists of a single contiguous block of coins where each coin touches exactly two coins from the row beneath it; see Figure 2.
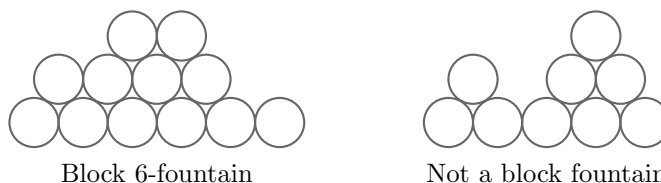

Block 6-fountain   Not a block fountain

Figure 2: Examples.

As shown in Wilf [12, Section 2.1, Example 7], there are F_{2n-1} block n-fountains.

A coin in a block fountain is called a *peak* if it does not touch any coin in a higher row. A block n-fountain with maximum number of coins will be called a *triangular n-stack*. Note that a triangular stack has only one peak.

Proposition 6. *There is a bijection between the set of block n-fountains and the set of* $(321, \underline{21}43)$-*avoiding permutations of size n.*

Proof. We will provide an algorithm to go from block fountains to permutations, illustrating the steps with an example. Given a block n-fountain, proceed as follows:

(i) Going from left to right, label the coins on the bottom row with the elements of $[n]$ in increasing order.

(ii) Identify the peaks of the fountain. For every peak not at the bottom, draw diagonals (of slope $\pm\sqrt{3}$) from the center of the peak to the two coins on the bottom row along those diagonals. An example is shown in Figure 3.
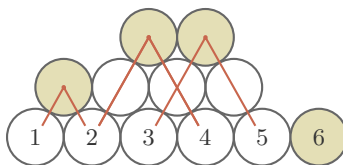
Figure 3: Labeled block 6-fountain

(iii) If a bottom coin i is a peak, we let $\sigma(i) = i$. Otherwise, if a peak at a higher row has diagonals pointing to the bottom coins i and j with $i < j$, we let $\sigma(i) = j$.

(iv) If the fountain has k peaks, the previous steps give positions $1 = i_1 < \cdots < i_k$ and corresponding values $\sigma(i_1) < \cdots < \sigma(i_k) = n$. We let σ be the unique permutation in $\mathcal{S}_n(321)$ having left-to-right maxima $\sigma(i_1), \ldots, \sigma(i_k)$.

For the example in Figure 3, we get the permutation $2\,4\,5\,1\,3\,6$.

Suppose σ contains a $\underline{21}43$ pattern at positions $(i, i+1, j, k)$ with $i+1 < j < k$. This means that $\sigma(i)$ and $\sigma(j)$ are left-to-right maxima, and $\sigma(i+1) < \sigma(i) < \sigma(k) < \sigma(j)$. In particular, $j - i \geq 2$ and $\sigma(j) - \sigma(i) \geq 2$. Without loss of generality, assume $j - i = 2$ and $\sigma(j) - \sigma(i) = 2$. In this case, part of the fountain must have two peaks and be of the form shown in Figure 4, which is not a block fountain because the coins at the top row are not contiguous. More generally, if $j - i > 2$, the gap between the peaks would be longer, and if $\sigma(j) - \sigma(i) > 2$, the peak along diagonal j would just be higher.

In conclusion, all permutations constructed as above avoid both 321 and $\underline{21}43$.

The inverse map is obtained by reversing the algorithm. Given $\sigma \in \mathcal{S}_n(321, \underline{21}43)$, let $i_1, \ldots, i_k$ be the locations of its left-to-right maxima, listed in increasing order. Note that for every $j \in \{1, \ldots, k\}$, we must have $\sigma(i_j) \geq i_j$. We start our fountain construction by placing a block of n coins as base, labeling them with the elements
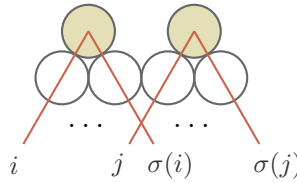
Figure 4: Not a block fountain.

of $[n]$ in increasing order. Then, for every j, we place the needed coins to make a triangular $(\sigma(i_j) - i_j + 1)$-stack whose base are the coins labeled i_j through $\sigma(i_j)$. As we did above, one can argue that any gap of coins in a row of the fountain would imply a $\underline{21}43$ pattern in σ. In other words, the fountain resulting from our construction must be a block n-fountain. $\square$

4. Boolean Permutations

Boolean permutations are those that avoid the patterns 321 and 3412. They are known to be enumerated by the odd-indexed Fibonacci numbers; see Tenner [8].

Proposition 7. *For $n \in \mathbb{N}$, we have $|\mathcal{S}_n(321, 3412)| = F_{2n-1}$.*

Proof. Let $g_n = |\mathcal{S}_n(321, 3412)|$. As stated before, it is clear that $g_1 = 1$ and $g_2 = 2$. Now, for $n \geq 3$ and $\sigma \in \mathcal{S}_n(321, 3412)$, we consider the disjoint cases $\sigma(1) = 1$, $\sigma(2) = 1$, or $\sigma(j) = 1$ for some $j > 2$. Every σ with $\sigma(1) = 1$ or $\sigma(2) = 1$ can be uniquely obtained from a $\sigma' \in \mathcal{S}_{n-1}(321, 3412)$ by increasing all the entries of σ' by one and inserting the 1 either at position 1 or position 2, respectively. Thus there are $2g_{n-1}$ such permutations.

If σ has the 1 at position $j > 2$, then $\sigma(1) = 2$; otherwise entry 2 would create either a 321 pattern (if it is to the left of 1, but not at position 1) or a 3412 pattern (if it is to the right of 1). Such a permutation can be uniquely obtained by inserting the 2 at position 1 into a permutation from $\mathcal{S}_{n-1}(321, 3412)$ that does not start with 1. There are $g_{n-1} - g_{n-2}$ permutations of this type.

In conclusion, $g_1 = 1$, $g_2 = 2$, and $g_n = 3g_{n-1} - g_{n-2}$ for $n \geq 3$. Consequently, $g_n = F_{2n-1}$. $\square$

Similar to Section 3, we consider the set $\mathcal{S}_n^{k \mapsto 1}(321, 3412)$ of permutations in $\mathcal{S}_n(321, 3412)$ having the 1 in position k, i.e., $\sigma(k) = 1$.

Proposition 8. *For $n > 1$, we have*

$$|\mathcal{S}_n^{1 \mapsto 1}(321, 3412)| = |\mathcal{S}_n^{2 \mapsto 1}(321, 3412)| = F_{2n-3},$$

and for $3 \leq k \leq n$, we have $|\mathcal{S}_n^{k \mapsto 1}(321, 3412)| = |\mathcal{S}_{n-1}^{k-1 \mapsto 1}(321, 3412)|$. Moreover, the function $g(x,t) = \sum\limits_{n \geq 1} \sum\limits_{k=1}^{n} |\mathcal{S}_n^{k \mapsto 1}(321, 3412)| \, t^k x^n$ satisfies

$$g(x,t) = \frac{tx - 2tx^2 + t^2x^3}{(1-tx)(1-3x+x^2)}.$$

Proof. This follows from the discussion in the proof of the previous proposition. First of all, every $\sigma \in \mathcal{S}_n(321, 3412)$ with $\sigma(1) = 1$ or $\sigma(2) = 1$ can be uniquely obtained from a permutation in $\mathcal{S}_{n-1}(321, 3412)$ by inserting the 1 at the corresponding position. This leads to the claimed formula for $k = 1$ and $k = 2$. For $k \geq 3$, every element of $\mathcal{S}_n^{k \mapsto 1}(321, 3412)$ can be uniquely obtained by inserting 2 at position 1 into an element of $\mathcal{S}_{n-1}^{k-1 \mapsto 1}(321, 3412)$. This proves the claimed formula for $k \geq 3$.

As in previous propositions, the claimed rational expression for $g(x,t)$ follows from straightforward algebraic manipulations. $\qquad\square$

$n \setminus k$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	2	2	1					
4	5	5	2	1				
5	13	13	5	2	1			
6	34	34	13	5	2	1		
7	89	89	34	13	5	2	1	
8	233	233	89	34	13	5	2	1

Table 3: Triangle for $|\mathcal{S}_n^{k \mapsto 1}(321, 3412)|$.

4.1. Noncrossing, Nonnesting Set Partitions

We finish this section with a discussion on noncrossing, nonnesting set partitions, and their representation via arc diagrams. Our goal is to provide a simple bijection between these objects and the set of $(321, 3412)$-avoiding permutations.

A *partition of the set* $[n] = \{1, \ldots, n\}$ is a set of disjoint nonempty sets (called blocks) whose union is $[n]$. Every partition π of $[n]$ can be represented by an arc diagram obtained by drawing an arc between each pair of integers that appear consecutively in the same block of π. An example is shown in Figure 5.

Two arcs (i_1, j_1) and (i_2, j_2) make a *crossing* if $i_1 < i_2 < j_1 < j_2$, and they make a *nesting* if $i_1 < i_2 < j_2 < j_1$. A *noncrossing/nonnesting partition* is a
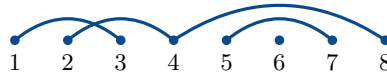
Figure 5: Representation of $\pi = \{\{1,3\}, \{2,4,8\}, \{5,7\}, \{6\}\}$.

partition with no crossings/nestings. Moreover, a partition of $[n]$ is said to be *indecomposable* if no subset of its blocks is a partition of $[k]$ with $k < n$. In other words, an indecomposable partition is one whose corresponding arc diagram cannot be separated into two disjoint arc diagrams with consecutive nodes. The partition $\{\{1,3\}, \{2,4,8\}, \{5,7\}, \{6\}\}$, shown above, is an example of an indecomposable partition with one crossing and one nesting.

We let $\mathcal{P}_{\mathrm{ncn}}(n)$ be the set of partitions of $[n]$ that are both noncrossing and nonnesting, and let $\mathcal{P}^i_{\mathrm{ncn}}(n)$ be the subset of such partitions that are indecomposable. Note that every partition in $\mathcal{P}^i_{\mathrm{ncn}}(n)$ consists of a block of size j for some $j \geq 2$, containing 1 and n, together with $n - j$ singletons. It is known that $|\mathcal{P}_{\mathrm{ncn}}(n)| = F_{2n-1}$; see Marberg [7, Example 4.2].

Proposition 9. *There is a bijection between the set of indecomposable, noncrossing, nonnesting set partitions of $[n]$ and the set of $(321, 3412)$-avoiding indecomposable permutations on $[n]$.[2]*

Proof. First, we map $\{1\} \mapsto 1$ and $\{1,2\} \mapsto 21$. Let $n \geq 3$ and $\pi \in \mathcal{P}^i_{\mathrm{ncn}}(n)$, say

$$\pi = \{\{u_0, \ldots, u_j\}, \{s_1\}, \ldots, \{s_k\}\}$$

with $j + 1 + k = n$, $u_0 = 1$, $u_j = n$, $u_0 < u_1 < \cdots < u_j$, and $2 \leq s_1 < \cdots < s_k \leq n-1$. We let σ_π be the unique element of $\mathcal{S}_n(321)$ with left-to-right maxima $u_1, \ldots, u_j$ at positions $1, u_1, \ldots, u_{j-1}$, respectively. In other words, if we let $s_0 = 1$, then

$$\sigma_\pi(1) = u_1, \quad \sigma_\pi(n) = s_k,$$

$$\sigma_\pi(u_{i-1}) = u_i \text{ for } i \in \{2, \ldots, j\},$$

$$\sigma_\pi(s_i) = s_{i-1} \text{ for } i \in \{1, \ldots, k\}.$$

For example, the partition $\{\{1,2,4,5,8\}, \{3\}, \{6\}, \{7\}\} \in \mathcal{P}^i_{\mathrm{ncn}}(8)$ corresponds to the permutation $2\,4\,1\,5\,8\,3\,6\,7$. This map can be visualized as follows:

 $\rightsquigarrow$ $2\,4\,1\,5\,8\,3\,6\,7.$
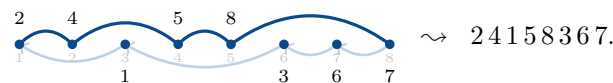
The elements of $\mathcal{P}^i_{\mathrm{ncn}}(5)$ are listed in Table 4 together with their corresponding permutations.

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[2]A permutation is called *indecomposable* if it is not a direct sum of two nonempty permutations.

Observe that for every $\pi \in \mathcal{P}^i_{\mathrm{ncn}}(n)$, the permutation σ_π consists of a single cycle, $\sigma_\pi = (1 \ u_1 \ \cdots \ u_j \ s_k \ \cdots \ s_1)$, and is therefore indecomposable.

If σ_π were to contain a 3412 pattern, then the 3 and 4 will have to be left-to-right maxima (otherwise σ_π would have a 321 pattern). Let u_{i-1} and u_i be nodes in the arc diagram of π connected by a single arc:

By our construction of σ_π (in one-line notation), there are $u_{i-1} - 1$ values to the left of and smaller than u_i, and $u_i - u_{i-1} - 1$ values of σ_π strictly between the entries u_i and u_{i+1} (all smaller than u_i). Thus, there are $(u_{i-1} - 1) + (u_i - u_{i-1} - 1) = u_i - 2$ numbers smaller than u_i to the left of u_{i+1}. Therefore, there can be at most one element smaller than u_i to the right of u_{i+1}, which means that a 3412 pattern is not possible.

In conclusion, σ_π is an indecomposable element of $\mathcal{S}_n(321, 3412)$.

This map is clearly invertible. Given an indecomposable $\sigma \in \mathcal{S}_n(321, 3412)$ with left-to-right maxima $u_1, \ldots, u_j$, listed in increasing order, we must have $u_1 > 1$, $\sigma^{-1}(u_1) = 1$, $u_j = n$, and $\sigma^{-1}(u_j) < n$. We then construct an arc diagram with n nodes, labeled 1 through n from left to right, by connecting the first and last nodes with consecutive arcs passing through the nodes labeled $1, u_1, \ldots, u_j$. The resulting diagram gives the indecomposable, noncrossing, nonnesting partition of $[n]$,

$$\pi_\sigma = \{\{1, u_1, \ldots, u_j\}, \{s_1\}, \ldots, \{s_{n-j-1}\}\},$$

where $s_1, \ldots, s_{n-j-1}$ are the labels of the $n-j-1$ isolated nodes of the diagram. $\square$

Corollary 1. *There is a bijection between the set of noncrossing, nonnesting set partitions of $[n]$ and the set of $(321, 3412)$-avoiding permutations on $[n]$.*

This follows by applying the map from Proposition 9 to each indecomposable component. If π has a decomposition $\pi_1|\pi_2|\cdots|\pi_k$, then we let

$$\sigma_\pi = \sigma_{\pi_1} \oplus \sigma_{\pi_2} \oplus \cdots \oplus \sigma_{\pi_k}.$$

For example, for $\pi = \{\{1, 2, 4\}, \{3\}, \{5\}, \{6, 8\}, \{7\}\}$, we have $\pi = \pi_1|\pi_2|\pi_3$ with

$$\pi_1 = \{\{1, 2, 4\}, \{3\}\}, \ \pi_2 = \{1\}, \ \text{and} \ \pi_3 = \{\{1, 3\}, \{2\}\}.$$

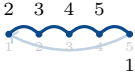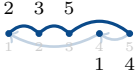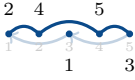Then $\sigma_{\pi_1} = 2\,4\,1\,3$, $\sigma_{\pi_2} = 1$, $\sigma_{\pi_3} = 3\,1\,2$, and $\sigma_\pi = 2\,4\,1\,3 \oplus 1 \oplus 3\,1\,2 = 2\,4\,1\,3\,5\,8\,6\,7$.
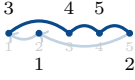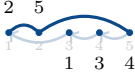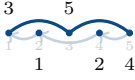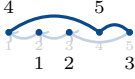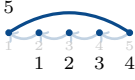
π	Diagram	σ_π
$\{\{1,2,3,4,5\}\}$		$2\,3\,4\,5\,1$
$\{\{1,2,3,5\},\{4\}\}$		$2\,3\,5\,1\,4$
$\{\{1,2,4,5\},\{3\}\}$		$2\,4\,1\,5\,3$
$\{\{1,3,4,5\},\{2\}\}$		$3\,1\,4\,5\,2$
$\{\{1,2,5\},\{3\},\{4\}\}$		$2\,5\,1\,3\,4$
$\{\{1,3,5\},\{2\},\{4\}\}$		$3\,1\,5\,2\,4$
$\{\{1,4,5\},\{2\},\{3\}\}$		$4\,1\,2\,5\,3$
$\{\{1,5\},\{2\},\{3\},\{4\}\}$		$5\,1\,2\,3\,4$

Table 4: Bijection between $\mathcal{P}^i_{\mathrm{ncn}}(5)$ and indecomposable elements of $\mathcal{S}_5(321, 3412)$.

Remark 2. If π is a partition of $[n]$ with blocks $B_1, \ldots, B_j$, we let

$$m(\pi) = \min\{\max(B_1), \ldots, \max(B_j)\}.$$

The kth column of Table 3 gives the number of partitions $\pi \in \mathcal{P}_{\mathrm{ncn}}(n)$ such that $m(\pi) = k$.

5. Permutations Avoiding 231 and 3124

As in previous sections, we start with a simple proof of the known fact that the elements of $\mathcal{S}_n(231, 3124)$ are counted by the odd-indexed Fibonacci numbers.

Proposition 10. *For $n \in \mathbb{N}$, we have $|\mathcal{S}_n(231, 3124)| = F_{2n-1}$.*

Proof. Let $g_n = |\mathcal{S}_n(231, 3124)|$. For $n \geq 3$ and $\sigma \in \mathcal{S}_n(231, 3124)$, we consider the disjoint cases $\sigma(1) = 1$, $\sigma(1) = n$, or $1 < \sigma(1) < n$. The first two types are of the

form $1 \oplus \sigma'$ and $1 \ominus \sigma'$, respectively, for some $\sigma' \in \mathcal{S}_{n-1}(231, 3124)$. Thus there are g_{n-1} permutations of each type for a combined total of $2g_{n-1}$.

Let $A_n = \{\tau \in \mathcal{S}_n(231, 3124) : 1 < \tau(1) < n\}$ and let $\sigma \in A_n$. If $\sigma(1) < \sigma(2)$, then $(\sigma(1), \sigma(2), 1)$ would form a forbidden 231 pattern, and thus $\sigma(1) - \sigma(2) > 0$. Now, if $\sigma(1) - \sigma(2) > 1$, then there would exist $a \in [n]$ with $\sigma(1) > a > \sigma(2)$. In that case, either $(\sigma(1), n, a)$ would form a 231 pattern or $(\sigma(1), \sigma(2), a, n)$ would form a 3124 pattern. Since both patterns are avoided by σ, we conclude that

$$\sigma(2) = \sigma(1) - 1.$$

This means that every permutation $\sigma \in A_n$ can be uniquely obtained from a permutation $\sigma' \in \mathcal{S}_{n-1}(231, 3124)$ with $\sigma'(1) < n - 1$ (there are $g_{n-1} - g_{n-2}$ of those) by 21-inflating its first entry. More precisely, for every σ' as above, we let $\sigma \in \mathcal{S}_n(231, 3124)$ be defined by $\sigma(1) = \sigma'(1) + 1$, and for $j \in \{2, \dots, n\}$, we let

$$\sigma(j) = \begin{cases} \sigma'(j-1) & \text{if } \sigma'(j-1) \leq \sigma'(1), \\ \sigma'(j-1) + 1 & \text{if } \sigma'(j-1) > \sigma'(1). \end{cases}$$

For example, the three permutations in $\mathcal{S}_4(231, 3124)$ not starting with 1 or 4 correspond to the three elements of $\mathcal{S}_3(231, 3124)$ not starting with 3:

$$1\,2\,3 \rightsquigarrow 2\,1\,3\,4, \quad 1\,3\,2 \rightsquigarrow 2\,1\,4\,3, \quad 2\,1\,3 \rightsquigarrow 3\,2\,1\,4.$$

In conclusion, $g_n = 3g_{n-1} - g_{n-2}$ for $n \geq 3$. Since $g_1 = 1$ and $g_2 = 2$, we have $g_n = F_{2n-1}$. $\qquad\square$

We now consider the subset $\mathcal{S}_n^{k \mapsto 1}(231, 3124) \subset \mathcal{S}_n(231, 3124)$ of permutations having the 1 in position k. Observe that the set $\mathcal{S}_n^{n \mapsto 1}(231, 3124)$ only contains the decreasing permutation $\sigma = n\,(n-1)\cdots 1$.

Proposition 11. *For $1 \leq k < n$, we have $|\mathcal{S}_n^{k \mapsto 1}(231, 3124)| = k \cdot |\mathcal{S}_{n-k}(231, 3124)|$.*
As in Proposition 2, the function $g(x, t) = \sum\limits_{n \geq 1} \sum\limits_{k=1}^{n} |\mathcal{S}_n^{k \mapsto 1}(231, 3124)| \, t^k x^n$ satisfies

$$g(x, t) = \frac{tx}{1 - tx} + \frac{tx}{(1 - tx)^2} \cdot \frac{x - x^2}{1 - 3x + x^2}.$$

Proof. Every $\sigma \in \mathcal{S}_n(231, 3124)$ that starts with 1 can be written as $\sigma = 1 \oplus \sigma'$, where σ' is an element of $\mathcal{S}_{n-1}(231, 3124)$. Thus $|\mathcal{S}_n^{1 \mapsto 1}(231, 3124)| = |\mathcal{S}_{n-1}(231, 3124)|$, as claimed.

Suppose that σ is such that $\sigma(k) = 1$ for $1 < k < n$. With the same argument used in the proof of Proposition 10, we deduce that, if $1 < \sigma(1) < n$, then σ must start with the elements of $[k]$ in decreasing order. This means, $\sigma = \delta_k \oplus \sigma''$, where δ_k is the decreasing permutation on $[k]$ and $\sigma'' \in \mathcal{S}_{n-k}(231, 3124)$.

On the other hand, if $\sigma(k) = 1$ for $1 < k < n$ and $\sigma(1) = n$, then the first k entries of σ must consist of two decreasing sequences of consecutive numbers. Indeed, if there were two gaps in the values of σ to the left of 1, there would exist entries a, b appearing to the right of 1 such that $n > a > \sigma(i) > b > \sigma(i+1)$ for some $i \in \{2, \ldots, k-1\}$. Now, if a is to the left of b, then $(\sigma(i), a, b)$ would form a 231 pattern, and if a is to the right of b, then $(\sigma(i), 1, b, a)$ would form a 3124 pattern. In other words, in this case, σ has exactly one gap in its first descending run. Hence $\sigma = \delta_j \ominus (\delta_{k-j} \oplus \sigma'')$ for some $j \in \{1, \ldots, k-1\}$, where δ_j and δ_{k-j} are the decreasing permutations on $[j]$ and $[k-j]$, and $\sigma'' \in \mathcal{S}_{n-k}(231, 3124)$. See Figure 6 for an illustration.
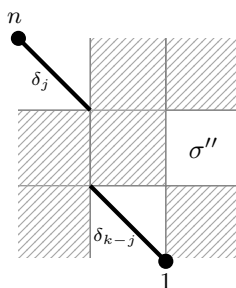

Figure 6: Typical $\sigma \in \mathcal{S}_n(231, 3124)$ with $\sigma(k) = 1$ and $\sigma(1) = n$.

In conclusion, for $1 < k < n$, every $\sigma'' \in \mathcal{S}_{n-k}(231, 3124)$ leads to k different permutations in $\mathcal{S}_n^{k \mapsto 1}(231, 3124)$. This finishes the proof. $\square$

Remark 3. Let δ_0 be the empty permutation. In the previous proof, we have shown that every $\sigma \in \mathcal{S}_n(231, 3124)$ with $\sigma(k_1) = 1$ is of the form $\sigma = \delta_{j_1} \ominus (\delta_{k_1 - j_1} \oplus \sigma_{k_1})$ for some $j_1 \in \{0, 1, \ldots, k_1 - 1\}$ and $\sigma_{k_1} \in \mathcal{S}_{n-k_1}(231, 3124)$. The argument can certainly be iterated. If σ_{k_1} has the 1 at position k_2, then $\sigma_{k_1} = \delta_{j_2} \ominus (\delta_{k_2 - j_2} \oplus \sigma_{k_2})$ for some $j_2 \in \{0, 1, \ldots, k_2 - 1\}$ and $\sigma_{k_2} \in \mathcal{S}_{n-k_1-k_2}(231, 3124)$, and therefore

$$\sigma = \delta_{j_1} \ominus (\delta_{k_1 - j_1} \oplus (\delta_{j_2} \ominus (\delta_{k_2 - j_2} \oplus \sigma_{k_2}))).$$

If σ has $d+1$ descending runs, then the above process can be iterated d times until σ_{k_d} is the decreasing permutation on $[n - k_1 - \cdots - k_d]$. In particular, this implies that all the descending runs of σ consist of one or two sequences of consecutive numbers.

5.1. Directed Column-Convex Polyominoes

We finish this section with a direct bijection between $\mathcal{S}_n(231, 3124)$ and the set of directed column-convex (DCC) polyominoes of area n. A *directed polyomino* is one that can be built by starting with a single cell and adding new cells on the right or

on the top of an existing cell. A polyomino is called *column-convex* if every column consists of contiguous cells. The *area* of a polyomino is the number of its cells. An example of a DCC polyomino is shown in Figure 7.
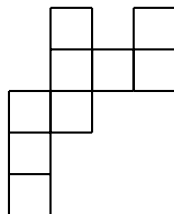

Figure 7: Directed column-convex polyomino of area 9.

It is known (see e.g., Delest and Delucq [3], Barcucci et. al. [2]) that the number of DCC polyominoes of area n is the Fibonacci number F_{2n-1}.

Proposition 12. *There is a bijection between the set of directed column-convex polyominoes of area n and the set of $(231, 3124)$-avoiding permutations of size n.*

Proof. Let P be a DCC polyomino with k columns and n cells. We will construct a corresponding permutation σ_P in $\mathcal{S}_n(231, 3124)$, having k descending runs. The steps of our algorithm are illustrated in Figure 8 using the polyomino from Figure 7.

(i) First, going through the columns of P from left to right, mark the cell that aligns with the bottom cell of the adjacent column to its right.

(ii) Going through the columns from left to right, use the elements of $[n]$ in consecutive order (starting with 1) to label the cells of each column from the top down to the cell that was marked in the previous step.

(iii) Finally, going through the columns of P from right to left, continue labeling the remaining cells in consecutive order from top to bottom. When all cells are filled with the elements of $[n]$, we build the permutation σ_P from left to right by reading the numbers in each column from bottom to top as its descending runs.

More examples are shown in Figure 9. By construction, the resulting permutation has the nesting properties discussed in Remark 3. So, $\sigma_P \in \mathcal{S}_n(231, 3124)$.

The above algorithm is reversible. Let $\sigma \in \mathcal{S}_n(231, 3124)$ with $\sigma = w_1 w_2 \cdots w_k$, where $w_1, \ldots, w_k$ are its descending runs. As discussed before, every w_j is of the form $w_j = u_j v_j$, where u_j and v_j are words consisting of decreasing consecutive numbers (allowing u_j to be the empty word). For every $j \in \{1, \ldots, k\}$, we let C_j be the polyomino consisting of $|w_j| = |u_j| + |v_j|$ cells in a single column. We now construct a corresponding polyomino P_σ with n cells by connecting $C_1, \ldots, C_k$ as
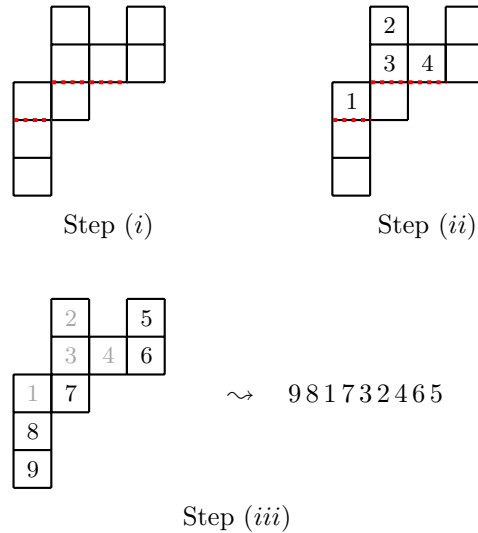
Figure 8: Example of bijection from Proposition 12.

follows. Once column C_j is placed, column C_{j+1} will be attached to its right in such a way that the base cell of C_{j+1} is adjacent to the $|v_j|$-th cell from the top of C_j. Clearly, P_σ is a DCC polyomino of area n. $\square$

Remark 4. In the above bijection, the elements of $\mathcal{S}_n^{k\mapsto 1}(231, 3124)$ correspond to DCC polyominoes of area n whose first column has exactly k cells.

Remark 5. Another class of combinatorial objects counted by the odd-indexed Fibonacci numbers are the nondecreasing Dyck paths. These are Dyck paths for which the sequence of the altitudes of the valleys is nondecreasing. A bijection between DCC polyominoes and nondecreasing Dyck paths can be found in Deutsch and Prodinger [4, Section 3]. Moreover, as discussed in Vella [10], the standard bijection[3] from 132-avoiding permutations to Dyck paths gives a bijection from (132, 3241)-avoiding permutations to nondecreasing Dyck paths.

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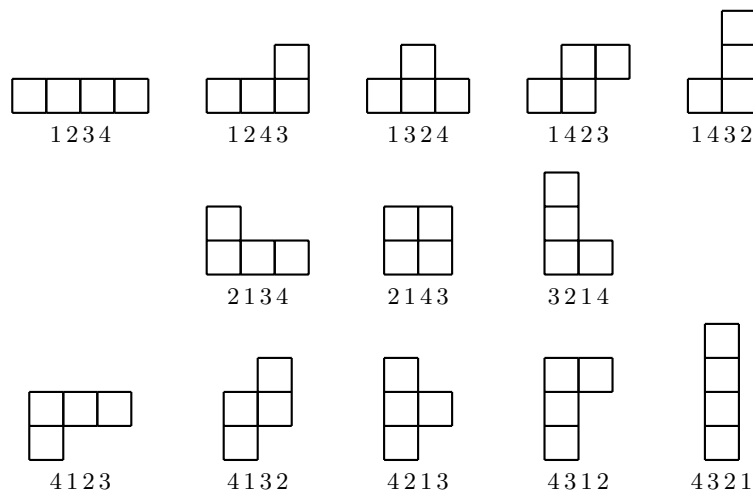
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[3]Also known as first-return bijection.

Figure 9: DCC polyominoes of area 4 and their corresponding $(231, 3124)$-avoiding permutations of size 4.

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